Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Question 1: Let u be a vector field in $\mathbb{R}^d$, (d is the space dimension). Let p be a scalar field in $\mathbb{R}^d$.
(a) Using the product rule, express $\text{div}(pu)$ in terms of $\nabla{\cdot}u$ and ∇p. (You may use $\text{div}u$ or $\nabla{\cdot}u$ to denote the divergence of u and ∇p or $\text{grad}p$ to denote the gradient of p. The space dimension is d.)

Using the product rule we have

$$\begin{aligned}\nabla{\cdot}(pu) &= \sum_{i=1}^{d} \partial_i(pu_i) = \sum_{i=1}^{d} u_i\partial_i p + p\partial_i u_i \\ &= u{\cdot}\nabla p + p\nabla{\cdot}u\end{aligned}$$

In conclusion $\nabla{\cdot}(pu) = u{\cdot}\nabla p + p\nabla{\cdot}u$.

(b) Let u be a smooth vector field in $\Omega \subset \mathbb{R}^d$ with zero divergence and zero normal component at the boundary of Ω. Let p be a smooth scalar field in Ω. Compute $\int_\Omega p(x)\nabla{\cdot}u(x)\mathrm{d}x$ and $\int_\Omega u(x){\cdot}\nabla p(x)\mathrm{d}x$.

(i) Using (a) and the fact that u is divergence free, we have

$$\int_\Omega u{\cdot}\nabla p\mathrm{d}x = \int_\Omega (\nabla{\cdot}(pu) - p\nabla{\cdot}u)\mathrm{d}x = \int_\Omega \nabla{\cdot}(pu)\mathrm{d}x.$$

The divergence theorem (fundamental theorem of calculus in d space dimension) implies that

$$\int_\Omega u{\cdot}\nabla p\mathrm{d}x = \int_{\partial\Omega} pu{\cdot}n\mathrm{d}s,$$

where $\partial\Omega$ denotes the boundary of Ω. Since $u{\cdot}n = 0$ at the boundary, we finally infer that $\int_\Omega u{\cdot}\nabla p\mathrm{d}x = 0$.

(ii) Note finally that $\int_\Omega p\nabla{\cdot}u\mathrm{d}x = 0$ since $\nabla{\cdot}u = 0$.

Question 2: Consider the vibrating beam equation $\partial_{tt}u(x,t)+\partial_{xxxx}u(x,t) = 0$, $x \in (-\infty,+\infty)$, $t > 0$ with $u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$, $\partial_{xx}u(\pm\infty,t) = 0$. Use the energy method to compute $\partial_t \int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + [\partial_{xx}u(x,t)]^2)\mathrm{d}x$. (Hint: test the equation with $\partial_t u(x,t)$).

Using the hint we have

$$0 = \int_{-\infty}^{+\infty} (\partial_{tt}u(x,t)\partial_t u(x,t) + \partial_{xxxx}u(x,t)\partial_t u(x,t))\mathrm{d}x$$

Using the product rule, $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_t u(x,t)$, and integrating by parts two times (i.e., applying the fundamental theorem of calculus) we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} (\frac{1}{2}\partial_t(\partial_t u(x,t))^2 - \partial_{xxx}u(x,t)\partial_t\partial_x u(x,t))\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (\partial_t \frac{1}{2}(\partial_t u(x,t))^2 + \partial_{xx}u(x,t)\partial_t\partial_{xx}u(x,t))\mathrm{d}x. \end{aligned}$$

We apply again the product rule $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_{xx}u(x,t)$,

$$0 = \int_{-\infty}^{+\infty} (\partial_t \frac{1}{2}(\partial_t u(x,t))^2 + \frac{1}{2}\partial_t(\partial_{xx}u(x,t))^2)\mathrm{d}x.$$

Switching the derivative with respect to t and the integration with respect to x, this finally gives

$$0 = \frac{1}{2}\partial_t \int_{-\infty}^{+\infty} ([\partial_t u(x,t)]^2 + [\partial_{xx}u(x,t)]^2)\mathrm{d}x.$$

Question 3: Let $k, f : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, $f(x) = 0$ if $x \in [-1,0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0,1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = -2$ and $T(1) = 2$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

On $[-1,0]$ we have $k^-(x) = 2$ and $f^-(x) = 0$ which implies $-\partial_{xx}T^-(x) = 0$. This in turn implies $T^-(x) = ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = -2 = -a + b$. This gives $a = b + 2$ and $T^-(x) = (b+2)x + b$.

We proceed similarly on $[0,+1]$ and we obtain $-\partial_{xx}T^-(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Dirichlet condition at $x = 1$ implies $T^+(1) = 2 = -1 + c + d$. This gives $c = 3 - d$ and $T^-(x) = -x^2 + (3-d)x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $2(b+2) = 3 - d$, respectively. In conclusion $b = -\frac{1}{3}$, $d = -\frac{1}{3}$ and

$$T(x) = \begin{cases} \frac{5}{3}x - \frac{1}{3} & \text{if } x \in [-1,0], \\ -x^2 + \frac{10}{3}x - \frac{1}{3} & \text{if } x \in [0,1]. \end{cases}$$

Question 4: Let $\mathrm{CS}(f) = \frac{\pi^2}{6} - 2(\cos(x) - \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} - \frac{\cos(4x)}{4^2} \ldots)$ be the Fourier cosine series of the function $f(x) := \frac{1}{2}x^2$ defined over $[0,+\pi]$.
(a) For which values of x in $[0,+\pi]$ does this series coincide with $f(x)$? (Explain).

The Fourier cosine series coincides with the function $f(x)$ over the entire interval $[0,+\pi]$ since f is smooth over $[0,+\pi]$ (recall that the Fourier cosine series is the Fourier series of the even extension of f over $[-\pi,+\pi]$).

(b) Compute the Fourier sine series, $\mathrm{SS}(x)$, of the function $g(x) := x$ defined over $[0,+\pi]$.

We know from class that it is always possible to obtain a Fourier sine series by differentiating term by term a Fourier cosine series, in other words

$$\mathsf{SS}(x) = \partial_x \mathsf{CS}(\frac{1}{2}x^2) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} \ldots \frac{\sin(nx)}{n} \ldots\right).$$

(c) For which values of $x \in [0, +\pi]$ does the Fourier sine series of g coincide with $g(x)$?.

The Fourier sine series coincides with the function $g(x) := x$ over the interval $[0, +\pi)$ since g is smooth over $[0, +\pi]$and $g(0) = 0$. The Fourier sine series of g is zero at $+\pi$, and thus differs from $g(+\pi)$.

Question 5: Using cylindrical coordinates and the method of separation of variables, solve the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{3}{2}\pi],\ r \in [0, 3]\}$, subject to the boundary conditions $u(r, 0) = 0$, $u(r, \frac{3}{2}\pi) = 0$, $u(3, \theta) = 18\sin(2\theta)$. (Give all the details.)

(1) We set $u(r, \theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{3}{2}\pi) = 0$, and $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}g(r)) = \lambda g(r)$.

(2) The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\frac{3}{2}\pi) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(3, \theta) = 18\sin(2\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

(3) The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{3}{2}\pi) = 0$ implies $\sqrt{\lambda}\frac{3}{2}\pi = n\pi$ with $n \in \mathbb{N}\backslash\{0\}$. This means $\sqrt{\lambda} = \frac{2}{3}n$, $n = 1, 2, \ldots$.

(4) From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $\frac{2}{3}n = \alpha = \sqrt{\lambda}$. The boundary condition at $r = 3$ gives $18\sin(2\theta) = c_2 3^{\frac{2}{3}n}\sin(\frac{2}{3}n\theta)$ for all $\theta \in [0, \frac{3}{2}\pi]$. This implies $n = 3$ and $c_2 = 2$.

(5) Finally, the solution to the problem is

$$u(r, \theta) = 2r^2\sin(2\theta).$$

Question 6: Let $p, q : [-1, +1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1, +1]$, where $q_0 \in \mathbb{R}$. Consider the eigenvalue problem $-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = \lambda\phi(x)$, supplemented with the boundary conditions $\phi(-1) = 0$ and $\phi(1) = 0$.

(a) Prove that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution, ϕ, to exist. (Hint: $q_0 \int_{-1}^{+1} \phi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x)\mathrm{d}x$.)

As usual we use the energy method. Let (ϕ, λ) be an eigenpair, then

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

After integration by parts and using the boundary conditions, we obtain

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

which, using the hint, can also be re-written

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathrm{d}x \leq \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Then

$$\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x \leq (\lambda - q_0) \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Assume that ϕ is non-zero, then

$$\lambda - q_0 \geq \frac{\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x}{\int_{-1}^{+1} \phi^2(x)\mathrm{d}x} \geq 0,$$

which proves that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution to exist.

(b) Assume that $p(x) \geq p_0 > 0$ for all $x \in [-1, +1]$ where $p_0 \in \mathbb{R}_+$. Show that $\lambda = q_0$ cannot be an eigenvalue, i.e., prove that $\phi = 0$ if $\lambda = q_0$. (Hint: $p_0 \int_{-1}^{+1} \psi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} p(x)\psi^2(x)\mathrm{d}x$.)

Assume that $\lambda = q_0$ is an eigenvalue. Then the above computation shows that

$$p_0 \int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x \leq \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x = 0,$$

which means that $\int_{-1}^{+1}(\partial_x\phi(x))^2\mathrm{d}x = 0$ since $p_0 > 0$. As a result $\partial_x\phi = 0$, i.e., $\phi(x) = c$ where c is a constant. The boundary conditions $\phi(-1) = 0 = \phi(1)$ imply that $c = 0$. In conclusion $\phi = 0$ if $\lambda = q_0$, thereby proving that (ϕ, q_0) is not an eigenpair.

Question 7: Use the Fourier transform technique to solve $\partial_t u(x,t)+\cos(t)\partial_x u(x,t)+(1+3t^2)u(x,t)=0$, $x\in\mathbb{R}$, $t>0$, with $u(x,0)=u_0(x)$. (Use the shift lemma: $\mathcal{F}(f(x-\beta))(\omega)=\mathcal{F}(f)(\omega)e^{i\omega\beta}$ and the definition $\mathcal{F}(f)(\omega):=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(x)e^{i\omega x}dx$)

Applying the Fourier transform to the equation gives

$$\partial_t\mathcal{F}(u)(\omega,t)+\cos(t)(-i\omega)\mathcal{F}(u)(\omega,t)+(1+3t^2)\mathcal{F}(u)(\omega,t)=0$$

This can also be re-written as follows:

$$\frac{\partial_t\mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)}=i\omega\cos(t)-(1+3t^2).$$

Then applying the fundamental theorem of calculus between 0 and t, we obtain

$$\log(\mathcal{F}(u)(\omega,t))-\log(\mathcal{F}(u)(\omega,0))=i\omega\sin(t)-(t+t^3).$$

This implies

$$\mathcal{F}(u)(\omega,t)=\mathcal{F}(u_0)(\omega)e^{i\omega\sin(t)}e^{-(t+t^3)}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t)=\mathcal{F}(u_0(x-\sin(t))(\omega)e^{-(t+t^3)}.$$

This finally gives

$$u(x,t)=u_0(x-\cos(t))e^{-(t+t^3)}.$$

Question 8: Consider the triangular domain $D=\{(x,y);x\ge0,\ y\ge0,1-x-y\ge0\}$. Let $f(x,y)=x^2-y^2-3$. Let $u\in\mathcal{C}^2(D)\cap\mathcal{C}^0(\overline{D})$ solve $-\Delta u=0$ in D and $u|_{\partial D}=f$. Compute $\min_{(x,y)\in\overline{D}}u(x,y)$ and $\max_{(x,y)\in\overline{D}}u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}}u(x,y)=\min_{(x,y)\in\partial D}f(x,y),\quad\text{and}\quad\max_{(x,y)\in\overline{D}}u(x,y)=\max_{(x,y)\in\partial D}f(x,y).$$

A point (x,y) is at the boundary of D if and only if $\{x=0$ and $y\in[0,1]\}$ or $\{y=0$ and $x\in[0,1]\}$, or $\{1-y-x=0$ and $x\in[0,1]\}$.

(i) In the first case, $x=0$ and $y\in[0,1]$, we have

$$f(x,y)=-y^2-3,\qquad y\in[0,1].$$

The maximum is -3 and the minimum is -4.

(ii) In the second case, $y=0$ and $x\in[0,1]$, we have

$$f(x,y)=x^2-3,\qquad x\in[0,1].$$

The maximum is -2 and the minimum is -3.

(iii) In the third case, $1-x=y$ and $x\in[0,1]$, we have

$$f(x,y)=x^2-(1-x)^2-3=2x-4,\qquad x\in[0,1].$$

The maximum is -2 and the minimum is -4.

We finally can conclude

$$\min_{(x,y)\in\partial D}f(x,y)=-4,\qquad\max_{(x,y)\in\partial D}f(x,y)=-2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}}u(x,y)=-4,\qquad\max_{(x,y)\in\overline{D}}u(x,y)=-2$$
