Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Question 1: Let $\Phi \in C^1(\mathbb{R}^3;\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^3;\mathbb{R})$ be defined by $\Phi(x_1,x_2,x_3) := x_1^3 - \sin(x_2-x_3)$ and $\Psi(x_1,x_2,x_3) := \log(1+x_1^2+x_2^4+x_3^6)+x_1$.
(a) Compute $\nabla\Phi$ and $\nabla\Psi$.

Applying the chain rule we obtain

$$\nabla\Phi(x_1,x_2,x_3) = \left(3x_1^2, -\cos(x_2-x_3), \cos(x_2-x_3)\right),$$

and

$$\nabla\Psi(x_1,x_2,x_3) = \left(1+\frac{2x_1}{1+x_1^2+x_2^4+x_3^6}, \frac{4x_2^3}{1+x_1^2+x_2^4+x_3^6}, \frac{6x_3^5}{1+x_1^2+x_2^4+x_3^6}\right).$$

Question 2: Let $\boldsymbol{A} \in C^0(\mathbb{R}^3;\mathbb{R}^3)$ and $\varphi \in C^0(\mathbb{R}^3;\mathbb{R})$. Let D be a subset of $\mathbb{R}^3$ with a smooth boundary ∂D and unit outward normal $\boldsymbol{n}$. Compute $\int_{\partial D} \boldsymbol{A}\cdot(\boldsymbol{n}\times(\varphi\boldsymbol{A}))\mathrm{d}s$. (*Hint:* recall that $(\boldsymbol{E}\times\boldsymbol{F})\cdot\boldsymbol{G} = \boldsymbol{E}\cdot(\boldsymbol{F}\times\boldsymbol{G})$.)

Using the definitions, and recalling that $\boldsymbol{E}\times\boldsymbol{E} = \boldsymbol{0}$ for all $\boldsymbol{E} \in \mathbb{R}^3$, we have

$$\int_{\partial D} \boldsymbol{A}\cdot(\boldsymbol{n}\times(\varphi\boldsymbol{A}))\mathrm{d}s = \int_{\partial D} (\boldsymbol{n}\times(\varphi\boldsymbol{A}))\cdot\boldsymbol{A}\mathrm{d}s = \int_{\partial D} \boldsymbol{n}\cdot((\varphi\boldsymbol{A})\times\boldsymbol{A})\mathrm{d}s = \int_{\partial D} (\boldsymbol{n}\cdot(\boldsymbol{A}\times\boldsymbol{A}))\varphi\mathrm{d}s = 0.$$

Question 3: Let $\nabla\times$ denote the curl operator acting on vector fields: i.e., let $\boldsymbol{A} = (A_1,A_2,A_3) \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ be a three-dimensional vector field over $\mathbb{R}^3$, then $\nabla\times\boldsymbol{A} := (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$.
(a) Let $\varphi \in C^2(\mathbb{R}^3;\mathbb{R})$. Compute $\nabla\times(\nabla\varphi^2)$. (*Hint:* Recall that $\partial_{ij}\psi = \partial_{ji}\psi$, for all $i,j \in \{1,2,3\}$ and all $\psi \in C^1(\mathbb{R}^3;\mathbb{R})$.)

The definitions imply that $\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$. Hence,

$$\nabla\times(\nabla\varphi^2) = (\partial_2\partial_3\varphi^2 - \partial_3\partial_2\varphi^2, \partial_3\partial_1\varphi^2 - \partial_1\partial_3\varphi^2, \partial_1\partial_2\varphi^2 - \partial_2\partial_1\varphi^2) = \partial_{31}\varphi^2 - \partial_{13}\varphi^2, \partial_{12}\varphi^2 - \partial_{21}\varphi^2) = \boldsymbol{0}$$

(b) Show that $2\psi\nabla\psi = \nabla(\psi^2)$ for all $\psi \in C^1(\mathbb{R}^3;\mathbb{R})$.

Let $\psi \in C^1(\mathbb{R}^3;\mathbb{R})$. Then by the chain rule

$$\nabla\psi^2 = (\partial_1\psi^2, \partial_2\psi^2, \partial_3\psi^2) = (2\psi\partial_1\psi, 2\psi\partial_2\psi, 2\psi\partial_3\psi) = 2\psi(\partial_1\psi, \partial_2\psi, \partial_3\psi) = 2\psi\nabla\psi.$$

(c) Let φ and ψ in $C^2(\mathbb{R}^3;\mathbb{R})$. Let D be a subset of $\mathbb{R}^3$ with a smooth boundary ∂D and unit outward normal $\boldsymbol{n}$. Compute $\int_{\partial D}(2\psi\nabla\varphi{\times}\nabla\psi){\cdot}\boldsymbol{n}\mathrm{d}s$. (*Hint:* Accept as a fact that $\nabla{\cdot}(\boldsymbol{A}{\times}\boldsymbol{B}) = \boldsymbol{B}{\cdot}\nabla{\times}\boldsymbol{A} - \boldsymbol{A}{\cdot}\nabla{\times}\boldsymbol{B}$ for all $\boldsymbol{A},\boldsymbol{B} \in C^1(\mathbb{R}^3;\mathbb{R}^3)$.)

We first observe that

$$\int_{\partial D}(2\psi\nabla\varphi{\times}\nabla\psi){\cdot}\boldsymbol{n}\mathrm{d}s = \int_{\partial D}(\nabla\varphi{\times}\nabla\psi^2){\cdot}\boldsymbol{n}\mathrm{d}s.$$

Then using the divergence theorem we infer that

$$\int_{\partial D}(\nabla\varphi{\times}\nabla\psi^2){\cdot}\boldsymbol{n}\mathrm{d}s = \int_D \nabla{\cdot}(\nabla\varphi{\times}\nabla\psi^2)\mathrm{d}x.$$

Then using the hint we have

$$\int_{\partial D}(\nabla\varphi{\times}\nabla\psi^2)\mathrm{d}s = \int_D \big(\nabla{\times}(\nabla\varphi){\cdot}\nabla\psi^2 - \nabla\varphi{\cdot}\nabla{\times}(\nabla\psi^2)\big)\mathrm{d}x.$$

But $\nabla{\times}(\nabla\varphi) = \mathbf{0}$ and $\nabla{\times}(\nabla\psi^2) = \mathbf{0}$. Hence

$$\int_{\partial D}(2\psi\nabla\varphi{\times}\nabla\psi){\cdot}\boldsymbol{n}\mathrm{d}s = 0.$$

Question 4: Let u solve $\partial_t u - \partial_x\big(\cos(\frac{\pi x}{2L})u + (\sin(\frac{\pi x}{L}) + 2)\partial_x u\big) = f(x)e^{-2t}$, $x \in (0,L)$, with $2\partial_x u(0,t) + u(0,t) = 2$, $\partial_x u(L,t) = 1$, $u(x,0) = u_0(x)$, where f and u_0 are two smooth functions.
(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over the domain $(0,L)$ and apply the fundamental Theorem of calculus:

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)dx &= \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_x\big(\cos(\pi x/2L)u + (\sin(\pi x/L)+2)\partial_x u\big)dx + e^{-2t}\int_0^L f(x)dx \\ &= \cos(\pi L/2L)u(L,t) + (\sin(\pi L/L)+2)\partial_x u(L,t) - \cos(0)u(0,t) - (\sin(0)+2)\partial_x u(0,t) + e^{-2t}\int_0^L f(x)dx \\ &= 2\partial_x u(L,t) - u(0,t) - 2\partial_x u(0,t) + e^{-2t}\int_0^L f(x)dx = 2 - 2 + e^{-2t}\int_0^L f(x)dx \\ &= e^{-2t}\int_0^L f(x)dx.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)dx = e^{-2t}\int_0^L f(x)dx.$$

(b) Use (a) to compute $\int_0^L u(x,T)dx$ as a function of the time T.

Applying the fundamental Theorem of calculus again gives

$$\begin{aligned}\int_0^L u(x,T)dx &= \int_0^L u(x,0)dx + \int_0^T \frac{d}{dt}\int_0^L u(x,t)dxdt \\ &= \int_0^L u_0(x)dx + \frac{1}{2}(1-e^{-2T})\int_0^L f(x)dx.\end{aligned}$$

(c) What is the limit of $\int_0^L u(x,T)dx$ as $T \to +\infty$?

The above formula gives

$$\lim_{T\to+\infty}\int_0^L u(x,T)dx = \int_0^L u_0(x)dx + \frac{1}{2}\int_0^L f(x)dx.$$

Question 5: Consider the vibrating beam equation $\partial_{tt}u(x,t)+\partial_{xx}\big(\frac{2+|x|}{1+|x|}\partial_{xx}u(x,t)\big)=0$, $x\in(-\infty,+\infty)$, $t>0$ with $u(\pm\infty,t)=0$, $\partial_x u(\pm\infty,t)=0$, $\partial_{xx}u(\pm\infty,t)=0$.

(a) Show that $\partial_t\int_{-\infty}^{+\infty}(\partial_t u(x,t))^2\,\mathrm{d}x=\int_{-\infty}^{+\infty}2\partial_{tt}u(x,t)\partial_t u(x,t)\mathrm{d}x$.

We have

$$\partial_t\int_{-\infty}^{+\infty}(\partial_t u(x,t))^2\,\mathsf{d}x=\int_{-\infty}^{+\infty}\partial_t\,(\partial_t u(x,t))^2\,\mathsf{d}x=\int_{-\infty}^{+\infty}2\partial_t u(x,t)\partial_{tt}u(x,t)\mathsf{d}x.$$

(b) Show that $\partial_t\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|}[\partial_{xx}u(x,t)]^2)\mathrm{d}x=\int_{-\infty}^{+\infty}2(\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\partial_{xx}(\partial_t u(x,t))\mathrm{d}x$.

We have

$$\partial_t\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|}[\partial_{xx}u(x,t)]^2)\mathsf{d}x=\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|})\partial_t[\partial_{xx}u(x,t)]^2\mathsf{d}x=\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|})2\partial_{xx}u(x,t)\partial_t\partial_{xx}u(x,t)\mathsf{d}x$$
$$=\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|})2\partial_{xx}u(x,t)\partial_{xx}(\partial_t u(x,t))\mathsf{d}x.$$

(c) Show that $\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\partial_{xx}(\partial_t u(x,t))\mathrm{d}x=\int_{-\infty}^{+\infty}\partial_{xx}\left((\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\right)\partial_t u(x,t)\mathrm{d}x$. (*Hint:* Integrate by parts two times and use the boundary conditions at infinity: $\partial_{tx}u(\pm\infty,t)=0$ and $\partial_t u(\pm\infty,t)=0$.)

We integrate by parts two times and use the boundary conditions at infinity.

$$\int_{-\infty}^{+\infty}(\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\partial_{xx}(\partial_t u(x,t))\mathsf{d}x=-\int_{-\infty}^{+\infty}\partial_x\left((\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\right)\partial_x(\partial_t u(x,t))\mathsf{d}x$$
$$\int_{-\infty}^{+\infty}\partial_{xx}\left((\frac{2+|x|}{1+|x|})\partial_{xx}u(x,t)\right)\partial_t u(x,t)\mathsf{d}x.$$

(d) Use the energy method to compute $\partial_t\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2+\frac{2+|x|}{1+|x|}[\partial_{xx}u(x,t)]^2)\mathrm{d}x$. Give all the details. (Hint: Multiply the equation by $\partial_t u(x,t)$, integrate over space and use (a), (c) and (b)).

Using the hint we have

$$0=\int_{-\infty}^{+\infty}\left(\partial_{tt}u(x,t)\partial_t u(x,t)+\partial_{xx}\left(\frac{2+|x|}{1+|x|}\partial_{xx}u(x,t)\right)\partial_t u(x,t)\right)\mathsf{d}x$$

Using (a) and (c) we obtain

$$0=\int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2+\left(\frac{2+|x|}{1+|x|}\right)\partial_{xx}u(x,t)\partial_t\partial_{xx}u(x,t))\mathsf{d}x.$$

Using (b) we obtain

$$0=\int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2+\frac{1}{2}\frac{2+|x|}{1+|x|}\partial_t(\partial_{xx}u(x,t))^2)\mathsf{d}x.$$

Switching the derivative with respect to t and the integration with respect to x, this finally gives

$$0=\frac{1}{2}\partial_t\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2+\frac{2+|x|}{1+|x|}[\partial_{xx}u(x,t)]^2)\mathsf{d}x.$$

Question 6: Let $k : [-1, +1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, if $x \in [-1, 0]$ and $k(x) = 1$ if $x \in (0, 1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $-\partial_x T(-1) + T(-1) = -1$ and $T(1) = 3$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1, +1]$.

On $[-1, 0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $-\partial_x T^-(-1) + T^-(-1) = -1 = -2b + a$. This gives $a = 2b - 1$ and $T^-(x) = 2b - 1 + bx$.

We proceed similarly on $[0, +1]$ and we obtain $T^+(x) = c + dx$. The Dirichlet boundary condition at $x = +1$ gives $T^+(1) = 3 = c + d$. This implies $c = 3 - d$ and $T^+(x) = 3 - d + dx$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b - 1 = 3 - d, \qquad \text{and} \qquad 2b = d.$$

This implies $d = 2$ and $b = 1$. In conclusion

$$T(x) = \begin{cases} x + 1 & \text{if } x \in [-1, 0], \\ 2x + 1 & \text{if } x \in [0, +1]. \end{cases}$$

Question 7: Consider the triangular domain $D = \{(x,y); x \geq 0,\ y \geq 0, 1-x-y \geq 0\}$. Let $f(x,y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. (a) Compute $\min_{(x,y)\in\overline{D}} f(x,y)$ and $\max_{(x,y)\in\overline{D}} f(x,y)$.

A point (x,y) is at the boundary of D if and only if $\{x = 0$ and $y \in [0,1]\}$ or $\{y = 0$ and $x \in [0,1]\}$, or $\{1-y-x = 0$ and $x \in [0,1]\}$.
(i) In the first case, $x = 0$ and $y \in [0,1]$, we have

$$f(x,y) = -y^2 - 3, \qquad y \in [0,1].$$

The maximum is -3 and the minimum is -4.
(ii) In the second case, $y = 0$ and $x \in [0,1]$, we have

$$f(x,y) = x^2 - 3, \qquad x \in [0,1].$$

The maximum is -2 and the minimum is -3.
(iii) In the third case, $1 - x = y$ and $x \in [0,1]$, we have

$$f(x,y) = x^2 - (1-x)^2 - 3 = 2x - 4, \qquad x \in [0,1].$$

The maximum is -2 and the minimum is -4.
We finally can conclude

$$\min_{(x,y)\in\partial D} f(x,y) = -4, \quad \max_{(x,y)\in\partial D} f(x,y) = -2.$$

(b) Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -4, \quad \max_{(x,y)\in\overline{D}} u(x,y) = -2$$

Question 8: Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi$, $t \in (0,\pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $5\phi(\pi) = -\phi'(\pi)$.

(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

(i) Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain.

$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) = \lambda \int_0^\pi \phi^2(t)dt.$$

Using the BCs, we infer

$$\int_0^\pi (\phi'(t))^2 dt + 5\phi(\pi)^2 = \lambda \int_0^\pi \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^\pi (\phi'(t))^2 dt = 0$ and $\phi(\pi)^2 = 0$, which implies that $\phi'(t) = 0$ and $\phi(\pi) = 0$. The fundamental theorem of calculus implies $\phi(t) = \phi(\pi) + \int_\pi^t \phi'(\tau)d\tau = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero solution to exist.

(b) Find the equation that λ must solve for the above problem to have a nonzero solution (do not try to solve it).

Since λ is positive, ϕ is of the following form

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition $\phi'(\pi) = -5\phi(\pi)$ implies

$$\sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) = -5c_2 \sin(\sqrt{\lambda}\pi).$$

The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, λ must solve the following equation

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + 5\sin(\sqrt{\lambda}\pi) = 0,$$

for a nonzero solution ϕ to exist.

Question 9: Let L be a positive real number. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} := \left(\int_{-L}^{L} f(t)^2 \text{d}t\right)^{\frac{1}{2}}$.

(a) Compute the best approximation of $h(t) = 5 + \pi^2 \cos(\pi t/L) + 5\sin(6\pi t/L)$ in $\mathbb{P}_1$.

Recall that the best approximation of h in $\mathbb{P}_1$, say p, is such that $p \in \mathbb{P}_1$ and $\int_{-L}^{L}(h(t) - p(t))p(t)\text{d}t = 0$ for all $p \in \mathbb{P}_1$.

The function $p(t) := 5 + \pi^2 \cos(\pi t/L)$ is in $\mathbb{P}_1$ and the function $h(t) - p(t) = h(t) - 5 - \pi^2 \cos(\pi t/L = 5\sin(6\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $p(t) = 5 + \pi^2 \cos(\pi t/L)$. In conclusion

$$p(t) = 5 + \pi^2 \cos(\pi t/L).$$

(b) Compute the best approximation of $1 + t^2$ in $\mathbb{P}_1$ with respect to the above norm. (Hint: $\int t^2 \cos(t)\text{d}t = 2t\cos(t) + (t^2 - 2)\sin(t)$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1 \cos(\pi t/L) + b_1 \sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(1+t^2)\text{d}t = 1 + \frac{2L^3}{6L} = 1 + \frac{1}{3}L^2,$$
$$a_1 = \frac{1}{L}\int_{-L}^{L}(1+t^2)\cos(\pi t/L)\text{d}t = \frac{1}{L}\frac{L^3}{\pi^3}\int_{-\pi}^{\pi} t^2\cos(t)\text{d}t = \frac{1}{L}\frac{L^3}{\pi^3}(-4\pi) = -4\frac{L^2}{\pi^2}$$
$$b_1 = \frac{1}{L}\int_{-L}^{L}(1+t^2)\sin(\pi t/L)\text{d}t = 0.$$

As a result

$$FS_1(t) = 1 + \frac{1}{3}L^2 - \frac{4L^2}{\pi^2}\cos(\pi t/L)$$

Question 10: Let $p, q : [-1, +1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1, +1]$, where $q_0 > 0$. Let $f \in C^0([-1, 1]; \mathbb{R})$ and consider the boundary value problem $-\partial_x(p(x)\partial_x u(x)) + q(x)u(x) = f(x)$, supplemented with the boundary conditions $\partial_x u(-1) = 0$ and $-\partial_x u(1) = 2u(1)$.
(a) Assume that the problem has a solution. Using the energy method, prove that this solution is unique. (*Hint:* $q_0 \int_{-1}^{+1} \phi^2(x) \mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x) \mathrm{d}x$.)

Let u_1 and u_2 be two solutions. Then letting $\phi := \phi_2 - \phi_1$, we have

$$-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = 0, \qquad \partial_x\phi(-1) = 0, \qquad -\partial_x\phi(1) = 2\phi(1).$$

As usual we use the energy method. We multiply the equation by ϕ and integrate over the domain:

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathsf{d}x = 0.$$

After integration by parts and using the boundary conditions, we obtain

$$\begin{aligned} 0 &= \int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathsf{d}x - 2p(x)\partial_x\phi(x)\phi(x)|_{-1}^{+1} \\ &= \int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathsf{d}x + 2p(1)\phi(1)^2 \end{aligned}$$

which, using the hint, can also be re-written

$$q_0 \int_{-1}^{+1} \phi^2(x)\mathsf{d}x \leq \int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathsf{d}x + 2p(1)\phi(1)^2 \leq 0.$$

Then using that $q_0 > 0$ we obtain

$$0 \leq \int_{-1}^{+1} \phi^2(x)\mathsf{d}x \leq 0,$$

which in turn implies that $\phi = 0$. Whence $u_1 = u_2$.
