Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**. Here are some results you may want to use:

Question 1:

Let $\gamma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a smooth scalar-valued function. Let $\delta(x,t) := \mathrm{e}^{\gamma(x,t)}$.
(a) Compute $\partial_t \delta(x,t)$.

Applying the chain rule, we obtain

$$\partial_t \delta(x,t) = \partial_t \gamma(x,t) \mathsf{e}^{\gamma(x,t)}.$$

(b) Compute $\partial_x \delta(x,t)$.

Applying the chain rule, we obtain

$$\partial_x \delta(x,t) = \partial_x \gamma(x,t) \mathsf{e}^{\gamma(x,t)}.$$

(c) Let $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a smooth integrable scalar-valued solution such that $\int_{-\infty}^{+\infty} |u(\xi,t)| \mathrm{d}\xi < \infty$ for all $t > 0$. Let $\nu > 0$ and $\phi(x,t) := \mathrm{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathrm{d}\xi}$. Compute $\partial_t \phi$, $\partial_x \phi$, and $\partial_{xx} \phi$.

The definition of ϕ, together with the chain rule, implies that

$$\begin{aligned}
\partial_t \phi(x,t) &= \partial_t \left(-\frac{1}{2\nu} \int_{-\infty}^{x} u(\xi,t) \mathsf{d}\xi \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi} \\
&= \left(-\frac{1}{2\nu} \int_{-\infty}^{x} \partial_t u(\xi,t) \mathsf{d}\xi \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi}
\end{aligned}$$

and

$$\begin{aligned}
\partial_x \phi(x,t) &= \partial_x \left(-\frac{1}{2\nu} \int_{-\infty}^{x} u(\xi,t) \mathsf{d}\xi \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi} \\
&= \left(-\frac{1}{2\nu} u(x,t) \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi}
\end{aligned}$$

and

$$\begin{aligned}
\partial_{xx} \phi(x,t) &= \left(-\frac{1}{2\nu} \partial_x u(x,t) \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi} + \left(-\frac{1}{2\nu} u(x,t) \right)^2 \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi} \\
&= \left(-\frac{1}{2\nu} \partial_x u(x,t) + \left(\frac{1}{2\nu} u(x,t) \right)^2 \right) \mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi}
\end{aligned}$$

(d) Compute $\partial_t \phi - \nu \partial_{xx}\phi$ in terms of u.

The above computations give

$$-\nu\partial_{xx}\phi(x,t) = -\frac{1}{2\nu}\left(-\nu\partial_x u(x,t) + \frac{1}{2}u^2(x,t)\right) \mathrm{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathrm{d}\xi}$$

In conclusion

$$\partial_t\phi - \nu\partial_{xx}\phi = -\frac{1}{2\nu}\left(\int_{-\infty}^{x} \partial_t u(\xi,t)\mathrm{d}\xi + \frac{1}{2}u^2(x,t) - \nu\partial_x u(x,t)\right) \mathrm{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathrm{d}\xi}.$$

(e) What equation the function u must solve so that $\partial_t\phi - \nu\partial_{xx}\phi = 0$ for all $x \in \mathbb{R}$ and all $t > 0$?

The above computation shows that claiming that $\partial_t\phi - \nu\partial_{xx}\phi = 0$ is equivalent to saying

$$\int_{-\infty}^{x} \partial_t u(\xi,t)\mathrm{d}\xi + \frac{1}{2}u^2(x,t) - \nu\partial_x u(x,t) = 0.$$

Remark: Notice in passing that taking the derivative of this equation with respect to x gives Burgers' equation

$$\partial_t u(x,t) + \partial_x(\frac{1}{2}u^2)(x,t) - \nu\partial_{xx}u(x,t) = 0.$$

The above technique is called Cole-Hopf transformation of Burger's equation.

Question 2: Let $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a smooth integrable scalar-valued function such that $\lim_{|X|\to\infty} u(X,t) = 0$, $\lim_{|X|\to\infty} \partial_x u(X,t) = 0$, $\lim_{|X|\to\infty} \partial_{xx} u(X,t) = 0$, $\lim_{|X|\to\infty} \partial_{xxx} u(X,t) = 0$.

(a) Compute $u\partial_x\left(\frac{1}{2}u^2\right) - \partial_x\left(\frac{1}{3}u^3\right)$.

Using the product rule, we obtain

$$u\partial_x\frac{1}{2}u^2(x,t) - \partial_x\frac{1}{3}u^3(x,t) = 2\frac{1}{2}u^2\partial_x u(x,t) - 3\frac{1}{3}u^2\partial_x u(x,t) = 0.$$

(b) Compute $\partial_x u\partial_{xx}u - \partial_x\left(\frac{1}{2}(\partial_x u)^2\right)$.

Using the product rule, we obtain

$$\partial_x u\partial_{xx}u = \partial_x u\partial_x(\partial_x u) = \partial_x\left(\frac{1}{2}(\partial_x u)^2\right).$$

Hence

$$\partial_x u\partial_{xx}u - \partial_x\left(\frac{1}{2}(\partial_x u)^2\right) = 0.$$

(c) Compute $\lim_{|X|\to\infty}\int_{-X}^{X} u(\xi,t)\partial_{xxx}u(\xi,t)\mathrm{d}\xi$.

Integrating by part once, we have

$$\int_{-X}^{X} u(\xi,t)\partial_{xxx}u(\xi,t)\mathsf{d}\xi = -\int_{-X}^{X} \partial_x u(\xi,t)\partial_{xx}u(\xi,t)\mathsf{d}\xi + [u(\xi,t)\partial_{xx}]_{-X}^{X}.$$

Now we notice that $\partial_x u(\xi,t)\partial_{xx}u(\xi,t) = \partial_x \frac{1}{2}(\partial_x u(\xi,t))^2$.

$$\begin{aligned}\int_{-X}^{X} u(\xi,t)\partial_{xxx}u(\xi,t)\mathsf{d}\xi &= -\int_{-X}^{X} \partial_x\left(\frac{1}{2}(\partial_x u(\xi,t))^2\right)\mathsf{d}\xi + [u(\xi,t)\partial_{xx}u(\xi,t)]_{-X}^{X} \\ &= [-\frac{1}{2}(\partial_x u(X,t))^2]_{-X}^{X} + [u(\xi,t)\partial_{xx}u(\xi,t)]_{-X}^{X}.\end{aligned}$$

Now taking the limit for $X\to+\infty$, we obtain

$$\int_{-X}^{X} u(\xi,t)\partial_{xxx}u(\xi,t)\mathsf{d}\xi = 0$$

(d) Assume now that u solves $\partial_t u(x,t) + \partial_x(\frac{1}{2}u^2)(x,t) - \kappa\partial_{xxx}u(x,t) = 0$ for all $x\in\mathbb{R}$ and all $t>0$ with $u(x,0)=u_0(x)$ where $\kappa\in\mathbb{R}$ and u_0 is a smooth integrable function. Compute $\int_{-\infty}^{\infty} u^2(\xi,t)\mathrm{d}\xi$ in terms of $\int_{-\infty}^{\infty} u_0^2(\xi)\mathrm{d}\xi$. (*Hint:* Energy method).

We apply the energy method. We multiply the equation by u and integrate over space and time. We start by integrating over space.

$$0 = \int_{-\infty}^{\infty} u\partial_t u(\xi,t) + u\partial_x(\frac{1}{2}u^2)(\xi,t) + \kappa u(\partial_{xxx}u)(\xi,t)\mathsf{d}\xi.$$

We use the results established above

$$\begin{aligned}&\int_{-\infty}^{\infty} u\partial_t u(\xi,t)\mathsf{d}\xi = \int_{-\infty}^{\infty}\partial_t(\frac{1}{2}u)(\xi,t) = \partial_t\int_{-\infty}^{\infty}\frac{1}{2}u(\xi,t) \\ &\int_{-\infty}^{\infty} u\partial_x(\frac{1}{2}u^2)(\xi,t)\mathsf{d}\xi = 0 \\ &\int_{-\infty}^{\infty}\kappa u(\partial_{xxx}u)(\xi,t)\mathsf{d}\xi = 0.\end{aligned}$$

Hence,

$$0 = \partial_t\int_{-\infty}^{\infty}\frac{1}{2}u(\xi,t).$$

This proves that

$$\int_{-\infty}^{\infty}\frac{1}{2}u(\xi,t)\mathsf{d}\xi = \int_{-\infty}^{\infty}\frac{1}{2}u(\xi,0)\mathsf{d}\xi = \int_{-\infty}^{\infty}\frac{1}{2}u_0(\xi)\mathsf{d}\xi.$$

Question 3: Let $k : [0, 2] \to \mathbb{R}$ be defined by $k(x) = \frac{1}{4}$ if $x \in [0, \frac{1}{2}]$ and $k(x) = \frac{1}{2}$ if $x \in (\frac{1}{2}, 1]$. Let $\mu : [0, 2] \to \mathbb{R}$ be defined by $\mu(x) = 1$ if $x \in [0, \frac{1}{2}]$ and $\mu(x) = 0$ if $x \in (\frac{1}{2}, 1]$. Let $T : [0, 1] \to \mathbb{R}$ be the solution to $\mu(x)T(x) - \partial_x(k\partial_x T)(x) = 0$ with $T(0) = 0$ and $\partial_x T(1) = \cosh(1)$. (*Hint:* do not try to simplify the expressions $\cosh(1)$ and $\sinh(1)$.)
(a) What should be the interface conditions at $x = \frac{1}{2}$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = \frac{1}{2}$. Let T^- denote the solution on $[0, \frac{1}{2}]$ and T^+ the solution on $[\frac{1}{2}, 1]$. One should have $T^-(\frac{1}{2}) = T^+(\frac{1}{2})$ and $k^-(\frac{1}{2})\partial_x T^-(\frac{1}{2}) = k^+(\frac{1}{2})\partial_x T^+(\frac{1}{2})$, where $k^-(\frac{1}{2}) = \frac{1}{4}$ and $k^+(\frac{1}{2}) = \frac{1}{2}$.

(b) Solve the problem, i.e., find $T(x)$ for all $x \in [0, 1]$.

(i) For all $x \in [0, \frac{1}{2})$ we have

$$\partial_{xx} T(x) = 4T(x).$$

The generic solution to this problem is

$$T(x) = a\cosh(2x) + b\sinh(2x).$$

The boundary condition at 0 implies $0 = a$. Hence

$$T(x) = b\sinh(2x), \qquad \forall x \in [0, \frac{1}{2}].$$

(ii) For all $x \in [\frac{1}{2}, 1]$, we have $\partial_x(\frac{1}{9}\partial_x T)(x) = 0$. Hence

$$\partial_{xx} T = 0.$$

This gives $T(x) = cx + d$. The boundary condition at 1 implies that $c = \cosh(1)$.

(iii) The interface conditions give

$$\begin{aligned} T(\frac{1}{2}^-) = T(\frac{1}{2}^+) &\Longrightarrow b\sinh(1) = \frac{1}{2}c + d \\ \frac{1}{4}\partial_x T(\frac{1}{2}^-) = \frac{1}{2}\partial_x T(\frac{1}{2}^+) &\Longrightarrow \frac{1}{4}\, b\, 2\cosh(1) = \frac{1}{2}c = \frac{1}{2}\cosh(1). \end{aligned}$$

Hence

$$b = 1, \quad d = \sinh(1) - \frac{1}{2}\cosh(1).$$

In conclusion

$$T(x) = \begin{cases} \sinh(2x) & x \in [0, \frac{1}{2}] \\ \cosh(1)(x - \frac{1}{2}) + \sinh(1) & \end{cases}$$

Question 4: Consider the following equation written in cylindrical coordinates $\frac{1}{r}\partial_r(r\partial_r u)+\frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{3}{2}\pi],\ r \in [0,3]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r, \frac{3}{2}\pi) = 0$, $u(3,\theta) = 18\sin(2\theta)$.

(a) Assuming that $u(r,\theta) = \phi(\theta)g(r)$, where $\phi \not\equiv 0$ and $g \not\equiv 0$, derive the equations that ϕ and g must solve.

We insert the expression $u(r,\theta) = \phi(\theta)g(r)$ into the equation and we obtain

$$r^2 \frac{1}{r}\frac{\partial_r(r\partial_r g)}{g(r)} = -\frac{\partial_{\theta\theta}\phi}{\phi(\theta)}.$$

As this equality must hold for all $r \in [0,3]$ and all $\theta \in [0, \frac{3\pi}{2})$, there must exist a constant λ so that This means $\partial_{\theta\theta}\phi = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{3}{2}\pi) = 0$, and $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}g(r)) = \lambda g(r)$.

(b) Use the energy method to determine the sign of λ in the following eigenvalue problem $\partial_{\theta\theta}\phi = -\lambda\phi$, $\phi(0) = 0$, $\phi(\frac{3}{2}\pi) = 0$, and solve the problem.

Using the usual energy method argument, we obtain that

$$-\lambda\int_0^{\frac{3}{2}\pi}\phi^2(\theta)\mathsf{d}\theta = \int_0^{\frac{3}{2}\pi}\phi\partial_{\theta\theta}\phi\mathsf{d}\theta = -\int_0^{\frac{3}{2}\pi}(\partial_\theta\phi)^2\mathsf{d}\theta + \phi(\frac{3}{2}\pi)\partial_\theta\phi(\frac{3}{2}\pi) - \phi(0)\partial_\theta\phi(0).$$

This shows that

$$\lambda = \frac{\int_0^{\frac{3}{2}\pi}(\partial_\theta\phi)^2\mathsf{d}\theta}{\int_0^{\frac{3}{2}\pi}\phi^2(\theta)\mathsf{d}\theta}.$$

That is, λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(3,\theta) = 18\sin(2\theta)$. Hence $\lambda > 0$.

The above argument proves that

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{3}{2}\pi) = 0$ implies $\sqrt{\lambda}\frac{3}{2}\pi = n\pi$ with $n \in \mathbb{N}\setminus\{0\}$. This means $\sqrt{\lambda} = \frac{2}{3}n$, $n = 1, 2, \ldots$ Hence

$$\phi(\theta) = c\sin(\frac{2}{3}n\theta).$$

(c) The generic solution to $r\partial_r(r\partial_r g) = \lambda g(r)$ is $c\,r^\alpha$ where $c \in \mathbb{R}$ is arbitrary. Compute α assuming that $\alpha \geq 0$.

From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $\frac{2}{3}n = \alpha = \sqrt{\lambda}$.

(d) The solution to the problem can be written in the form $u(r,\theta) = cr^{\frac{2}{3}n}\sin(\frac{2}{3}n\theta)$. Compute c and n.

The boundary condition at $r = 3$ gives $18\sin(2\theta) = c_2 3^{\frac{2}{3}n}\sin(\frac{2}{3}n\theta)$ for all $\theta \in [0, \frac{3}{2}\pi]$. This implies $n = 3$ and $c_2 = 2$.

Finally, the solution to the problem is

$$u(r,\theta) = 2r^2\sin(2\theta).$$

Question 5: Give the definition of a scalar-valued function being harmonic in a domain D in $\mathbb{R}^2$.

We say that $u : D \to \mathbb{R}$ is harmonic if $\Delta u = 0$ where Δ denotes the Laplace operator.

Question 6: State the maximum and the minumum principle for smooth harmonic functions.

Let $u : D \to \mathbb{R}$ be a smooth harmonic function. Then

$$\min_{x\in\partial D} u(\boldsymbol{x}) \leq \min_{x\in D} u(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in D} u(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x}).$$

Question 7: Let $D \subset \mathbb{R}^2$ be defined in cylindrical coordinate by $D := \{(r,\theta) \in \mathbb{R}^2; 0 \leq r \leq 1, \quad \frac{1}{4}\pi \leq \theta \leq \frac{7}{8}\pi\}$. Let u be the unique smooth function that solves $\Delta u = 0$ with $u_{|\partial D} = f$ with $f(r,\theta) = r(r-1)\sin(\theta)$. Compute the maximum and the minumum of u in D.

As u is a smooth harmonic function, we can apply maximum principle. We have

$$f(1,\theta) = 0, \qquad f(r, \frac{1}{4}\pi) = \sin(\frac{\pi}{4})r(r-1), \qquad f(r, \frac{7\pi}{8}) = \sin(\frac{7}{8}\pi)r(r-1).$$

Notice that $\frac{1}{\sqrt{2}} = \sin(\frac{1}{4}\pi) > \sin(\frac{1}{8}\pi) = \sin(\frac{7\pi}{8}) > 0$. We then have

$$\min_{\frac{1}{4}\pi\leq\theta\leq\frac{7}{8}\pi} f(1,\theta) = 0, \qquad \max_{\frac{1}{4}\pi\leq\theta\leq\frac{7}{8}\pi} f(1,\theta) = 0$$

$$\min_{0\leq r\leq 1} f(r, \frac{1}{4}\pi) = -\frac{1}{4}\sin(\frac{1}{4}\pi), \qquad \max_{0\leq r\leq 1} f(r, \frac{1}{4}\pi) = 0,$$

$$\min_{0\leq r\leq 1} f(r, \frac{7\pi}{8}) = -\frac{1}{4}\sin(\frac{1}{8}\pi), \qquad \max_{0\leq r\leq 1} f(r, \frac{7}{8}\pi) = 0.$$

Hence

$$\min_{\boldsymbol{x}\in D} u(\boldsymbol{x}) = -\frac{1}{4\sqrt{2}}, \qquad \max_{\boldsymbol{x}\in D} u(\boldsymbol{x}) = 0.$$

Question 8: Let $p, q : [-1, +1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1, +1]$, where $q_0 \in \mathbb{R}$. Consider the eigenvalue problem $-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = \lambda\phi(x)$, supplemented with the boundary conditions $\phi(-1) = 0$ and $\phi(1) = 0$.
(a) Prove if a non-zero (smooth) eigenvector exists, say ϕ, then $\lambda \geq q_0$. (Hint: observe that $q_0 \int_{-1}^{+1} \phi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x)\mathrm{d}x$.)

As usual we use the energy method. Let (ϕ, λ) be an eigenpair, then

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

After integration by parts and using the boundary conditions, we obtain

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

which, using the hint, can also be re-written

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathrm{d}x \leq \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Then

$$\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x \leq (\lambda - q_0) \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Assume that ϕ is non-zero, then

$$\lambda - q_0 \geq \frac{\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x}{\int_{-1}^{+1} \phi^2(x)\mathrm{d}x} \geq 0,$$

which proves that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution to exist.

(b) Assume that $p(x) \geq p_0 > 0$ for all $x \in [-1, +1]$ where $p_0 \in \mathbb{R}_+$. Show that $\lambda = q_0$ cannot be an eigenvalue, i.e., prove that $\phi = 0$ if $\lambda = q_0$. (Hint: observe that $p_0 \int_{-1}^{+1} \psi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} p(x)\psi^2(x)\mathrm{d}x$.)

Assume that $\lambda = q_0$ is an eigenvalue. Then the above computation shows that

$$p_0 \int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x \leq \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x = 0,$$

which means that $\int_{-1}^{+1}(\partial_x\phi(x))^2\mathrm{d}x = 0$ since $p_0 > 0$. As a result $\partial_x\phi = 0$, i.e., $\phi(x) = c$ where c is a constant. The boundary conditions $\phi(-1) = 0 = \phi(1)$ imply that $c = 0$. In conclusion $\phi = 0$ if $\lambda = q_0$, thereby proving that (ϕ, q_0) is not an eigenpair.