Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Question 1: Let u solve $\partial_t u - \partial_x(\mu(x,t)\partial_x u) = g(x)e^{-t}$, $x \in (0,L)$, $t > 0$, with $\mu(0,t)\partial_x u(0,t) = 1$, $\mu(L,t)\partial_x u(L,t) = 1 + 2e^{-t}$, $u(x,0) = f(x)$, where $\mu > 0$, f and g are three smooth functions.
(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over the domain $(0,L)$ and apply the fundamental Theorem of calculus:

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\mathrm{d}x = \int_0^L \partial_t u(x,t)\mathrm{d}x &= \int_0^L \partial_x(\mu(x,t)\partial_x u)\mathrm{d}x + e^{-t}\int_0^L g(x)\mathrm{d}x \\ &= \mu(L,t)\partial_x u(L) - \mu(0,t)\partial_x u(0) + e^{-t}\int_0^L g(x)\mathrm{d}x \\ &= 1 + 2e^{-t} - 1 + e^{-t}\int_0^L g(x)\mathrm{d}x.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)\mathrm{d}x = e^{-t}\left(\int_0^L g(x)\mathrm{d}x + 2\right).$$

(b) Use (a) to compute $\int_0^L u(x,t)dx$ as a function of t.

Applying the fundamental Theorem of calculus again gives

$$\begin{aligned}\int_0^L u(x,T)\mathrm{d}x &= \int_0^L u(x,0)\mathrm{d}x + \int_0^T e^{-t}\mathrm{d}t\left(\int_0^L g(x)\mathrm{d}x + 2\right). \\ &= \int_0^L f(x)dx + (1 - e^{-T})\left(\int_0^L g(x)dx + 2\right).\end{aligned}$$

(c) What is the limit of $\int_0^L u(x,t)dx$ as $t \to +\infty$?

The above formula gives

$$\lim_{T\to+\infty}\int_0^L u(x,T)dx = \int_0^L f(x)dx + \int_0^L g(x)dx + 2.$$

Question 2: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t \in (0,1)$, supplemented with the boundary condition $\phi(0) = 0$, $\phi(1) = 0$.

(a) Prove that it is necessary that λ be positive for a non-zero smooth solution to exist.

(i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = 0$, which implies that $\phi'(t) = 0$ for all $t \in (0,1]$. The fundamental theorem of calculus applied between t and 1 implies $\phi(t) = \phi(1) + \int_1^t \phi'(\tau)d\tau = 0$ since $\phi(1) = 0$ and $\phi'(\tau) = 0$ for all $\tau \in (t,1]$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.

(b) The general solution to $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$ is $\phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda})$ for $\lambda \geq 0$. Find all the eigenvalues $\lambda \geq 0$ and the associated nonzero eigenfunctions.

Since $\lambda \geq 0$ by hypothesis, ϕ is of the following form

$$\phi(t) = \phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda}).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\phi(1) = 0 = c_2\sin(2\sqrt{\lambda})$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $2\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. In conclusion

$$\lambda = (n\pi)^2/4, \quad n = 1, 2, \ldots, \qquad \phi(t) = c\sin(n\pi\sqrt{t}).$$

Question 3: Using cylindrical coordinates and the method of separation of variables, solve the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta} u = 0$, inside the domain $D = \{\theta \in [0, \frac{3\pi}{2}],\ r \in [0,1]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r, \frac{3\pi}{2}) = 0$, $u(1,\theta) = \sin(\frac{4}{3}\theta)$. (Give all the details.)

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(1) We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{3\pi}{2}) = 0$, and $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}g(r)) = \lambda g(r)$.

(2) The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\frac{3\pi}{2}) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(1,\theta) = \sin(\frac{4}{3}\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

(3) The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{3\pi}{2}) = 0$ implies $\sqrt{\lambda}\frac{3\pi}{2} = n\pi$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = \frac{2}{3}n$.

(4) From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $\frac{2}{3}n = \alpha$. The boundary condition at $r = 1$ gives $\sin(\frac{4}{3}\theta) = 1^{\frac{2}{3}n}\sin(\frac{2}{3}n\theta)$ for all $\theta \in [0, \frac{3\pi}{2}]$. This implies $n = 2$.

(5) Finally, the solution to the problem is

$$u(r,\theta) = r^{\frac{4}{3}}\sin(\frac{4}{3}\theta).$$

Question 4: Let $k : [-1, +1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, if $x \in [-1, 0]$ and $k(x) = 3$ if $x \in (0, 1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $\partial_x T(-1) = T(-1) + 3$ and $-\partial_x T(1) = T(1) - 7$.

(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 3$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1, +1]$.

On $[-1, 0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $\partial_x T^-(-1) - T^-(-1) = 3 = 2b - a$. This gives $a = 2b - 3$ and $T^-(x) = 2b - 3 + bx$.

We proceed similarly on $[0, +1]$ and we obtain $T^+(x) = c + dx$. The Robin boundary condition at $x = +1$ gives $-\partial_x T^+(+1) - T^+(1) = -7 = -2d - c$. This implies $c = -2d + 7$ and $T^+(x) = -2d + 7 + dx$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b - 3 = -2d + 7, \qquad \text{and} \qquad 2b = 3d.$$

This implies $d = 2$ and $b = 3$. In conclusion

$$T(x) = \begin{cases} 3x + 3 & \text{if } x \in [-1, 0], \\ 2x + 3 & \text{if } x \in [0, +1]. \end{cases}$$

Question 5: Consider the square $D = (-1,+1)\times(-1,+1)$. Let $f(x,y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\nabla^2 u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2 = 1$ and $y \in (-1,1)$ or $y^2 = 1$ and $x \in (-1,1)$. In the first case, $x^2 = 1$ and $y \in (-1,1)$, we have

$$f(x,y) = 1 - y^2 - 3, \qquad y \in (-1,1).$$

The maximum is -2 and the minimum is -3. In the second case, $y^2 = 1$ and $x \in (-1,1)$, we have

$$f(x,y) = x^2 - 1 - 3, \qquad x \in (-1,1).$$

The maximum is -3 and the minimum is -4. We finally can conclude

$$\min_{(x,y)\in\partial D} f(x,y) = \min_{-1\le x\le 1} x^2 - 4, = -4, \qquad \max_{(x,y)\in\partial D} f(x,y) = \max_{-1\le y\le 1} -2 - y^2 = -2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -4, \qquad \max_{(x,y)\in\overline{D}} u(x,y) = -2$$

Question 6: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.

(b) For what values of $x \in \mathbb{R}$ is $FS(f)$ equal to x^4? (Explain)

The periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is piecewise smooth and globally continuous since $f(L) = f(-L)$. This means that the Fourier series is equal to x^4 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x^3)$ by differentiating $\frac{1}{4}FS(x^4)$ term by term? (Explain)

Yes it is possible since the periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is continuous and piecewise smooth.

Question 7: Let L be a positive real number. Let $P_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2 \text{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $1 + t$ in V with respect to the above norm. (Hint: $\int_{-L}^{L} t \sin(\pi t/L)\text{d}t = 2L^2/\pi$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1 \cos(\pi t/L) + b_1 \sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L} \int_{-L}^{L} (1 + t)\text{d}t = 1,$$

$$a_1 = \frac{1}{L} \int_{-L}^{L} (1 + t) \cos(\pi t/L)\text{d}t = 0$$

$$b_1 = \frac{1}{L} \int_{-L}^{L} (1 + t) \sin(\pi t/L)\text{d}t = \frac{1}{L} \int_{-L}^{L} t \sin(\pi t/L)\text{d}t = -2\cos(\pi)\frac{L}{\pi} = \frac{2L}{\pi}.$$

As a result

$$FS_1(t) = 1 + \frac{2L}{\pi} \sin(\pi t/L)$$

(b) Compute the best approximation of $h(t) = 2\cos(2\pi t/L) - 5\sin(3\pi t/L)$ in P_1.

The function $h(t)2\cos(2\pi t/L) - 5\sin(3\pi t/L)$ is orthogonal to all the members of P_1 since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in P_1 is zero

$$FS_1(h) = 0.$$