Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

**Question 1:** Let $\nabla\times$ be the curl operator acting on vector fields: i.e., let $A = (A_1, A_2, A_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a three-dimensional vector field over $\mathbb{R}^3$, then $\nabla\times A = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$. Accept as a fact that $\nabla\cdot(A\times B) = B\cdot\nabla\times A - A\cdot\nabla\times B$ for all smooth vector fields $A$ and $B$. Let $\Omega$ be a subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Find an integration by parts formula for $\int_\Omega B\cdot\nabla\times A \mathrm{d}x$.

Using the divergence Theorem we infer that

$$\int_\Omega (B\cdot\nabla\times A - A\cdot\nabla\times B)\mathsf{d}x = \int_\Omega \nabla\cdot(A\times B) = \int_{\partial\Omega} (A\times B)\cdot n\mathsf{d}s.$$

which implies that

$$\int_\Omega B\cdot\nabla\times A\mathsf{d}x = \int_\Omega A\cdot\nabla\times B\mathsf{d}x + \int_{\partial\Omega} (A\times B)\cdot n\mathsf{d}s.$$

**Question 2:** Let $u, f : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions of class $C^1$. (a) Compute $\partial_x f(u(x))$.

Using the chain rule we obtain

$$\partial_x f(u(x)) = f'(u(x))\partial_x u.$$

where $f'$ denotes the derive of $f$.

(b) Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be functions of class $C^1$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $F(v) = \int_0^v f'(t)\psi'(t)\mathrm{d}t$. Use (a) to compute $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x))$.

Using the chain rule we obtain

$$\partial_x(F(u(x)) = F'(u(x))\partial_x u(x) = f'(u(x))\psi'(u(x))\partial_x u(x) = \partial_x(f(u(x)))\psi'(u(x)).$$

This means that $\partial_x(F(u(x)) = \partial_x(f(u(x)))\psi'(u(x))$.

(c) Using the notation of (a) and (b), assume that $u(\pm\infty) = 0$ and compute $\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\mathrm{d}x$.

Using (b) and $u(\pm\infty) = 0$ we have

$$\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\mathsf{d}x = \int_{-\infty}^{+\infty} \partial_x(F(u(x)))\mathsf{d}x = F(u(x))|_{-\infty}^{+\infty} = F(0) - F(0) = 0.$$

**Question 3:** Let $u$ solve $\partial_t u - \partial_x((\sin(x)+2)\partial_x u) = g(x)e^{-t}$, $x \in (0,L)$, with $\partial_x u(0,t) = \sin(L)+2$, $\partial_x u(L,t) = 2$, $u(x,0) = f(x)$, where $f$ and $g$ are two smooth functions.

(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of $t$.

Integrate the equation over the domain $(0,L)$ and apply the fundamental Theorem of calculus:

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)dx &= \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_x((\sin(x)+2)\partial_x u)dx + e^{-t}\int_0^L g(x)dx \\ &= (\sin(L)+2)\partial_x u(L) - (\sin(0)+2)\partial_x u(0) + e^{-t}\int_0^L g(x)dx \\ &= (\sin(L)+2)2 - 2(\sin(L)+2) + e^{-t}\int_0^L g(x)dx \\ &= e^{-t}\int_0^L g(x)dx.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)dx = e^{-t}\int_0^L g(x)dx.$$

(b) Use (a) to compute $\int_0^L u(x,t)dx$ as a function of $t$.

Applying the fundamental Theorem of calculus again gives

$$\begin{aligned}\int_0^L u(x,T)dx &= \int_0^L u(x,0)dx + \int_0^T \frac{d}{dt}\int_0^L u(x,t)dxdt \\ &= \int_0^L f(x)dx + (1-e^{-T})\int_0^L g(x)dx.\end{aligned}$$

(c) What is the limit of $\int_0^L u(x,t)dx$ as $t \to +\infty$?

The above formula gives

$$\lim_{T\to+\infty}\int_0^L u(x,T)dx = \int_0^L f(x)dx + \int_0^L g(x)dx.$$

**Question 4:** Consider the vibrating beam equation $\partial_{tt}u(x,t) + \partial_{xx}\big(\frac{x^2+\cos(x)}{1+x^2}\partial_{xx}u(x,t)\big) = 0$, $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$, $x \in (-\infty,+\infty)$, $t > 0$ with $u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$, $\partial_{xx}u(\pm\infty,t) = 0$. Use the energy method to compute $\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + \frac{x^2+\cos(x)}{1+x^2}[\partial_{xx}u(x,t)]^2)\mathrm{d}x$ in terms of $f$ and $g$. Give all the details. (Hint: test the equation with $\partial_t u(x,t)$).

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Using the hint we have

$$0 = \int_{-\infty}^{+\infty}(\partial_{tt}u(x,t)\partial_t u(x,t) + \partial_{xx}\left(\frac{x^2+\cos(x)}{1+x^2}\partial_{xx}u(x,t)\right))\mathrm{d}x$$

Using the product rule, $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_t u(x,t)$, and integrating by parts two times (i.e., applying the fundamental theorem of calculus) we obtain

$$\begin{aligned}0 &= \int_{-\infty}^{+\infty}(\frac{1}{2}\partial_t(\partial_t u(x,t))^2 - \partial_x\left(\frac{x^2+\cos(x)}{1+x^2}\partial_{xx}u(x,t)\right)\partial_t\partial_x u(x,t))\mathrm{d}x\\ &= \int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2 + \left(\frac{x^2+\cos(x)}{1+x^2}\right)\partial_{xx}u(x,t)\partial_t\partial_{xx}u(x,t))\mathrm{d}x.\end{aligned}$$

We apply again the product rule $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_{xx}u(x,t)$,

$$0 = \int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2 + \frac{1}{2}\frac{x^2+\cos(x)}{1+x^2}\partial_t(\partial_{xx}u(x,t))^2)\mathrm{d}x.$$

Switching the derivative with respect to $t$ and the integration with respect to $x$, this finally gives

$$0 = \frac{1}{2}\partial_t\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + \frac{x^2+\cos(x)}{1+x^2}[\partial_{xx}u(x,t)]^2)\mathrm{d}x.$$

In other words,

$$\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + \frac{x^2+\cos(x)}{1+x^2}[\partial_{xx}u(x,t)]^2)\mathrm{d}x = \int_{-\infty}^{+\infty}(g(x)^2 + \frac{x^2+\cos(x)}{1+x^2}[\partial_x f(x)]^2)\mathrm{d}x.$$

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**Question 5:** Let $k, f : [-1, +1] \longrightarrow \mathbb{R}$ be such that $k(x) = 3$, $f(x) = -6$ if $x \in [-1, 0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0, 1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 1$ and $\partial_x T(1) = 1$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function $T$ and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let $T^-$ denote the solution on $[-1, 0]$ and $T^+$ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 3$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find $T(x)$, $x \in [-1, +1]$. Give all the details.

On $[-1, 0]$ we have $k^-(x) = 3$ and $f^-(x) = -6$ which implies $-3\partial_{xx}T^-(x) = -6$. This in turn implies $T^-(x) = x^2 + ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = 1 = 1 - a + b$. This gives $a = b$ and $T^-(x) = x^2 + bx + b$.

We proceed similarly on $[0, +1]$ and we obtain $-\partial_{xx}T^-(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Neumann condition at $x = 1$ implies $T^+(1) = 1 = -2 + c$. This gives $c = 3$ and $T^-(x) = -x^2 + 3x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $3b = 3$, respectively. In conclusion $b = 1$, $d = 1$ and

$$T(x) = \begin{cases} x^2 + x + 1 & \text{if } x \in [-1, 0], \\ -x^2 + 3x + 1 & \text{if } x \in [0, 1]. \end{cases}$$

**Question 6:** (a) Compute the coefficients of the sine series of $f(x) = x$ for $x \in [0, +\pi]$. (Recall that by definition $\mathrm{SS}(f)(x) = \sum_{m=1}^{+\infty} b_m \sin(mx)$ with $b_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\mathrm{d}x$.)

The definition of $\mathsf{SS}(f)(x)$ implies that

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^\pi x\sin(mx)\mathrm{d}x = -\frac{2}{\pi}\int_0^\pi -\frac{1}{m}\cos(mx)\mathrm{d}x + \frac{2}{\pi}[-x\frac{1}{m}\cos(mx)]_0^\pi \\ &= \frac{2}{m}(-1)^{m+1}. \end{aligned}$$

As a result $\mathsf{SS}(f)(x) = \sum_{m=1}^{+\infty} \frac{2}{m}(-1)^{m+1}\sin(mx)$.

(b) For which values of $x$ in $[0, +\pi]$ does the sine series coincide with $f(x)$? (Explain).

The sine series coincides with the function $f(x)$ over the entire interval $[0, +\pi)$ since $f(0) = 0$ and $f$ is smooth over $[0, +\pi)$. The series does not coincide with $f(+\pi)$ since $f(+\pi) \neq 0$.

(c) The sine series of $x^2$ over $[0, +\pi]$ is $\mathrm{SS}(x^2)(x) = \sum_{m=1}^{+\infty}(\frac{4}{m^3\pi}((-1)^m - 1) + \frac{2\pi}{m}(-1)^{m+1})\sin(mx)$. Compute the sine series of $h(x) = x(\pi - x)$. (Hint: use (a))

Let $h(x) = x(\pi - x)$. Note that by linearity of the sine series we have

$$\mathsf{SS}(h)(x) = \mathsf{SS}(\pi x)(x) - \mathsf{SS}(x^2)(x),$$

as a result $b_m(h) = \pi b_m(x) - b_m(x^2)$, i.e.,

$$b_m(h) = \pi\frac{2}{m}(-1)^{m+1} - (\frac{4}{m^3\pi}((-1)^m - 1) + \frac{2\pi}{m}(-1)^{m+1}) = \frac{4}{m^3\pi}(1 + (-1)^{m+1}).$$

In conclusion

$$\mathsf{SS}(h)(x) = \sum_{m=1}^{\infty} \frac{4}{m^3\pi}(1 + (-1)^{m+1})\sin(mx).$$

(d) Compute the cosine series of the function $g(x) := \pi - 2x$ defined over $[0, +\pi]$. (Hint: $\partial_x(x(\pi - x)) = \pi - 2x$.)

Observe that $h(0) = h(\pi) = 0$; as a result the sine series of $h$ is continuous at $0$ and $+\pi$. This in turn implies that it is the legitimate to differentiate the sine series of $h$ term by term to obtain the cosine series of $h'(x) = g(x)$. In other words,

$$\mathsf{CS}(g)(x) = \partial_x \mathsf{SS}(h)(x) = \sum_{m=1}^{\infty} \frac{4}{m^2\pi}(1 + (-1)^{m+1})\cos(mx).$$

(e) Compute the sine series of $h(x) = \sin(x)$ for $x \in [0, +\pi]$.

Obviously

$$\mathsf{SS}(h)(x) = \sin(x), \quad \forall x \in \mathbb{R}.$$

**Question 7:** Using cylindrical coordinates and the method of separation of variables, solve the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{3}{2}\pi],\ r \in [0,3]\}$, subject to the boundary conditions $\partial_\theta u(r,0) = 0$, $u(r, \frac{3}{2}\pi) = 0$, $u(3,\theta) = 9\cos(\theta)$. (Give all the details of all the steps.)

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(1) We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi'(0) = 0$ and $\phi(\frac{3}{2}\pi) = 0$, and $r\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}}{\mathrm{d}r}g(r)) = \lambda g(r)$.

(2) The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi'(0) = 0, \qquad \phi(\frac{3}{2}\pi) = 0,$$

implies that $\lambda$ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(3,\theta) = 9\sin(2\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

(3) The boundary condition $\phi'(0) = 0$ implies $c_2 = 0$. The boundary condition $\phi(\frac{3}{2}\pi) = 0$ implies that $\cos(\sqrt{\lambda}\frac{3}{2}\pi) = 0$, i.e., $\sqrt{\lambda}\frac{3}{2}\pi = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = \frac{1}{3}(2n+1)$, $n = 0, 1, 2, \ldots$.

(4) From class we know that $g(r)$ is of the form $r^\alpha$, $\alpha \geq 0$. The equality $r\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}}{\mathrm{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $\frac{1}{3}(2n+1) = \alpha = \sqrt{\lambda}$. The boundary condition at $r = 3$ gives $9\cos(\theta) = c_1 3^{\frac{1}{3}(2n+1)}\cos(\frac{1}{3}(2n+1)\theta)$ for all $\theta \in [0, \frac{3}{2}\pi]$. This implies $n = 1$ and $c_1 = 3$.

(5) Finally, the solution to the problem is

$$u(r,\theta) = 3r\cos(\theta).$$

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**Question 8:** Let $p, q : [-1, +1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1, +1]$, where $q_0 \in \mathbb{R}$. Consider the eigenvalue problem $-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = \lambda\phi(x)$, supplemented with the boundary conditions $\partial_x\phi(-1) = 0$ and $-\partial_x\phi(1) = 2\phi(1)$.

(a) Prove that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution, $\phi$, to exist. (Hint: $q_0 \int_{-1}^{+1} \phi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x)\mathrm{d}x$.)

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As usual we use the energy method. Let $(\phi, \lambda)$ be an eigenpair, then

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

After integration by parts and using the boundary conditions, we obtain

$$\begin{aligned}\lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x &= \int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathrm{d}x - 2p(x)\partial_x\phi(x)\phi(x)|_{-1}^{+1} \\ &= \int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathrm{d}x + 2p(1)\phi(1)^2\end{aligned}$$

which, using the hint, can also be re-written

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathrm{d}x + 2p(1)\phi(1)^2 \leq \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Then

$$2p(1)\phi(1)^2 + \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x \leq (\lambda - q_0) \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Assume that $\phi$ is non-zero, then

$$\lambda - q_0 \geq \frac{\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x + 2p(1)\phi(1)^2}{\int_{-1}^{+1} \phi^2(x)\mathrm{d}x} \geq 0,$$

which proves that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution to exist.

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(b) Assume that $p(x) \geq p_0 > 0$ for all $x \in [-1, +1]$ where $p_0 \in \mathbb{R}_+$. Show that $\lambda = q_0$ cannot be an eigenvalue, i.e., prove that $\phi = 0$ if $\lambda = q_0$. (Hint: $p_0 \int_{-1}^{+1} \psi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} p(x)\psi^2(x)\mathrm{d}x$.)

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Assume that $\lambda = q_0$ is an eigenvalue. Then the above computation shows that

$$p_0 \int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x \leq \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x = 0,$$

which means that $\int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x = 0$ since $p_0 > 0$. As a result $\partial_x\phi = 0$, i.e., $\phi(x) = c$ where $c$ is a constant. The boundary condition $-\partial\phi(1) = 2\phi(1)$ implies that $c = 0$. In conclusion $\phi = 0$ if $\lambda = q_0$, thereby proving that $(\phi, q_0)$ is not an eigenpair.

**Question 9:** Use the Fourier transform technique to solve $\partial_t u(x,t) - \partial_{xx}u(x,t) + \cos(t)\partial_x u(x,t) + (1+2t)u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$. (Hint: use the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{i\omega x}\mathrm{d}x$), the result $\mathcal{F}(\mathrm{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\frac{\omega^2}{4\alpha}}$, the convolution theorem and the shift lemma: $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)\mathrm{e}^{i\omega\beta}$. Go slowly and give all the details.)

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Applying the Fourier transform to the equation gives

$$\partial_t\mathcal{F}(u)(\omega,t) + \omega^2\mathcal{F}(u)(\omega,t) + \cos(t)(-i\omega)\mathcal{F}(u)(\omega,t) + (1+2t)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t\mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = -\omega^2 + i\omega\cos(t) - (1+2t).$$

Then applying the fundamental theorem of calculus between $0$ and $t$, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = -\omega^2 t + i\omega\sin(t) - (t+t^2).$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)\mathsf{e}^{-\omega^2 t}\mathsf{e}^{i\omega\sin(t)}\mathsf{e}^{-(t+t^2)}.$$

Using the result $\mathcal{F}(\mathsf{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathsf{e}^{-\frac{\omega^2}{4\alpha}}$ where $\alpha = \frac{1}{4t}$, this implies that

$$\begin{aligned}\mathcal{F}(u)(\omega,t) &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0)(\omega)\mathcal{F}(\mathsf{e}^{-\frac{x^2}{4t}})(\omega)\mathsf{e}^{i\omega\sin(t)}\mathsf{e}^{-(t+t^2)} \\ &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0 * \mathsf{e}^{-\frac{x^2}{4t}})(\omega)\mathsf{e}^{i\omega\sin(t)}\mathsf{e}^{-(t+t^2)}.\end{aligned}$$

Then setting $g = u_0 * \mathsf{e}^{-\frac{x^2}{4t}}$ the convolution theorem followed by the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\mathcal{F}(g(x-\sin(t)))(\omega)\mathsf{e}^{-(t+t^2)}.$$

This finally gives

$$u(x,t) = \sqrt{\frac{1}{4\pi t}}g(x-\sin(t))\mathsf{e}^{-(t+t^2)} = e^{-(t+t^2)}\sqrt{\frac{1}{4\pi t}}\int_{-\infty}^{+\infty} u_0(y)\mathsf{e}^{-\frac{(x-\sin(t)-y)^2}{4t}}\mathsf{d}y.$$

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