M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, November 14, 2008
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{3}$$

Question 1

Solve the wave equation on the semi-infinite domain $(0, +\infty)$,

$$\partial_{tt} w - 4\partial_{xx} w = 0, \quad x \in (0, +\infty),\ t > 0$$
$$w(x,0) = (1+x^2)^{-1}, \quad x \in (0,+\infty); \qquad \partial_t w(x,0) = 0, \quad x \in (0,+\infty); \quad \text{and} \quad \partial_x w(0,t) = 0, \quad t > 0.$$

(Hint: Consider a particular extension of w over $\mathbb{R}$)

We define $f(x) = (1+x^2)^{-1}$ and its even extension $f_e(x)$ on $-\infty, +\infty$. Let w_e be the solution to the wave equation over the entire real line with f_e as initial data:

$$\partial_{tt} w_e - 4\partial_{xx} w_e = 0, \quad x \in \mathbb{R},\ t > 0$$
$$w_e(x,0) = f_e(x), \quad x > 0, \qquad \partial_t w_e(x,0) = 0, \quad x \in \mathbb{R}.$$

The solution to this problem is given by the D'Alembert formula

$$w_e(x,t) = \frac{1}{2}(f_e(x-2t) + f_e(x+2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Let x be positive. Then $w(x,t) = w_e(x,t)$ for all $x \in (0,+\infty)$, since by construction $\partial_x w_e(0,t) = 0$ for all times.

Case 1: If $x - 2t > 0$, $f_e(x-2t) = f(x-2t)$; as a result

$$w(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t), \quad \text{If } x - 2t > 0.$$

Case 2: If $x - 2t < 0$, $f_e(x-2t) = f(-x+2t)$; as a result

$$w(x,t) = \frac{1}{2}(f(-x+2t) + f(x+2t)), \quad \text{If } x - 2t < 0.$$

Note that actually $f_e(x) = (1+x^2)^{-1}$; as a result, the solution can also be re-written as follows:

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right).$$

Question 2

Solve the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 \leq x \leq 1, \quad 0 < t, \\ &\partial_x u(0,t) = 0, \quad \partial_x u(1,t) = 0 && 0 < t, \\ &u(x,0) = \cos(\pi x), \quad u_t(x,0) = 0, && 0 < x < +1. \end{aligned}$$

(Hint: Consider the periodic extension over $\mathbb{R}$ of a particular extension of u over $[-1+1]$).

The even extension of u over $[-1+1]$, say u_e, satisfies the PDE and the initial conditions, and always satisfies $\partial_x u_e(0,t) = 0$, $\partial_x u_e(1,t) = 0$. Since $\partial u_e(1,t)$, we deduce $\partial u_e(-1,t) = 0$. This means that the periodic extension of u_e, says u_p, is smooth and also satisfies the PDE plus the initial conditions. By construction $\partial_x u_p(0,t) = 0$ and $\partial_x u_e(1,t) = 0$. As a result, we can obtain u by computing the solution of the wave equation on $\mathbb{R}$ using the periodic extension over $\mathbb{R}$ of the even extension of the initial data over $[-1+1]$, i.e., $u = u_p|_{[0,1]}$

We have to define the even extension of $\cos(\pi x)$ on $(-1,+1)$. Clearly $\cos(\pi x)$ is the even extension. Now we define the periodic extension of $\cos(\pi x)$ over the entire real line. Clearly $\cos(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\cos(\pi(x-t)) + \cos(\pi(x+t)) \\ &= \tfrac{1}{2}((\cos(\pi t)\cos(\pi x) + \sin(\pi t)\sin(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\cos(\pi x) - \sin(\pi t)\sin(\pi x)) \\ &= \cos(\pi t)\cos(\pi x). \end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\cos(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 3

Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = -s$ and $t_\Gamma(s) = -s$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 3u_x + 2u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } x = t \end{cases} \quad \text{for all } (x,t) \text{ in } \Gamma.$$

We define the characteristics by

$$\frac{dx(t,s)}{dt} = 3, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t \phi(t,s) + 2\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-2t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}, \qquad x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = -s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$. This means $x(t,s) = -s + 3(t+s)$ and we obtain $s = \frac{1}{2}(x - 3t)$, which means

$$u(x,t) = 2e^{-2(\frac{1}{2}(x-3t)-t)} = 2e^{t-x}, \qquad \text{if } x - 3t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$. This means $x(t,s) = s + 3t$ and we obtain $s = x - 3t$, which means

$$u(x,t) = e^{-2t}, \qquad \text{if } x - 3t > 0.$$

Question 4

Solve the integral equation: $f(x) + \frac{3}{2}\int_{-\infty}^{+\infty} e^{-|x-y|} f(y)dy = e^{-|x|}$, for all $x \in (-\infty, +\infty)$.

The equation can be re-written

$$f(x) + \frac{3}{2} e^{-|x|} * f = e^{-|x|}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (2))

$$\mathcal{F}(f) + \frac{3}{2} 2\pi \mathcal{F}(e^{-|x|})\mathcal{F}(f) = \mathcal{F}(e^{-|x|}).$$

Using (3), we obtain

$$\mathcal{F}(f) + 3\pi \frac{1}{\pi} \frac{1}{1+\omega^2} \mathcal{F}(f) = \frac{1}{\pi} \frac{1}{1+\omega^2},$$

which gives

$$\mathcal{F}(f) \frac{\omega^2 + 4}{1+\omega^2} = \frac{1}{\pi} \frac{1}{1+\omega^2}.$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{\pi} \frac{1}{4+\omega^2} = \frac{1}{2} \mathcal{F}(e^{-2|x|}).$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{2} e^{-2|x|}$.

Question 5

Consider the following conservation equation

$$\partial_t \rho + \partial_x (q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = q'(\rho) = 2(1-\rho(X(t,x_0),t)), \quad X(0,x_0) = x_0.$$

Set $\phi(t) = \rho(X(t,x_0),t)$ and insert in the equation. We obtain that $\partial_t \phi(t,x_0) = 0$; meaning that $\phi(t,x_0) = \phi(0,x_0)$, i.e., ρ is constant along the characteristics: $\rho(X(t,x_0),t) = \rho(x_0,0) = \rho_0(x_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t,x_0) = 2(1-\rho_0(x_0))t + x_0$. The implicit representation of the solution is

$$X(t,x_0) = 2(1-\rho_0(x_0))t + x_0; \quad \rho(X(t,x_0),t) = \rho_0(x_0)$$

We then have two cases depending whether x_0 is positive or negative.
Case 1: $x_0 < 0$, then $\rho_0(x_0) = 2$ and $X(t,x_0) = 2(1-2)t + x_0 = -2t + x_0$. This means $x_0 = X(t,x_0) + 2t$ and

$$\rho(x,t) = 2 \quad \text{if} \quad x < -2t.$$

Case 2: $x_0 > 0$, then $\rho_0(x_0) = 1$ and $X(t,x_0) = 2(1-1)t + x_0 = x_0$. This means $x_0 = X(t,x_0)$ and

$$\rho(x,t) = 1 \quad \text{if} \quad 0 < x.$$

We see that there is a gap in the region $\{-2t < x < 0\}$. This implies that there is an expansion wave. We have to consider a third case $x_0 = 0$ and $\rho_0 \in (1,2)$.
Case 3: $x_0 = 0$, then $X(t,x_0) = 2(1-\rho_0)t$, i.e., $\rho_0 = 1 - \frac{X(t,x_0)}{2t}$. This means that

$$\rho(x,t) = 1 - \frac{x}{2t}, \quad \text{if} \quad -2t < x < 0.$$
