

M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, October 26, 2009
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded.**

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)e^{i\omega x} dx, \quad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x} d\omega, \quad (1)$$

$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \quad (2)$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi} \frac{\alpha}{\omega^2 + \alpha^2}, \quad \mathcal{F}\left(\frac{2\alpha}{x^2 + \alpha^2}\right)(\omega) = e^{-\alpha|\omega|}, \quad (3)$$

$$\mathcal{F}(f(x - \beta)) = e^{i\omega\beta} \mathcal{F}(f) \quad (4)$$

Question 1

Use the Fourier transform method to solve the equation $\partial_t u + \cos(t)\partial_x u = 0$, $u(x, 0) = \sin(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\cos(t)\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \cos(t)\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega \cos(t)\mathcal{F}(u). \end{aligned}$$

This is an ODE. The solution is

$$\mathcal{F}(u)(\omega, t) = K(\omega)e^{i\omega \sin(t)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega, 0) = K(\omega).$$

The shift lemma (see formula (4)) implies

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0)(\omega)e^{i\omega \sin(t)} = \mathcal{F}(u_0(x - \sin(t))),$$

where $u_0(x) := \sin(x)$. Applying the inverse Fourier transform finally gives

$$u(x, t) = \sin(x - \sin(t)).$$

Question 2

Solve the integral equation: $\int_{-\infty}^{+\infty} f(y)f(x-y)dy = \frac{4}{x^2+4}$, for all $x \in (-\infty, +\infty)$. How many solutions did you find?

The equation can be re-written

$$f * f = \frac{4}{x^2 + 4}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (2))

$$2\pi\mathcal{F}(f)^2 = \mathcal{F}\left(\frac{4}{x^2 + 4}\right)$$

Using (3), we obtain

$$2\pi\mathcal{F}(f)^2 = e^{-2|\omega|}.$$

which gives

$$\mathcal{F}(f) = \pm \frac{1}{\sqrt{2\pi}} e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain

$$f(x) = \pm \frac{1}{\sqrt{2\pi}} \frac{2}{x^2 + 1}.$$

We found two solutions: a positive one and a negative one.

Question 3

Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0, 4)$, $t > 0$, with

$$w(x, 0) = f(x), \quad x \in (0, 4), \quad \partial_t w(x, 0) = 0, \quad x \in (0, 4), \quad \text{and} \quad w(0, t) = 0, \quad w(4, t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1, 2]$, $f(x) = 3 - x$, if $x \in [2, 3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f . Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its odd extension, say $f_{o,p}$, where

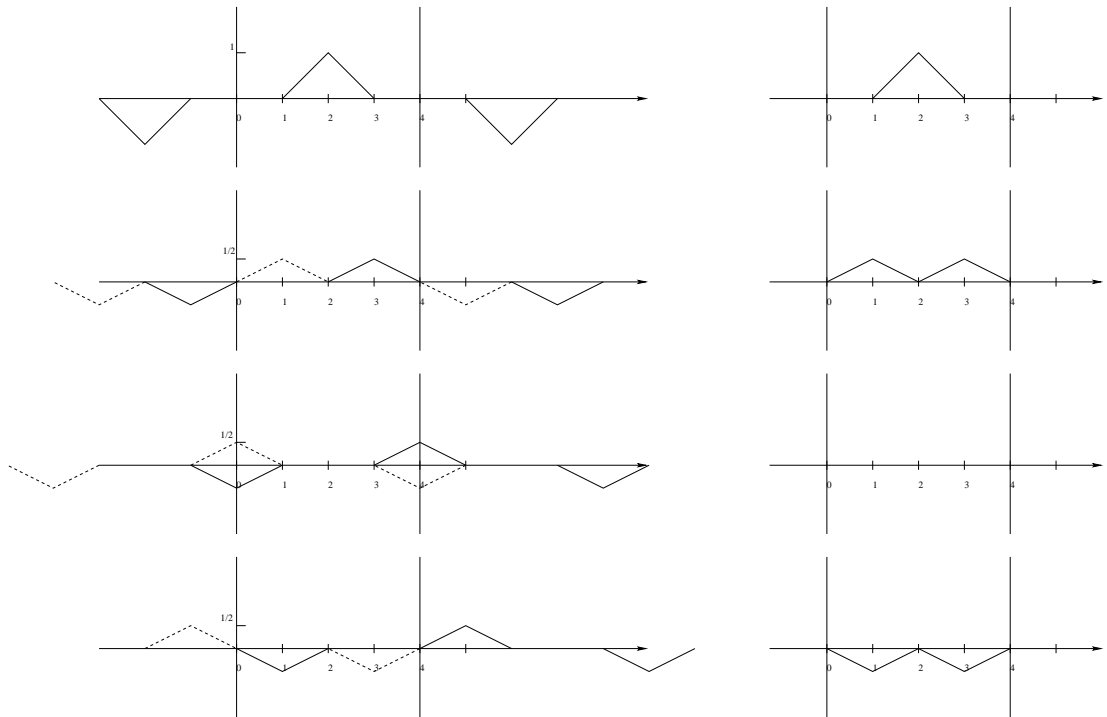
$$f_{o,p}(x+8) = f_{o,p}(x), \quad \forall x \in \mathbb{R}$$

$$f_{o,p}(x) = \begin{cases} f(x) & \text{if } x \in [0, 4] \\ -f(-x) & \text{if } x \in [-4, 0) \end{cases}$$

The solution is

$$u(x, t) = \frac{1}{2}(f_{o,p}(x-t) + f_{o,p}(x+t)).$$

I draw on the left of the figure the graph of $f_{o,p}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.

Solution in domain $[0, 4]$ at $t = 0, 1, 2, 3$

Question 4

Compute the Fourier transform of $e^{-a|x|} \cos(bx)$ (Hint: use $\cos(bx) = \frac{1}{2}(e^{ibx} + e^{-ibx})$), where a and b are real numbers and $a > 0$ (computation should be short and simple).

$$\begin{aligned}\mathcal{F}(e^{-a|x|} \cos(bx))(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a|x|} \frac{1}{2} (e^{ibx} + e^{-ibx}) e^{i\omega x} dx \\ &= \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (e^{-a|x|} e^{ibx} e^{i\omega x} + e^{-a|x|} e^{-ibx} e^{i\omega x}) dx \\ &= \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (e^{-a|x|} e^{i(b+\omega)x} + e^{-a|x|} e^{i(\omega-b)x}) dx \\ &= \frac{1}{2} (\mathcal{F}(e^{-a|x|})(b+\omega) + \mathcal{F}(e^{-a|x|})(\omega-b))\end{aligned}$$

Now we use formula (3) to obtain

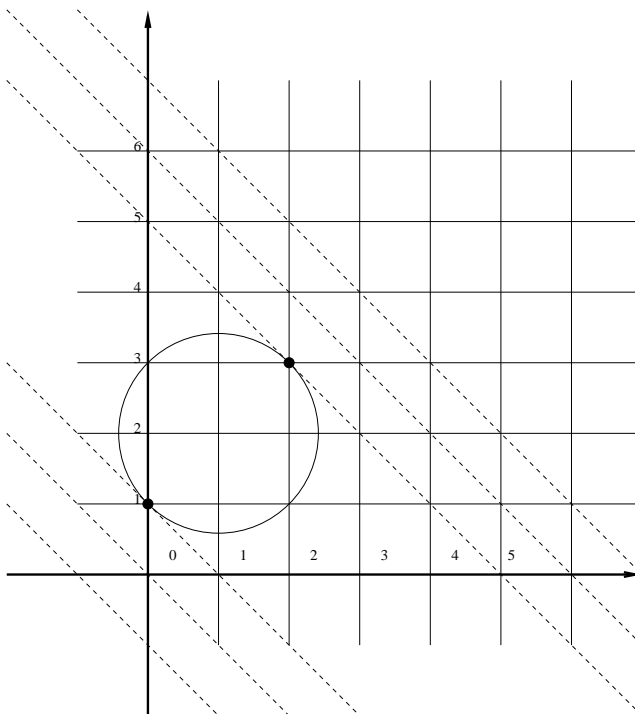
$$\mathcal{F}(e^{-a|x|} \cos(bx))(\omega) = \frac{1}{2\pi} \left(\frac{a}{(b+\omega)^2 + a^2} + \frac{a}{(b-\omega)^2 + a^2} \right).$$

Question 5

Let D be the open disk of radius $\sqrt{2}$ centered at $(1, 2)$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ be a harmonic function in the disk D . Assume that $u(x, y) = (x + y)^2$ on the boundary of disk. Compute $\min_{(x,y) \in \overline{D}} u(x, y)$ and $\max_{(x,y) \in \overline{D}} u(x, y)$. You can give a geometric answer.

Since u is in $\mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ and is harmonic, we can apply the maximum principle (Theorem). This theorem says that the maximum and minimum of u are attained at the boundary of the disk. The problem then amounts to finding the maximum and minimum of $(x+y)^2$ over the circle of radius $\sqrt{2}$ centered at $(1, 2)$. The iso-values of the function $(x+y)^2$ are parallel lines of slope -1 . These iso-line are perpendicular to the gradient of $(x+y)^2$ which is the vector $(2(x+y), 2(x+y)) = 2(x+y)(1, 1)$. We must find the two tangent lines to the circle that are perpendicular to the vector $(1, 1)$. One easily verify that $(0, 1)$ and $(2, 3)$ are the tangent points (verify that the segment connecting these two points is parallel to the vector $(1, 1)$ and passes through the center of the circle). As a result

$$\min_{(x,y) \in \overline{D}} u(x, y) = 1 \quad \text{and} \quad \max_{(x,y) \in \overline{D}} u(x, y) = 25.$$



The dashed lines are iso-lines of $(x + y)^2$.