Mid term 2. **Notes, books, and calculators are not authorized.** Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here are some formulae that you may want to use:

$$\text{Solution to } y'(t) + g(t)y(t) = 0 \text{ is } y(t) = y(0)e^{-\int_0^t g(\tau)\mathrm{d}\tau}. \tag{1}$$

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{2}$$

$$\mathcal{F}(f*g)(\omega) = 2\pi\mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega), \qquad \mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta} \tag{3}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \qquad \sqrt{\frac{\pi}{\alpha}}\mathcal{F}(e^{-\frac{x^2}{4\alpha}}) = e^{-\alpha\omega^2}. \tag{4}$$

$$FS(f)(x) = \sum_{n=0}^{+\infty} a_n \cos(n\pi x/L) + \sum_{n=1}^{+\infty} b_n \sin(n\pi x/L), \tag{5}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi\frac{x}{L})\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi\frac{x}{L})\mathrm{d}x \tag{6}$$

Question 1: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^2$. (a) Sketch the graph of the Fourier series of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.

(b) For what values of x is $FS(f)$ equal to x^2?

The periodic extension of $f(x) = x^2$ over $\mathbb{R}$ is piecewise smooth and globally continuous. This means that the Fourier series is equal to x^2 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x)$ by differentiating $\frac{1}{2}FS(f)$ term by term?

Yes it is possible since the periodic extension of $f(x) = x^2$ over $\mathbb{R}$ is continuous and piecewise smooth.

Question 2: Let $f(x) = x$, $x \in [-L, L]$. (a) Sketch the graph of the Fourier series of f.

The Fourier series is equal to the periodic extension of f, except at the points $(2n+1)L$, $n \in \mathbb{Z}$ where it is equal to $0 = \frac{1}{2}(1-1)$.

(b) Compute the coefficients of the Fourier series of f. (Hint: $\int_a^b tg(t)\mathrm{d}t = [t\int g]_a^b - \int_a^b(\int g)(t)\mathrm{d}t$).

f is odd, hence the cosine coefficients are zero. The sine coefficients b_n are obtained by integration by parts

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin(\frac{n\pi x}{L})dx = -\frac{1}{L}\frac{L}{n\pi}[x\cos(n\pi\frac{x}{L})]_{-L}^{+L} + \frac{L}{n\pi}\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})dx.$$

As a result $b_n = -2\cos(n\pi)\frac{L}{n\pi} = 2(-1)^{n+1}\frac{L}{n\pi}$ and

$$FS(f)(x) = \frac{2L}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi x}{L}).$$

Question 3: Let L be a positive real number. Let $V = \mathrm{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2\mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $1+t$ in V with respect to the above norm.

We know from class that the truncated Fourier series

$$S_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(1+t)\mathrm{d}t = 1,$$
$$a_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\cos(\pi t/L)\mathrm{d}t = 0$$
$$b_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\sin(\pi t/L)\mathrm{d}t = \frac{1}{L}\int_{-L}^{L}t\sin(\pi t/L)\mathrm{d}t = -2\cos(\pi)\frac{L}{\pi} = \frac{2L}{\pi}.$$

As a result

$$S_1(t) = 1 + \frac{2L}{\pi}\sin(\pi t/L)$$

(b) Compute the best approximation of $3 + 2\cos(\pi t/L) - 5\sin(\pi t/L)$ in V.

The function $3 + 2\cos(\pi t/L) - 5\sin(\pi t/L)$ is a member of V; as a result, The best approximation is the function itself.

Question 4: Use the Fourier transform technique to solve $\partial_t u(x,t) + t\partial_x u(x,t) + 2u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$.

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + t(-i\omega)\mathcal{F}(u)(\omega,t) + 2\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega t - 2.$$

Then applying the fundamental theorem of calculus we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = i\omega\frac{1}{2}t^2 - 2t.$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\frac{1}{2}t^2}e^{-2t}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x - \frac{1}{2}t^2)(\omega)e^{-2t}.$$

This finally gives

$$u(x,t) = u_0(x - \frac{1}{2}t^2)e^{-2t}.$$

Question 5: Solve $\partial_t u(x,t) - 2\partial_{xx} u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$ and $u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$.

Applying the Fourier transform with respect to the x-variable:

$$\partial_t \mathcal{F}(u)(\omega,t) - 2(-i\omega)^2 \mathcal{F}(u)(\omega,t) = 0.$$

This give the ODE

$$\partial_t \mathcal{F}(u)(\omega,t) + 2\omega^2 \mathcal{F}(u)(\omega,t) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega,t) = a(\omega)e^{-2\omega^2 t}.$$

The initial condition implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{-2\omega^2 t} = \sqrt{\frac{\pi}{2t}}\mathcal{F}(u_0)(\omega)\mathcal{F}(e^{-\frac{x^2}{8t}})(\omega,t).$$

The convolution theorem gives

$$\mathcal{F}(u)(\omega,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{2t}}\mathcal{F}(u_0 * e^{-\frac{x^2}{8t}})(\omega,t).$$

As a result

$$u(x,t) = \sqrt{\frac{1}{8\pi t}} \int_{-\infty}^{+\infty} u_0(y) e^{-\frac{(x-y)^2}{8t}} \mathrm{d}y.$$

Question 6: Solve the following integral equation (Hint: $(a+b)^2 = a^2 + 2ab + b^2$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\int_{-\infty}^{+\infty} \frac{2}{y^2+1} f(x-y)\mathrm{d}y + 2\pi\frac{4}{x^2+4} = 0 \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 2(\frac{2}{x^2+1}) * f + 2\pi\frac{4}{x^2+4} = 0.$$

We take the Fourier transform and use the convolution theorem (3) together with (4) to obtain

$$\begin{aligned}2\pi\mathcal{F}(f)^2 - 4\pi\mathcal{F}(f)e^{-|\omega|} - 2\pi e^{-2|\omega|} &= 0\\ \mathcal{F}(f)^2 - 2\mathcal{F}(f)e^{-|\omega|} + e^{-2|\omega|} &= 0\\ (\mathcal{F}(f) - e^{-|\omega|})^2 &= 0\end{aligned}$$

This implies

$$\mathcal{F}(f) = e^{-|\omega|}.$$

Taking the inverse Fourier transform, we obtain

$$f(x) = \frac{2}{x^2+1}.$$

Question 7: Prove that if there exists a smooth solution to the Klein-Gordon equation then it is unique: $\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + m^2u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$ and $u(\pm\infty,t) = 0$, $\partial_t u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$. (Hint: test with $\partial_t u(x,t)$ and use $\phi\phi' = (\frac{1}{2}\phi^2)'$.)

Let u_1 and u_2 be two solutions. Then setting $\phi = u_1 - u_2$, we obtain that ϕ solves the homogeneous problem $\partial_{tt}\phi - c^2\partial_{xx}\phi + m^2\phi = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = \phi(x)$, $\partial_t\phi(x,0) = 0$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$, we obtain

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \partial_x\phi\partial_t\partial_x\phi\mathsf{d}x + m^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_x\phi)^2\mathsf{d}x + m^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x \\
&= d_t\left(\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_x\phi)^2\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x\right) \\
&= d_t\left(\int_{-\infty}^{+\infty} \left((\partial_t\phi)^2 + c^2(\partial_x\phi)^2 + m^2\phi^2\right)\mathsf{d}x\right).
\end{aligned}$$

Introduce

$$E(t) = \int_{-\infty}^{+\infty} \left((\partial_t\phi)^2 + c^2(\partial_x\phi)^2 + m^2\phi^2\right)\mathsf{d}x.$$

Then

$$d_t E(t) = 0.$$

The fundamental Theorem of calculus gives

$$E(t) = E(0) = 0.$$

This means

$$\int_{-\infty}^{+\infty} \left((\partial_t\phi(x,t))^2 + c^2(\partial_x\phi(x,t))^2 + m^2\phi^2(x,t)\right)\mathsf{d}x = 0, \quad \text{for all } t \geq 0.$$

This implies

$$0 = (\partial_t\phi(x,t))^2 + c^2(\partial_x\phi(x,t))^2 + m^2\phi^2(x,t), \quad \text{for all } t \geq 0,\ x \in \mathbb{R},$$

i.e. $\phi = 0$, thereby proving the uniqueness.
