M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, March 26, 2012
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{3}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{4}$$

$$\mathcal{F}(f(x-\beta)) = e^{i\omega\beta}\mathcal{F}(f) \tag{5}$$

Question 1: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (-\infty,+\infty)$, $t>0$, with initial data $u(x,0) = \frac{1}{1+x^2}$, $\partial_t u(x,0) = \frac{2x}{(1+x^2)^2}$. Compute the solution $w(x,t)$.

The wave speed is 2. The solution is given by the D'Alembert formula,

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) + \frac{1}{2}\int_{x-t}^{x+t}\frac{2\tau}{(1+\tau^2)^2}\mathsf{d}\tau$$

After integration, we obtain

$$= \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) - \frac{1}{2}\left[\frac{1}{(1+\tau^2)}\right]_{x-t}^{x+t},$$

which finally gives

$$w(x,t) = \frac{1}{1+(x-t)^2}.$$

Question 2: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2}\partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\frac{2t}{1+t^2}\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2}\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega\frac{2t}{1+t^2}\mathcal{F}(u). \end{aligned}$$

This is a first-order linear ODE:

$$\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega\frac{2t}{1+t^2} = i\omega\frac{\mathsf{d}}{\mathsf{d}t}(\log(1+t^2))$$

The solution is

$$\mathcal{F}(u)(\omega,t) = K(\omega)e^{i\omega\log(1+t^2)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega,0) = K(\omega).$$

The shift lemma (see formula (5)) implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\log(1+t^2)} = \mathcal{F}(u_0(x-\log(1+t^2))),$$

Applying the inverse Fourier transform finally gives

$$u(x,t) = u_0(x-\log(1+t^2)).$$

Question 3: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad \partial_x w(0,t) = 0, \quad \partial_x w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f. Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

We know from class that with homogeneous Neumann boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its even extension, say $f_{\mathsf{e,p}}$, where

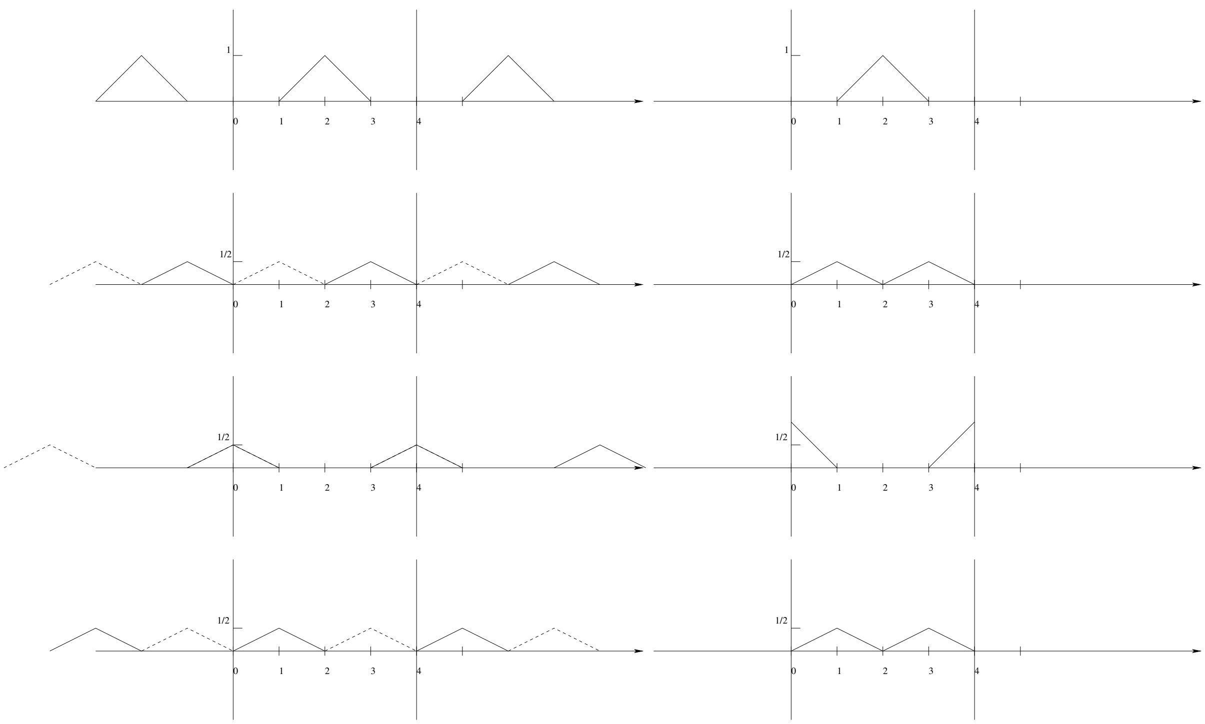
$$f_{\mathsf{e,p}}(x+8) = f_{\mathsf{e,p}}(x), \qquad \forall x \in \mathbb{R}$$

$$f_{\mathsf{e,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ f(-x) & \text{if } x \in [-4,0) \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{e,p}}(x-t) + f_{\mathsf{e,p}}(x+t)).$$

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.



(a) Initial data + periodic extension of the even extension at $t = 0, 1, 2, 3$. Solid line waves move to the right, dotted line waves move to the left

(b) Solution in domain $(0,4)$ at $t = 0, 1, 2, 3$

Question 4: Solve the integral equation:

$$\int_{-\infty}^{+\infty} \left(f(y) - \sqrt{2}e^{-\frac{y^2}{2\pi}} - \frac{1}{1+y^2} \right) f(x-y)dy = -\int_{-\infty}^{+\infty} \frac{\sqrt{2}}{1+y^2} e^{-\frac{(x-y)^2}{2\pi}} \mathrm{d}y, \qquad \forall x \in (-\infty, +\infty).$$

(Hint: there is an easy factorization after applying the Fourier transform.)

The equation can be re-written

$$f * (f - \sqrt{2}e^{-\frac{x^2}{2\pi}} - \frac{1}{1+x^2}) = -\frac{1}{1+x^2} * \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (2))

$$2\pi\mathcal{F}(f) \left(\mathcal{F}(f) - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) - \mathcal{F}(\frac{1}{1+x^2}) \right) = -2\pi\mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})$$

Using (3), (4) we obtain

$$\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) = \sqrt{2}\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}} e^{-\frac{\omega^2}{4\frac{1}{2\pi}}} = e^{-\frac{\pi\omega^2}{2}}$$

$$\mathcal{F}(\frac{1}{1+x^2}) = \frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f) \left(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}} - \frac{1}{2}e^{-|\omega|} \right) = -\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}.$$

This equation can also be re-written as follows

$$\mathcal{F}(f)^2 - \mathcal{F}(f)e^{-\frac{\pi\omega^2}{2}} - \mathcal{F}(f)\frac{1}{2}e^{-|\omega|} + \frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}} = 0,$$

and can be factorized as follows:

$$(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}})(\mathcal{F}(f) - \frac{1}{2}e^{-|\omega|}) = 0.$$

This means that either $\mathcal{F}(f) = e^{-\frac{\pi\omega^2}{2}}$ or $\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}$. Taking the inverse Fourier transform, we finally obtain two solutions

$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = \frac{1}{1+x^2}.$$

Another solution consists of observing that the equation can also be re-written

$$\mathcal{F}(f)^2 - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})\mathcal{F}(f) - \mathcal{F}(\frac{1}{1+x^2})\mathcal{F}(f) + \mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) = 0$$

Question 5: Consider the equation $u'(x) + u = f(x)$ for $x \in (0, 1)$ with $u(0) = a$. Let $G(x, x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

The Green's function is defined by

$$-G'(x, x_0) + G(x, x_0) = \delta(x - x_0), \quad G(1, x_0) = 0.$$

We multiply the equation by u and we integrate over $(0, 1)$ (in the distribution sense),

$$\int_0^1 -G'(x, x_0)u(x)\mathrm{d}x + \int_0^1 G(x, x_0)u(x)\mathrm{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x, x_0)(u'(x) + u(x))\mathrm{d}x - G(1, x_0)u(1) + G(0, x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0)f(x)\mathrm{d}x + aG(0, x_0). \quad \forall x_0 \in (0, 1).$$

(b) Compute $G(x, x_0)$.

For $x < x_0$ and $x_0 > x$ we have

$$-G'(x, x_0) + G(x, x_0) = 0.$$

The solution is

$$G(x, x_0) = \begin{cases} \alpha e^x & \text{for } x < x_0 \\ \beta e^x & \text{for } x > x_0. \end{cases}$$

The boundary condition $G(1, x_0) = 0$ implies $\beta = 0$.

For every $\epsilon > 0$ we have

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} (-G'(x, x_0) + G(x, x_0))\mathrm{d}x \\ &= G(x_0 - \epsilon, x_0) - G(x_0 + \epsilon, x_0) + \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0)\mathrm{d}x \end{aligned}$$

The term $R_\epsilon = \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0)\mathrm{d}x$ can be bounded as follows:

$$|R_\epsilon| \le 2\epsilon \max_{x\in[0,1]} |G(x, x_0)| = 2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1 = G(x_0^-, x_0) - G(x_0^+, x_0) = \alpha e^{x_0}.$$

This implies

$$\alpha = e^{-x_0}.$$

Finally

$$G(x, x_0) = \begin{cases} e^{x-x_0} & \text{for } x < x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$

Question 6: Prove that if there exists a smooth solution to the Klein-Gordon equation then it is unique: $\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + m^2u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$ and $u(\pm\infty,t) = 0$, $\partial_t u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$. (Hint: test with $\partial_t u(x,t)$ and use $\phi\phi' = (\frac{1}{2}\phi^2)'$.)

Let u_1 and u_2 be two solutions. Then setting $\phi = u_1 - u_2$, we obtain that ϕ solves the homogeneous problem $\partial_{tt}\phi - c^2\partial_{xx}\phi + m^2\phi = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = \phi(x)$, $\partial_t\phi(x,0) = 0$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$, we obtain

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \partial_x\phi\partial_t\partial_x\phi\mathsf{d}x + m^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_x\phi)^2\mathsf{d}x + m^2 d_t\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x \\
&= d_t\left(\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_x\phi)^2\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \frac{1}{2}\phi^2\mathsf{d}x\right) \\
&= d_t\left(\int_{-\infty}^{+\infty} \left((\partial_t\phi)^2 + c^2(\partial_x\phi)^2 + m^2\phi^2\right)\mathsf{d}x\right).
\end{aligned}$$

Introduce

$$E(t) = \int_{-\infty}^{+\infty} \left((\partial_t\phi)^2 + c^2(\partial_x\phi)^2 + m^2\phi^2\right)\mathsf{d}x.$$

Then

$$d_t E(t) = 0.$$

The fundamental Theorem of calculus gives

$$E(t) = E(0) = 0.$$

This means

$$\int_{-\infty}^{+\infty} \left((\partial_t\phi(x,t))^2 + c^2(\partial_x\phi(x,t))^2 + m^2\phi^2(x,t)\right)\mathsf{d}x = 0, \quad \text{for all } t \geq 0.$$

This implies

$$0 = (\partial_t\phi(x,t))^2 + c^2(\partial_x\phi(x,t))^2 + m^2\phi^2(x,t), \quad \text{for all } t \geq 0,\ x \in \mathbb{R},$$

i.e. $\phi = 0$, thereby proving the uniqueness.