Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f(x-\beta))(\omega) = e^{i\beta\omega}\mathcal{F}(f)(\omega), \tag{2}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{3}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{4}$$

Question 1: Consider the equation $\partial_x\left(\frac{1}{1+3x^2}\partial_x u(x)\right) = f(x)$, $x\in(0,1)$, $\partial_x u(0) = a$, $u(1) = b$. Let $G(x,x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by G.

The operator is clearly self-adjoint. Then for all $x\neq x_0$ we have

$$\partial_x\left(\frac{1}{1+3x^2}\partial_x G(x,x_0)\right) = \delta_{x-x_0}, \quad \partial_x G(0,x_0) = 0, \quad G(1,x_0) = 0.$$

(ii) Give the integral representation of $u(x_0)$ for all $x_0\in(0,1)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

Multiply the equation defining G by u and integrate over $(0,1)$,

$$\langle\delta_{x-x_0},u\rangle = u(x_0) = \int_0^1 \partial_x\left(\frac{1}{1+3x^2}\partial_x G(x,x_0)\right)u(x)\mathsf{d}x.$$

We integrate by parts and we obtain

$$\begin{aligned} u(x_0) &= -\int_0^1 \frac{1}{1+3x^2}\partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x + \left[\frac{1}{1+3x^2}\partial_x G(x,x_0)u(x)\right]_0^1 \\ &= \int_0^1 G(x,x_0)\partial_x\left(\frac{1}{1+3x^2}\partial_x u(x)\right)\mathsf{d}x + \frac{1}{4}\partial_x G(1,x_0)u(1) - \left[G(x,x_0)\frac{1}{1+3x^2}\partial_x u(x)\right]_0^1 \\ &= \int_0^1 G(x,x_0)\partial_x\left(\frac{1}{1+3x^2}\partial_x u(x)\right)\mathsf{d}x + \frac{1}{4}\partial_x G(1,x_0)u(1) + G(0,x_0)\partial_x u(0). \end{aligned}$$

Now, using the boundary conditions and the fact that $\partial_x((1+3x^2)^{-1}\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_0^1 G(x,x_0)f(x)\mathsf{d}x + \frac{1}{4}\partial_x G(1,x_0)b + G(0,x_0)a.$$

(iii) Compute $G(x, x_0)$ for all $x, x_0 \in (0,1)$.

For all $x \neq x_0$ we have

$$\partial_x \left(\frac{1}{1+3x^2} \partial_x G(x,x_0) \right) = \delta_{x-x_0}, \quad \partial_x G(0,x_0) = 0, \quad G(1,x_0) = 0.$$

The generic solution is

$$G(x,x_0) = \begin{cases} a(x+x^3)+b & \text{if } 0 \le x < x_0 \\ c(x+x^3)+d & \text{if } x_0 < x \le 1. \end{cases}$$

The boundary conditions give

$$\partial_x G(0,x_0) = 0 = a, \qquad G(1,x_0) = 0 = 2c + d.$$

As a result

$$G(x,x_0) = \begin{cases} b & \text{if } 0 \le x < x_0 \\ c(x+x^3) - 2c & \text{if } x_0 < x \le 1. \end{cases}$$

G must be continuous at x_0,

$$b = c(x_0 + x_0^3) - 2c$$

and must satisfy the gap condition

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x \left(\frac{1}{1+3x^2} \partial_x G(x,x_0) \right) \mathsf{d}x = 1, \qquad \forall \epsilon > 0.$$

This gives

$$\frac{1}{1+3x_0^2} (\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0)) = 1,$$

i.e. $\partial_x G(x_0^+, x_0) = 1 + 3x_0^2 = c(1+3x_0^3)$. In conclusion $c = 1$ and $b = x_0 + x_0^3 - 2$. In other words,

$$G(x,x_0) = \begin{cases} x_0 + x_0^3 - 2 & \text{if } 0 \le x < x_0 \\ x + x^3 - 2 & \text{if } x_0 < x \le 1. \end{cases}$$

Question 2: Consider the operator $L : \phi \longmapsto -\partial_x(x^{\frac{1}{2}}\partial_x\phi(x)) - \frac{\pi^2}{4}x^{-\frac{1}{2}}\phi(x)$, with domain $D = \{v \in \mathcal{C}^2(1,4);\ v(1) = 0,\ v(4) = 0\}$.
(i) What is the Null space of L? (Hint: The general solution to $-\partial_x(x^{\frac{1}{2}}\partial_x\phi(x)) - \lambda x^{-\frac{1}{2}}\phi(x) = 0$ is $\phi(x) = c_1\cos(2\sqrt{x}\sqrt{\lambda}) + c_2\sin(2\sqrt{x}\sqrt{\lambda})$ for all $\lambda \geq 0$.)

Let ϕ be a member of the null space of L, say $\mathsf{N}(L)$. Then

$$-\partial_x(x^{\frac{1}{2}}\partial_x\phi(x)) - \frac{\pi^2}{4}x^{-\frac{1}{2}}\phi(x) = 0.$$

In other words, using the hint, $\phi(x) = c_1\cos(2\sqrt{x}\sqrt{\lambda}) + c_2\sin(2\sqrt{x}\sqrt{\lambda})$ with $\lambda = \frac{\pi^2}{4}$. The boundary conditions imply that

$$\phi(1) = 0 = -c_1, \quad \text{and} \quad \phi(4) = 0 = c_2\sin(2\pi).$$

In conclusion $\mathsf{N}(L) = \mathsf{span}\{\sin(\pi\sqrt{x})\}$, i.e., $\mathsf{N}(L)$ is a one-dimensional vector space.

(ii) Consider the problem $-\partial_x(x^{\frac{1}{2}}\partial_x\phi(x)) - \frac{\pi^2}{4}x^{-\frac{1}{2}}\phi(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $x \in (1,4)$, with $\phi(1) = 0$, $\phi(4) = 0$. Does this problem have a solution? (Hint: $\mathrm{d}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}\mathrm{d}x$.)

We are in the second case of the Fredholm alternative, since the null space of the operator L is not reduced to $\{0\}$. We must verify that $\frac{1}{2}x^{-\frac{1}{2}}$ is orthogonal to $\sin(\pi\sqrt{x})$. Using the hint and the change of variable $x^{\frac{1}{2}} = z$, we have

$$\int_1^4 \sin(\pi\sqrt{x})\frac{1}{2}x^{-\frac{1}{2}}\mathsf{d}x = \int_1^4 \sin(\pi x^{\frac{1}{2}})\mathsf{d}(x^{\frac{1}{2}}) = \int_1^2 \sin(\pi z)\mathsf{d}z = -\frac{1}{\pi}\left[\cos(\pi z)\right]_1^2 = -\frac{2}{\pi}.$$

Hence $\int_1^4 \sin(\pi\sqrt{x})\frac{1}{2}x^{-\frac{1}{2}}\mathsf{d}x \neq 0$, which means that the above problem does not have a solution.

Question 3: Solve the following integral equation (Hint: $x^2 - 3xa + 2a^2 = (x-a)(x-2a)$):

$$\int_{-\infty}^{+\infty} (f(y) - 3\sqrt{2}e^{-\frac{y^2}{2\pi}}) f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 3\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use (4) to obtain

$$2\pi\mathcal{F}(f)^2 - 2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} = -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}}$$

$$\mathcal{F}(f)^2 - 3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + 2e^{-\omega^2\pi} = 0$$

$$(\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f) - 2e^{-\omega^2\frac{\pi}{2}}) = 0.$$

This implies

$$\text{either} \quad \mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}, \quad \text{or} \quad \mathcal{F}(f) = 2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either} \quad f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = 2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 4: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) + c\partial_x u(x,t) + \gamma u(x,t) = 0,$$

for all $x \in (-\infty, +\infty)$, $t > 0$, with $u(x,0) = u_0(x)$ for all $x \in (-\infty, +\infty)$.

By taking the Fourier transform of the PDE, one obtains

$$\partial_t \mathcal{F}(u) - i\omega c\mathcal{F}(y) + \gamma\mathcal{F}(y) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega, t) = c(\omega)e^{i\omega ct - \gamma t}.$$

The initial condition implies that $c(\omega) = \mathcal{F}(u_0)(\omega)$:

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0)(\omega)e^{i\omega ct}e^{-\gamma t}.$$

The shift lemma in turn implies that

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0(x-ct))(\omega)e^{-\gamma t} = \mathcal{F}(u_0(x-ct)e^{-\gamma t})(\omega).$$

Applying the inverse Fourier transform gives:

$$u(x,t) = u_0(x-ct)e^{-\gamma t}.$$

Question 5: Solve the following PDE by the method of characteristics:

$$\partial_t w + 3\partial_x w = 0, \quad x > 0,\ t > 0$$
$$w(x,0) = f(x), \quad x > 0, \quad \text{and} \quad w(0,t) = h(t), \quad t > 0.$$

First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 0 & \text{if } s < 0, \\ s, & \text{if } s \geq 0. \end{cases} \quad \text{and} \quad t_\Gamma(s) = \begin{cases} -s & \text{if } s < 0, \\ 0, & \text{if } s \geq 0. \end{cases}$$

The we define the characteristics by

$$\partial_t X(s,t) = 3, \quad \text{with} \qquad X(s, t_\Gamma(s)) = x_\Gamma(s).$$

The general solution is $X(s,t) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$. Now we make the change of variable $\phi(s,t) = w(X(s,t),t)$ and we compute $\partial_t\phi(s,t)$,

$$\partial_t\phi(s,t) = \partial_t w(X(s,t),t) + \partial_x w(X(s,t),t)\partial_t X(s,t) = \partial_t w(X(s,t),t) + 3\partial_x w(X(s,t),t) = 0.$$

This means that $\phi(s,t) = \phi(s, t_\Gamma(s))$. In other words

$$w(X(s,t),t) = w(X(s,t_\Gamma(s)), t_\Gamma(s)) = w(x_\Gamma(s), t_\Gamma(s)).$$

<u>Case 1:</u> If $s < 0$, then $X(s,t) = 3(t - t_\Gamma(s))$. This implies $t_\Gamma(s) = t - X/3$. The condition $s < 0$ and the definition $t_\Gamma(s) = -s$ imply $t - X/3 \geq 0$. Moreover we have

$$w(X,t) = w(0, t_\Gamma(s)) = h(t_\Gamma(s)).$$

In conclusion

$$w(X,t) = h(t - X/3), \qquad \text{if} \qquad 3t > X.$$

<u>Case 2:</u> If $s \geq 0$, then $X(s,t) = 3t + x_\Gamma(s)$. This implies $x_\Gamma(s) = X - 3t$. The condition $s \geq 0$ and the definition $x_\Gamma(s) = s$ imply $X - 3t \geq 0$. Moreover we have

$$w(X,t) = w(x_\Gamma(s), 0) = f(x_\Gamma(s)).$$

In conclusion

$$w(X,t) = f(X - 3t), \qquad \text{if} \qquad X \geq 3t.$$
