Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

**Question 1:** Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad w(0,t) = 0, \quad w(4,t) = 0, \quad t > 0.$$

where $f(x) = 1$, if $x \in [1,2]$ and $f(x) = 0$ otherwise. (i) Give a simple expression of the solution in terms of an extension of $f$.

We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where $f$ must be replaced by the periodic extension (of period $8$) of its odd extension, say $f_{\mathsf{o,p}}$, where

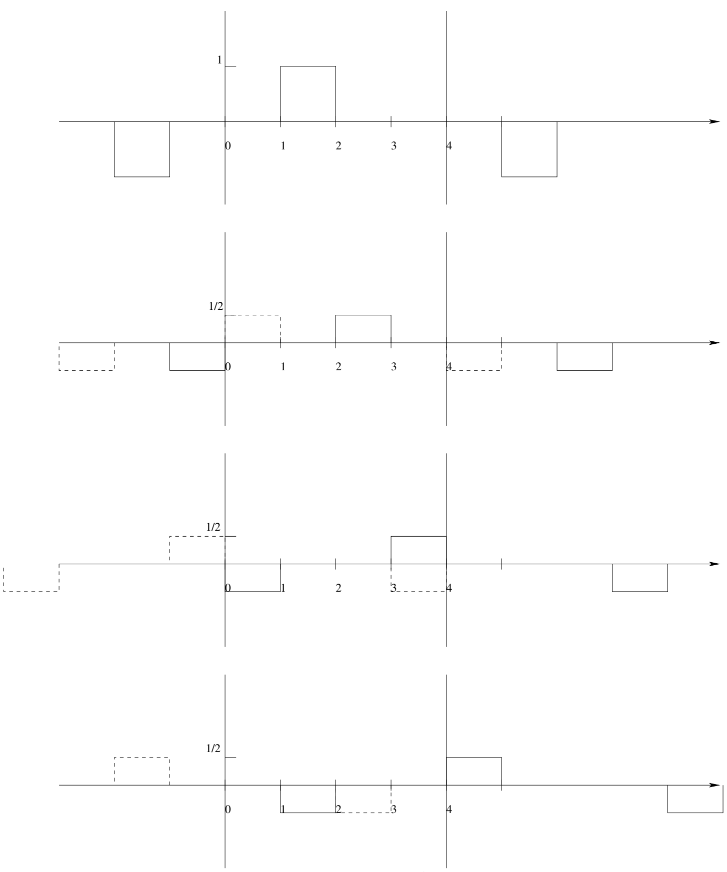
$$f_{\mathsf{o,p}}(x+8) = f_{\mathsf{o,p}}(x), \qquad \forall x \in \mathbb{R}$$
$$f_{\mathsf{o,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ -f(-x) & \text{if } x \in [-4,0) \end{cases}$$

The solution is

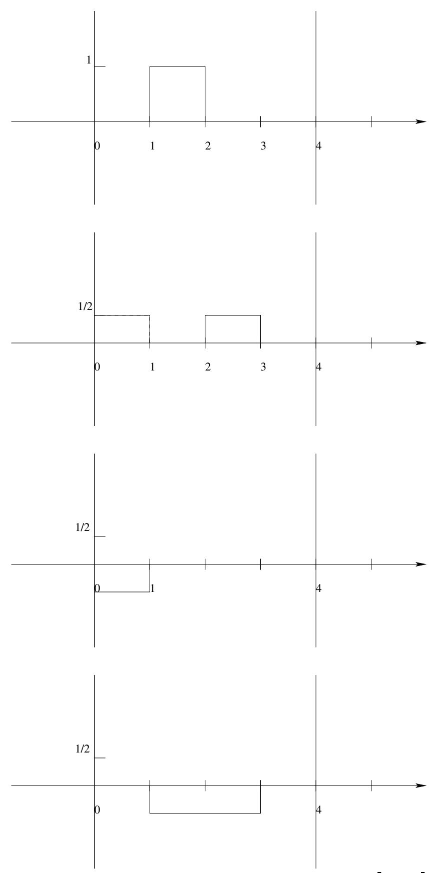
$$u(x,t) = \frac{1}{2}(f_{\mathsf{o,p}}(x-t) + f_{\mathsf{o,p}}(x+t)).$$

(ii) Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed $1$, the other half moves to the left with speed $1$.



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.



Solution in domain $[0,4]$ at $t = 0, 1, 2, 3$

**Question 2:** Consider the equation $\partial_{tt}u - 4\partial_{xx}u = 0$, $x \in (0,2)$, $t > 0$, with $u(0,t) = 0$, $u(2,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = \sin(\pi x) + \sin(4\pi x)$. Compute the solution for all $t > 0$ and all $x \in (0,2)$. (Hint: $\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$)

Notice first that the wave speed, say $c$, is equal to 2. Since we have homogeneous Dirichlet boundary conditions at both ends of the domain, we need to consider the periodic extension of the even extension of the data. Let us set $g(x) = \sin(\pi x) + \sin(4\pi x)$, the even extension is $g_e = \sin(\pi x) + \sin(4\pi x)$ since $g$ is even, and finally the periodic extension of $g_e$ is $g_{e,p} = \sin(\pi x) + \sin(4\pi x)$ since $g_e$ is periodic of period 4. Then

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_{x-ct}^{c+ct} g_{e,p}(\xi)\mathrm{d}\xi = \frac{1}{2c}\int_{x-ct}^{c+ct} (\sin(\pi\xi) + \sin(4\pi\xi))\mathrm{d}\xi \\
&= -\frac{1}{2c}\left(\frac{1}{\pi}\cos(\pi\xi) + \frac{1}{4\pi}\cos(4\pi\xi)\right)\Big|_{x-ct}^{x+ct} \\
&= -\frac{1}{2c\pi}\Big(\cos(\pi(x+ct) - \cos(\pi(x-ct)) + \frac{1}{4}(\cos(4\pi(x+ct) - \cos(4\pi(x-ct))\Big) \\
&= \frac{2}{2c\pi}\left(\sin(\pi x)\sin(\pi ct) + \frac{1}{4}\sin(4\pi x)\sin(4\pi ct)\right).
\end{aligned}
$$

in conclusion, using $c = 2$ we have

$$
u(x,t) = \frac{1}{4\pi}\sin(\pi x)\sin(2\pi t) + \frac{1}{8\pi}\sin(4\pi x)\sin(8\pi t).
$$

**Question 3:** Consider the following conservation equation

$$
\partial_t\rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty,+\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}
$$

(i) What is the wave speed for this problem when $q(\rho) = \rho(2 - \sin(\rho))$ (and $\rho(x,t)$ is the conserved quantity).?

The wave speed if the quantity $q'(\rho) = 2 - \sin(\rho) - \rho\cos(\rho)$.

(ii) What is the wave speed for this problem when $q(\rho) = 2\rho + \cos(\rho^2)$ (and $\rho(x,t)$ is the conserved quantity).

The wave speed is the quantity $q'(\rho) = 2 - 2\rho\sin(\rho^2)$.

**Question 4:** Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid x > 0,\ x + 3t > 0\}$. Use the method of characteristics to solve the equation $\partial_t u + 4\partial_x u + 2u = 0$ for $(x,t) \in \Omega$ and $u(x,0) = x + 4$, for $x > 0$, $u(-3t,t) = t + 4$, for $t > 0$.

(i) We first parameterize the boundary of $\Omega$ by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 3s & s < 0 \\ s & s > 0, \end{cases} \qquad t_\Gamma(s) = \begin{cases} -s & s < 0 \\ 0 & s > 0. \end{cases}$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 4, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s))$.
(iii) Set $\Phi(t,s) = u(X(t,s),t)$. Then

$$\begin{aligned} \partial_t \Phi(t,s) &= \partial_x u(X(t,s),t)\partial_t X(t,s) + \partial_t u(X(t,s),t)\partial_t t \\ &= 4\partial_x u(X(t,s),t) + \partial_t u(X(t,s),t) = -2u(X(t,s),t) = -2\Phi(s,t) \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{-2(t-t_\Gamma(s))}$, i.e., $u(X(t,s)) = u(X(t_\Gamma(s),s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))} = u(x_\Gamma(s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))}$.
(iv) The implicit representation of the solution is

$$X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s)), \quad u(X(t,s)) = u(x_\Gamma(s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))}.$$

(v) The explicit representation is obtained by replacing the parameterization $(t,s)$ by $(X,t)$. Using the definitions of $x_\Gamma(s)$ and $t_\Gamma(s)$, we have two cases:
Case 1: $s < 0$. The definition of $X(t,s)$ gives $X(s,t) = 3s + 4(t+s)$, i.e., $s = (X - 4t)/7$. Then

$$\begin{aligned} u(X,t) = (t_\Gamma(s) + 4)e^{-2(t-t_\Gamma(s))} &= (-s+4)e^{-2(t+s)} = (4 - (X-4t)/7)e^{-2(t+(X-4t)/7)} \\ &= (4 + \frac{4t - X}{7})e^{-\frac{2}{7}(3t+X)} \end{aligned}$$

i.e., $\boxed{u(X,t) = (4 + \frac{4t-X}{7})e^{-\frac{2}{7}(3t+X)} \text{ if } X < 4t}$.
Case 2: $s > 0$. The definition of $X(t,s)$ gives $X(s,t) = s + 4t$, i.e., $s = X - 4t$. Then

$$u(X,t) = (x_\Gamma(s) + 4)e^{-2(t-t_\Gamma(s))} = (s+4)e^{-2t} = (4 + X - 4t)e^{-2t}.$$

i.e., $\boxed{u(X,t) = (4 + X - 4t)e^{-2t} \text{ if } X > 4t}$.

**Question 5:** Consider the equation $\partial_x\left(\frac{1}{2+\sin(x)}\partial_x u(x)\right) = f(x)$, $x \in (0,\pi)$, $u(0) = a$, $\partial_x u(\pi) = b$. Let $G(x,x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by $G$.

The operator is clearly self-adjoint. Then we have

$$\partial_x\left(\frac{1}{2+\sin(x)}\partial_x G(x,x_0)\right) = \delta_{x-x_0}, \quad G(0,x_0) = 0, \quad \partial_x G(\pi,x_0) = 0.$$

(ii) Give the integral representation of $u(x_0)$ for all $x_0 \in (0,\pi)$ in terms of $G$, $f$, and the boundary data. (Do not compute $G$ in this question).

Multiply the equation defining $G$ by $u$ and integrate over $(0,\pi)$,

$$\langle \delta_{x-x_0}, u\rangle = u(x_0) = \int_0^\pi \partial_x\left(\frac{1}{2+\sin(x)}\partial_x G(x,x_0)\right)u(x)\mathsf{d}x.$$

We integrate by parts and we obtain

$$\begin{aligned} u(x_0) &= -\int_0^\pi \frac{1}{2+\sin(x)}\partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x + \left[\frac{1}{2+\sin(x)}\partial_x G(x,x_0)u(x)\right]_0^\pi \\ &= \int_0^\pi G(x,x_0)\partial_x\left(\frac{1}{2+\sin(x)}\partial_x u(x)\right)\mathsf{d}x - \frac{1}{2}\partial_x G(0,x_0)u(0) - \left[G(x,x_0)\frac{1}{2+\sin(x)}\partial_x u(x)\right]_0^\pi \\ &= \int_0^\pi G(x,x_0)\partial_x\left(\frac{1}{2+\sin(x)}\partial_x u(x)\right)\mathsf{d}x - \frac{1}{2}\partial_x G(0,x_0)u(0) - \frac{1}{2}G(\pi,x_0)\partial_x u(\pi). \end{aligned}$$

Now, using the boundary conditions and the fact that $\partial_x((2+\sin(x))^{-1}\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_0^1 G(x,x_0)f(x)\mathsf{d}x - \frac{1}{2}\partial_x G(0,x_0)a - \frac{1}{2}G(\pi,x_0)b.$$

(iii) Compute $G(x, x_0)$ for all $x, x_0 \in (0, 1)$. (Hint: go slowly and do not skip details.)

For all $x \neq x_0$ we have

$$\partial_x \left( \frac{1}{2+\sin(x)} \partial_x G(x, x_0) \right) = \delta_{x-x_0}, \quad \partial_x G(0, x_0) = 0, \quad G(1, x_0) = 0.$$

The generic solution is

$$G(x, x_0) = \begin{cases} a(2x - \cos(x)) + b & \text{if } 0 \leq x < x_0 \\ c(2x - \cos(x)) + d & \text{if } x_0 < x \leq \pi. \end{cases}$$

The boundary conditions give

$$G(0, x_0) = 0 = -a + b, \qquad \partial_x G(\pi, x_0) = 0 = 2c.$$

As a result

$$G(x, x_0) = \begin{cases} a(1 + 2x - \cos(x)) & \text{if } 0 \leq x < x_0 \\ d & \text{if } x_0 < x \leq \pi. \end{cases}$$

$G$ must be continuous at $x_0$,

$$d = a(1 + 2x_0 - \cos(x_0))$$

and must satisfy the gap condition

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x \left( \frac{1}{2+\sin(x)} \partial_x G(x, x_0) \right) \mathrm{d}x = 1, \qquad \forall \epsilon > 0.$$

This gives

$$\frac{1}{2+\sin(x_0)} (\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0)) = 1,$$

i.e. $\partial_x G(x_0^-, x_0) = -2 - \sin(x_0) = a(2 + \sin(x_0))$. In conclusion $a = -1$ and $d = -(1 + 2x_0 - \cos(x_0))$. In other words,

$$G(x, x_0) = \begin{cases} -(1 + 2x - \cos(x)) & \text{if } 0 \leq x < x_0 \\ -(1 + 2x_0 - \cos(x_0)) & \text{if } x_0 < x \leq \pi. \end{cases}$$

**Question 6:** Consider the operator $L : \phi \longmapsto x^2\partial_{xx}\phi(x)) + \frac{1}{4}\phi(x)$, with domain $D = \{v \in \mathcal{C}^2(1,2);\ v(1) = 0,\ v(2) = 0\}$.
(i) What is the Null space of $L$? (Hint: The general solution to $x^2\partial_{xx}\phi(x)) + \frac{1}{4}\phi(x) = 0$ is $\phi(x) = c_1\sqrt{x}\log(x) + c_2\sqrt{x}$.)

Let $\phi$ be a member of the null space of $L$, say $\mathsf{N}(L)$. Then

$$x^2\partial_{xx}\phi(x)) + \frac{1}{4}\phi(x) = 0$$

In other words, using the hint, $\phi(x) = c_1\sqrt{x}\log(x) + c_2\sqrt{x}$. The boundary conditions imply that

$$\phi(1) = 0 = c_2, \quad \text{and} \quad \phi(2) = 0 = c_1\sqrt{2}\log(2).$$

In conclusion $\mathsf{N}(L) = \{0\}$.

(ii) Does the problem $x^2\partial_{xx}\phi(x)) + \frac{1}{4}\phi(x) = \cos(x)$, $x \in (1,2)$, with $u(1) = 0$, $u(2) = 0$ have a solution? If yes, is it unique? Why?

We are in the first case of the Fredholm alternative; there is a unique solution to this problem.

(iii) Give the formal adjoint of $L$ and its domain.

Let $u \in D$ and $v \in D^*$, then

$$\begin{aligned}\int_1^2 (Lu(x))v(x)\mathsf{d}x &= \int_1^2 (x^2\partial_{xx}\phi(x) + \frac{1}{4}\phi(x)v(x))\mathsf{d}x \\ &= \int_1^2 -\partial_x\phi(x)\partial_x(x^2v(x))\mathsf{d}x + \int_1^2 \frac{1}{4}\phi(x))v(x)\mathsf{d}x + 4\partial_x\phi(2)v(2) - \partial_x\phi(1)v(1)\end{aligned}$$

We enforce $v(1) = v(2) = 0$ to get rid of the boundary terms. Then

$$\begin{aligned}\int_1^2 (Lu(x))v(x)\mathsf{d}x &= \int_1^2 \phi(x)\partial_{xx}(x^2v(x))\mathsf{d}x + \int_1^2 \frac{1}{4}\phi(x))v(x)\mathsf{d}x \\ &= \int_1^2 \phi(x)(\partial_{xx}(x^2v(x)) + v(x))\mathsf{d}x.\end{aligned}$$

This means that $D* = \{v \in \mathcal{C}^2(1,2);\ v(1) = 0,\ v(2) = 0\} = D$ and $L^*v = \partial_{xx}(x^2v) + v$.

**Question 7:** Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty), \ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 1 & \text{if } x \le 0, \\ 1-x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 \le x. \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \le t \le 1$.

The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \le 0$, we have $u_0(x_0) = 1$ and $X(t,x_0) = t + x_0$; as a result, $x_0 = X - t$, and

$$u(x,t) = 1, \qquad \text{if } x \le t.$$

Case 2: If $0 \le x_0 \le 1$, we have $u_0(x_0) = 1 - x_0$ and $X(t,x_0) = t(1-x_0) + x_0$; as a result $x_0 = (X-t)/(1-t)$, and

$$u(x,t) = 1 - (t-x)/(t-1), \qquad \text{if } 0 \le x - t \le 1 - t,$$

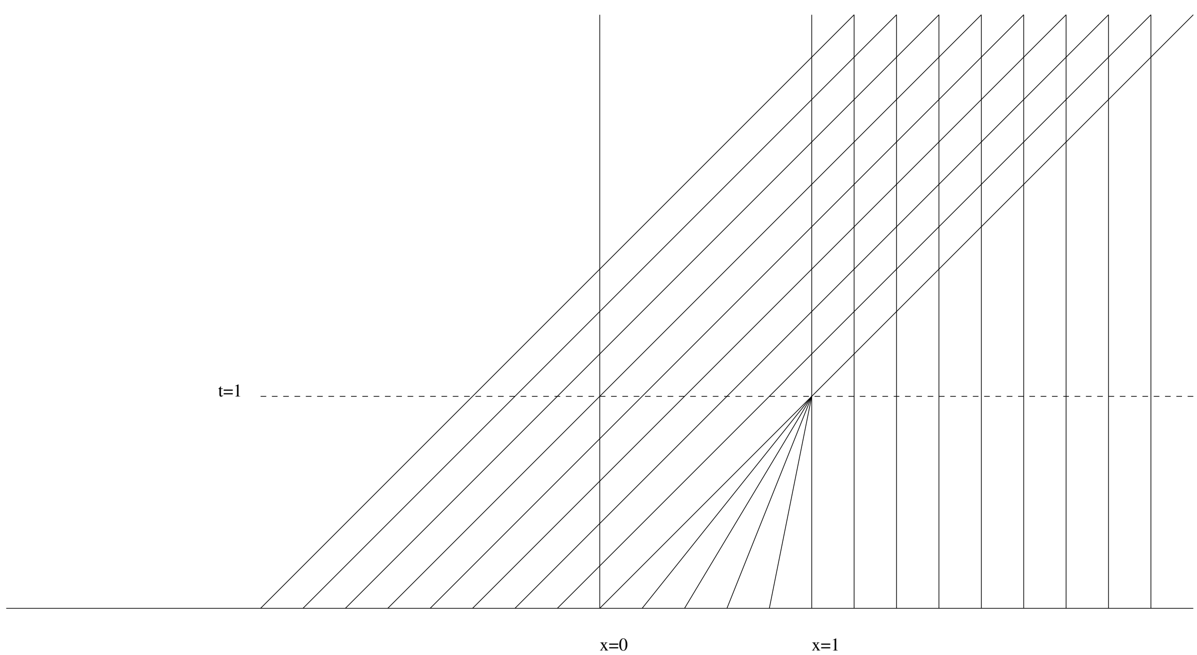
which can also be re-written

$$u(x,t) = \frac{x-1}{t-1}, \qquad \text{if } t \le x \le 1.$$

case 3: If $1 \le x_0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result

$$u(x,t) = 0, \qquad \text{if } 1 \le x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.



**Question 8:** Use the Fourier transform technique to solve $\partial_t u(x,t) - \partial_{xx} u(x,t) + \sin(t)\partial_x u(x,t) + (2+3t^2)u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$. (Hint: use the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{i\omega x}\mathrm{d}x$), the result $\mathcal{F}(\mathrm{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\frac{\omega^2}{4\alpha}}$, the convolution theorem and the shift lemma: $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)\mathrm{e}^{i\omega\beta}$. Go slowly and give all the details.)

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Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \omega^2 \mathcal{F}(u)(\omega,t) + \sin(t)(-i\omega)\mathcal{F}(u)(\omega,t) + (2+3t^2)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = -\omega^2 + i\omega\sin(t) - (2+3t^2).$$

Then applying the fundamental theorem of calculus between $0$ and $t$, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = -\omega^2 t - i\omega(\cos(t)-1) - (2t+t^3).$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)\mathsf{e}^{-\omega^2 t}\mathsf{e}^{-i\omega(\cos(t)-1)}\mathsf{e}^{-(2t+t^3)}.$$

Using the result $\mathcal{F}(\mathsf{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathsf{e}^{-\frac{\omega^2}{4\alpha}}$ where $\alpha = \frac{1}{4t}$, this implies that

$$\begin{aligned}\mathcal{F}(u)(\omega,t) &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0)(\omega)\mathcal{F}(\mathsf{e}^{-\frac{x^2}{4t}})(\omega)\mathsf{e}^{-i\omega(\cos(t)-1)}\mathsf{e}^{-(2t+t^3)}\\ &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0 * \mathsf{e}^{-\frac{x^2}{4t}})(\omega)\mathsf{e}^{-i\omega(\cos(t)-1)}\mathsf{e}^{-(2t+t^3)}.\end{aligned}$$

Then setting $g = u_0 * \mathsf{e}^{-\frac{x^2}{4t}}$ the convolution theorem followed by the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\mathcal{F}(g(x+\cos(t)-1))(\omega)\mathsf{e}^{-(2t+t^3)}.$$

This finally gives

$$u(x,t) = \sqrt{\frac{1}{4\pi t}}g(x+\cos(t)-1)\mathsf{e}^{-(2t+t^3)} = e^{-(2t+t^3)}\sqrt{\frac{1}{4\pi t}}\int_{-\infty}^{+\infty} u_0(y)\mathsf{e}^{-\frac{(x+\cos(t)-1-y)^2}{4t}}\mathsf{d}y.$$

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