Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad \mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{1}$$

Question 1: (a) Solve the following equation by using an extension technique (give all the details):

$$\partial_{tt}w - 4\partial_{xx}w = 0, \quad x \in (0,+\infty),\ t>0$$
$$w(x,0) = x(1+x^2)^{-1}, \quad x\in(0,+\infty); \qquad \partial_t w(x,0) = 0, \quad x\in(0,+\infty); \quad \text{and} \quad w(0,t)=0, \quad t>0.$$

Since we have a homogeneous Dirichlet boundary condition at $x=0$, we define $f(x) = x(1+x^2)^{-1}$ and its odd extension $f_o(x)$ over $(-\infty,+\infty)$. Let w_o be the solution to the wave equation over the entire real line with f_0 as initial data:

$$\partial_{tt}w_o - 4\partial_{xx}w_o = 0, \quad x\in\mathbb{R},\ t>0$$
$$w_o(x,0) = f_o(x), \quad x>0, \qquad \partial_t w_o(x,0) = 0, \quad x\in\mathbb{R}.$$

The solution to this problem is given by the D'Alembert formula

$$w_o(x,t) = \frac{1}{2}(f_o(x-2t) + f_o(x+2t)), \qquad \text{for all } x\in\mathbb{R} \text{ and } t\geq 0.$$

Let x be positive. Then $w(x,t) = w_o(x,t)$ for all $x\in(0,+\infty)$, since by construction $w_o(0,t) = f_o(-2t)+f_o(2t) = 0$ for all times.

Case 1: If $x-2t>0$, $f_o(x-2t) = f(x-2t)$; as a result

$$w(x,t) = \frac{1}{2}(f(x-2t)+f(x+2t)), \quad \text{if } x-2t>0.$$

Case 2: If $x-2t<0$, $f_o(x-2t) = -f(-x+2t)$; as a result

$$w(x,t) = \frac{1}{2}(-f(-x+2t)+f(x+2t)), \quad \text{if } x-2t<0.$$

We can get a more compact form by observing that actually $f_o(z) = z(1+z^2)^{-1}$; as a result, the solution can also be re-written as follows:

$$w(x,t) = \frac{1}{2}\left(\frac{x-2t}{1+(x-2t)^2} + \frac{x+2t}{1+(x+2t)^2}\right).$$

(b) Compute the solution at $x=2$ and $t=2$.

We have $x-2t = 2-4<0$ hence

$$w(x,t) = \frac{1}{2}(-f(-x+2t)+f(x+2t)) = \frac{1}{2}(-f(2)+f(6)) = \frac{1}{2}\left(-\frac{2}{1+4}+\frac{6}{1+36}\right).$$

Question 2: (a) Let M be a positive real number and let $\omega \in \mathbb{R}$. Compute $\lim_{M\to\infty} e^{(-1+i\omega)M}$. (Justify all the steps.)

We have

$$|e^{(-1+i\omega)M}| = |e^{-M}||e^{i\omega)M}| = e^{-M}.$$

Hence $\lim_{M\to\infty} |e^{(-1+i\omega)M}| = 0$, which in turn implies that $\lim_{M\to\infty} e^{(-1+i\omega)M} = 0$ since the function $\mathbb{C} \ni z \longmapsto |z| \in \mathbb{R}$ is continuous.

(b)Compute the Fourier transform of $f(x) = H(x)e^{-x}$ where H is the Heavide function.

Using the definitions we have

$$\begin{aligned}\mathcal{F}(f)(\omega) &= \lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} H(x)e^{-x}e^{i\omega x}\mathsf{d}x = \lim_{M\to\infty} \frac{1}{2\pi} \int_{0}^{M} e^{(-1+i\omega)x}\mathsf{d}x \\ &= \frac{1}{2\pi}\frac{1}{1-i\omega}.\end{aligned}$$

Note that here we used that $\lim_{M\to\infty} e^{(-1+i\omega)M} = 0$

Question 3: (a) Draw the graph of the function $-1 + 2H(x)$ where $H(x)$ is the Heaviside function.

(b) Compute the derivative in the distribution sense of $f(x) = |x|$.

Let $\psi \in C_c^\infty(\mathbb{R})$ be a function of class C^∞ with compact support (it means that there exists $M \geq 0$ such that $\psi(x) = 0$ if $|x| \geq M$) Then

$$\begin{aligned}\langle \partial_x f, \psi\rangle &:= -\int_{\mathbb{R}} f(x)\partial_x\psi(x)\mathsf{d}x = \int_{-\infty}^{0} x\partial_x\psi(x)\mathsf{d}x - \int_{0}^{\infty} x\partial_x\psi(x)\mathsf{d}x \\ &= -\int_{-\infty}^{0} \psi(x)\mathsf{d}x + \int_{0}^{\infty} \psi(x)\mathsf{d}x \\ &= \int_{-\infty}^{0} (-1+2H(x))\psi(x)\mathsf{d}x + \int_{0}^{\infty} (-1+2H(x))\psi(x)\mathsf{d}x = \int_{-\infty}^{+\infty} (-1+2H(x))\psi(x)\mathsf{d}x.\end{aligned}$$

Hence,

$$\langle \partial_x f, \psi\rangle = \int_{-\infty}^{+\infty} (-1+2H(x))\psi(x)\mathsf{d}x, \qquad \forall \psi \in C_c^\infty(\mathbb{R}).$$

This means that $\partial_x f(x) = -1 + 2H(x)$.

(c) Compute the second derivative in the distribution sense of $f(x) = |x|$.

Solution 1: We have already established that $\partial_x f(x) = -1 + 2H(x)$, hence $\partial_{xx} f(x) = 2\partial_x H(x)$. But we know from class that $\partial_x H(x)$ is the Dirac measure at 0, hence

$$\partial_{xx} f(x) = 2\delta_0.$$

Solution 2: We apply the definition of the second derivative. Let $\psi \in C_c^\infty(\mathbb{R})$ be a function of class C^∞ with compact support, then

$$\begin{aligned}\langle \partial_{xx} f, \psi\rangle &:= -\langle \partial_x f, \partial_x\psi\rangle := \int_{\mathbb{R}} f(x)\partial_{xx}\psi(x)\mathsf{d}x = -\int_{-\infty}^{0} x\partial_{xx}\psi(x)\mathsf{d}x + \int_{0}^{\infty} x\partial_{xx}\psi(x)\mathsf{d}x \\ &= \int_{-\infty}^{0} \partial_x\psi(x)\mathsf{d}x - \int_{0}^{\infty} \partial_x\psi(x)\mathsf{d}x \\ &= \psi(0) + \psi(0) = 2\psi(0) = 2\langle \delta_0, \psi\rangle\end{aligned}$$

whence

$$\langle \partial_{xx} f, \psi\rangle = 2\langle \delta_0, \psi\rangle, \qquad \forall \psi \in C_c^\infty(\mathbb{R}).$$

This means $\partial_{xx} f = 2\delta_0$.

Question 4: Solve the integral equation:

$$\int_{-\infty}^{+\infty}\left(f(y)-\sqrt{2}e^{-\frac{y^2}{2\pi}}-\frac{1}{1+y^2}\right)f(x-y)dy=-\int_{-\infty}^{+\infty}\frac{\sqrt{2}}{1+y^2}e^{-\frac{(x-y)^2}{2\pi}}\mathrm{d}y,\qquad \forall x\in(-\infty,+\infty).$$

(Hint: there is an easy factorization after applying the Fourier transform and $\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})=e^{-\frac{\pi\omega^2}{2}}$ and $\mathcal{F}(\frac{1}{1+x^2})=\frac{1}{2}e^{-|\omega|}$.)

The equation can be re-written

$$f*(f-\sqrt{2}e^{-\frac{x^2}{2\pi}}-\frac{1}{1+x^2})=-\frac{1}{1+x^2}*\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem

$$2\pi\mathcal{F}(f)\left(\mathcal{F}(f)-\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})-\mathcal{F}(\frac{1}{1+x^2})\right)=-2\pi\mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})$$

Solution 1: Using the hint we obtain

$$\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})=\sqrt{2}\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\frac{\omega^2}{4\frac{1}{2\pi}}}=e^{-\frac{\pi\omega^2}{2}}$$

$$\mathcal{F}(\frac{1}{1+x^2})=\frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)\left(\mathcal{F}(f)-e^{-\frac{\pi\omega^2}{2}}-\frac{1}{2}e^{-|\omega|}\right)=-\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}.$$

This equation can also be re-written as follows

$$\mathcal{F}(f)^2-\mathcal{F}(f)e^{-\frac{\pi\omega^2}{2}}-\mathcal{F}(f)\frac{1}{2}e^{-|\omega|}+\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}=0,$$

and can be factorized as follows:

$$(\mathcal{F}(f)-e^{-\frac{\pi\omega^2}{2}})(\mathcal{F}(f)-\frac{1}{2}e^{-|\omega|})=0.$$

This means that either $\mathcal{F}(f)=e^{-\frac{\pi\omega^2}{2}}$ or $\mathcal{F}(f)=\frac{1}{2}e^{-|\omega|}$. Taking the inverse Fourier transform, we finally obtain two solutions

$$f(x)=\sqrt{2}e^{-\frac{x^2}{2\pi}},\quad \text{or}\quad f(x)=\frac{1}{1+x^2}.$$

Solution 2: Another solution consists of remarking that the equation with the Fourier transform can be rewritten as follows:

$$\mathcal{F}(f)^2-\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}})\mathcal{F}(f)-\mathcal{F}(\frac{1}{1+x^2})\mathcal{F}(f)+\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}})\mathcal{F}(\frac{1}{1+x^2})=0,$$

which can factorized as follows:

$$(\mathcal{F}(f)-\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}}))(\mathcal{F}(f)-\mathcal{F}(\frac{1}{1+x^2}))=0.$$

Then either $\mathcal{F}(f)=\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}})$ or $\mathcal{F}(f)=\mathcal{F}(\frac{1}{1+x^2})$. The conclusion follows easily.

Question 5: Let $h \in C^\infty(\mathbb{R})$ be a non-negative function.
(a)Let $\phi \in C^2(\mathbb{R}\times\mathbb{R})$. Compute $\partial_t(h(\phi(x,t))$.

We apply the chain rule:

$$\partial_t(h(\phi(x,t))) = h'(\phi(x,t))\partial_t\phi(x,t).$$

(b) Consider the quasilinear Klein-Gordon equation: $\partial_{tt}\phi(x,t) - c^2\partial_{xx}\phi(x,t) + m^2\phi(x,t) + \beta^2 h'(\phi(x,t)) = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = f(x)$, $\partial_t\phi(x,0) = g(x)$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Find an energy $E(t)$ that is invariant with respect to time (Hint: test with $\partial_t\phi(x,t)$ and use (a).)

Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$, we obtain

$$\begin{aligned}
0 = &\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x + \beta^2\int_{-\infty}^{+\infty} h'(\phi)\partial_t\phi\mathsf{d}x \\
&\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x + \beta^2\int_{-\infty}^{+\infty} \partial_t(h(\phi))\mathsf{d}x \\
&= \partial_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \partial_x\phi\partial_t\partial_x\phi\mathsf{d}x + \partial_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \beta^2 h(\phi))\mathsf{d}x \\
&= \partial_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + \partial_t\int_{-\infty}^{+\infty} \frac{c^2}{2}(\partial_x\phi)^2\mathsf{d}x + \partial_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \beta^2 h(\phi))\mathsf{d}x \\
&= \partial_t\int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \beta^2 h(\phi)\right)\mathsf{d}x.
\end{aligned}$$

Introduce

$$E(t) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \beta^2 h(\phi)\right))\mathsf{d}x.$$

Then

$$\partial_t E(t) = 0.$$

The fundamental Theorem of calculus gives

$$E(t) = E(0).$$

In conclusion the quantity $E(t)$ is invariant with respect to time, as requested.

Question 6: Let $L : D(L) \longrightarrow C^0(0,1)$ with $Lu = \partial_x((1+x^2)u(x)) - 2xu(x)$ and $D(L) = \{v \in C^1(0,1) \mid v(0) = 0\}$.
(a) Compute the formal adjoint of L and its domain.

Let $u \in D(L)$ and $v \in D(L^\mathsf{T})$, then

$$\begin{aligned}\int_0^1 Lu(x)v(x)\mathsf{d}x &= \int_0^1 (\partial_x((1+x^2)u(x)) - 2xu(x))v(x)\mathsf{d}x \\ &= \int_0^1 (-(1+x^2)u(x)\partial_x v(x) - 2xu(x)v(x))\mathsf{d}x + (1+x^2)u(x)v(x)|_0^1 \\ &= \int_0^1 u(x)(-(1+x^2)\partial_x v(x) - 2xv(x))\mathsf{d}x + 2u(1)v(1).\end{aligned}$$

Since we do not know $u(1)$ we get rid of the term $2u(1)v(1)$ by selecting v so that $v(1) = 0$; whence

$$D(L^\mathsf{T}) = \{v \in C^1(0,1) \mid v(1) = 0\}, \qquad L^\mathsf{T} v = -(1+x^2)\partial_x v(x) - 2xv(x)$$

(b) Compute the Green's function associated with the operator L.
(*Hint:* $\lim_{\epsilon\to 0+} \int_{x_0-\epsilon}^{x_0+\epsilon} (1+x^2)\partial_x G(x,x_0)\mathrm{d}x = (1+x_0^2)(G(x_0^+,x_0) - G(x_0^-,x_0))$.)

Let $G(x,x_0)$, be the Green's function, $x \in [0,1]$, $x_0 \in (0,1)$. We know from class that $G(x,x_0) \in D(L^\mathsf{T})$ and satisfies $L^\mathsf{T} G(x,x_0) = \delta_{x_0}$, i.e., $-(1+x^2)\partial_x G(x,x_0) - 2xG(x,x_0) = \delta_{x_0}$ and $G(0,x_0) = 0$.

Case 1: $x < x_0$, then $-(1+x^2)\partial_x G(x,x_0) - 2xG(x,x_0) = 0$, which implies that $\partial_x((1+x^2)G(x,x_0)) = 0$. Then the fundamental theorem of calculus gives

$$(1+x^2)G(x,x_0) = a, \qquad x < x_0$$

Case 2: $x_0 < x$, then $-(1+x^2)\partial_x G(x,x_0) - 2xG(x,x_0) = 0$, which implies that $\partial_x((1+x^2)G(x,x_0)) = 0$. Then the fundamental theorem of calculus gives

$$(1+x^2)G(x,x_0) = b.$$

Now we can apply the boundary condition $G(1,x_0) = 0 = \frac{1}{2}b$, hence $G(x,x_0) = 0$ if $x_0 < x$.

The jump condition gives (with obvious abuse of notation)

$$\lim_{\epsilon\to 0+} \int_{x_0-\epsilon}^{x_0+\epsilon} (-(1+x^2)\partial_x G(x,x_0) - 2xG(x,x_0))\mathsf{d}x = 1$$

hence

$$1 = -(1+x_0^2)(\partial_x G(x,x_0^+) - G(x,x_0^-)) = (1+x_0^2)G(x,x_0^-) = a.$$

In conclusion

$$G(x,x_0) = \begin{cases} \frac{1}{1+x^2} & \text{if } x < x_0 \\ 0 & \text{otherwise} \end{cases}$$

(c) Use the Green's function to solve the equation $\partial_x((1+x^2)u(x)) - 2xu(x) = f(x)$ with $u(0) = a$ and $f \in C^0(0,1)$.

We multiply the PDE by $G(x,x_0)$ and integrate by parts.

$$\begin{aligned}\int_0^1 (\partial_x((1+x^2)u(x)) - 2xu(x))G(x,x_0)\mathsf{d}x &= \int_0^1 u(x)(-(1+x^2)\partial_x G(x,x_0) - 2xG(x,x_0))\mathsf{d}x + 2u(1)G(1,x_0) - u(0)G(0,x_0) \\ &= u(x_0) - a\end{aligned}$$

hence

$$u(x_0) = a + \int_0^1 \frac{f(x)}{1+x^2}\mathsf{d}x$$

Of course we could have observed from the start that $(1+x^2)\partial_x u(x) = f(x)$ and the fundamental theorem of calculus gives the solution. The purpose of this question was to see whether you understand the Green's function theory and the notion of formal adjoint.

Question 7: Let us denote $\alpha = e^{\frac{\pi}{2}}$. Consider the operator

$$L : D(L) := \{v \in \mathcal{C}^2(1,\alpha) |\ v(1) = 0,\ v(\alpha) = 0\} \ni u \longmapsto 13u - 5x\partial_x u + x^2\partial_{xx}u \in \mathcal{C}^0(1,\alpha).$$

(a) Compute the formal adjoint of L, L^{T}, and its domain, $D(L^{\mathsf{T}})$.

Let w be a smooth function, say $w \in \mathcal{C}^2(1,\alpha)$. Then

$$\begin{aligned}
\int_1^\alpha Lv(x)w(x)\mathrm{d}x &= \int_1^\alpha (13v(x) - 5x\partial_x v(x) + x^2\partial_{xx}v(x))w(x)\mathrm{d}x \\
&= \int_1^\alpha (13v(x)w(x) + v(x)\partial_x(5xw(x)))\mathrm{d}x - 5xv(x)w(x)|_1^\alpha - \int_1^\alpha \partial_x v(x)\partial_x(x^2w(x))\mathrm{d}x + x^2\partial_x v(x)w(x)|_1^\alpha \\
&= \int_1^\alpha v(x)(13w(x) + \partial_x(5xw(x)))\mathrm{d}x + \int_1^\alpha v(x)\partial_{xx}(x^2w(x))\mathrm{d}x + x^2\partial_x v(x)w(x)|_1^\alpha - v(x)\partial_x(x^2w(x))|_1^\alpha \\
&= \int_1^\alpha v(x)(13w(x) + \partial_x(5xw(x))\partial_{xx}(x^2w(x)))\mathrm{d}x + x^2\partial_x v(x)w(x)|_1^\alpha.
\end{aligned}$$

We get rid of the boundary term by enforcing $w(1) = 0$ and $w(\alpha) = 0$. Hence the formal adjoint is defined by

$$L^{\mathsf{T}} : D(L^{\mathsf{T}}) = \{v \in \mathcal{C}^2(1,\alpha) |\ v(1) = 0,\ v(\alpha) = 0\} \ni w \longmapsto 13w(x) + \partial_x(5xw(x)) + \partial_{xx}(x^2w(x)) \in \mathcal{C}^0(1,\alpha).$$

(b) The general solution to $L(v) = 0$ is $v(x) = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$, $a, b \in \mathbb{R}$ and the general solution to $L^{\mathsf{T}}(w) = 0$ is $cx^{-4}\cos(2\log(x)) + dx^{-4}\sin(2\log(x))$. Compute the null spaces of L and L^{T}.

Let $v \in N(L)$, i.e., $L(v) = 0$ and $v \in D(L)$, then $v(x) = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$. The boundary conditions gives

$$v(1) = 0, \quad v(\alpha) = a\alpha^3\cos(2\tfrac{\pi}{2}) + bx^3\sin(2\tfrac{\pi}{2}) = -a\alpha^3 = 0,$$

which in turn implies that $a = 0$, i.e., $N(L) = \mathrm{span}\{x^3\sin(2\log(|x|))\}$.

Let $w \in N(L^{\mathsf{T}})$, i.e., $L^{\mathsf{T}}(w) = 0$ and $w \in D(L^{\mathsf{T}})$, then $w(x) = cx^{-4}\cos(2\log(x)) + dx^{-4}\sin(2\log(x))$. The boundary conditions give

$$w(1) = 0, \quad w(\alpha) = c\alpha^{-4}\cos(2\tfrac{\pi}{2}) + dx^{-4}\sin(2\tfrac{\pi}{2}) = -c\alpha^{-4} = 0,$$

which implies $c = 0$, i.e., $N(L^{\mathsf{T}}) = \mathrm{span}\{x^{-4}\sin(2\log(x))\}$.

(c) Under which condition does the following problem have a solution: $Lu = f$, $u(1) = 0$, $u(\alpha) = 0$? Give all the details.

We are in the second case of the Fredholm alternative since $N(L) \neq \{0\}$. Hence this problem has a solution only if f is orthogonal to $N(L^{\mathsf{T}})$, that is to say $\int_1^\alpha f(x)x^{-4}\sin(2\log(|x|))\mathrm{d}x = 0$.

(d) Does the problem $Lu = 1$, $u(1) = 0$, $u(\alpha) = 0$, have a solution?Give all the details.

Let $I = \int_1^\alpha x^{-4}\sin(2\log(|x|))\mathrm{d}x$. Since $\alpha = e^{\frac{\pi}{2}}$ we have $I = \int_1^{e^{\frac{\pi}{2}}} x^{-4}\sin(2\log(|x|))\mathrm{d}x$. For $x \in (1, e^{\frac{\pi}{2}})$ the values of $2\log(|x|)$ are in $(0,\pi)$, hence the values of $\sin(2\log(|x|))$ are in $(0,1)$. Hence I is positive. In conclusion the problem has no solution.