Last name: ***name:***

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Question 1: Let $L > 0$. Let $h : [0, L] \to \mathbb{R}$ be an integrable function.
(a) Recall the definition of the coefficients of the cosine series of h, say $a_n(h)$, for all $n \in \mathbb{N}$. (No justification is needed. Just give the formula.)

The definition of $a_n(f)$ is

$$a_n(f) = \begin{cases} \frac{1}{L}\int_0^L h(t)\mathsf{d}t & n = 0, \\ \frac{2}{L}\int_0^L h(t)\cos(n\pi\frac{t}{L})\mathsf{d}t & n \geq 1. \end{cases}$$

(b) Recall the definition of the coefficients of the sine series of h, say $b_n(h)$, for all $n \in \mathbb{N}$, $n \geq 1$. (No justification is needed. Just give the formula.)

The definition of $b_n(f)$ is

$$b_n(f) = \frac{2}{L}\int_0^L h(t)\sin(n\pi\tfrac{t}{L})\mathsf{d}t.$$

Question 2: Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x$.
(a) Compute $\int_0^{n\pi} f(t)\cos(t)\mathrm{d}t$ (*Hint:* integrate by parts).

Using the hint, we have

$$\int_0^{n\pi} f(t)\cos(t)\mathsf{d}t = \int_0^{n\pi} f(t)\cos(t)\mathsf{d}t = -\int_0^{n\pi}\sin(t)\mathsf{d}t + n\pi\sin(n\pi)$$
$$= \cos(n\pi) - 1 = (-1)^n - 1.$$

(b) Compute $\int_0^1 f(t)\cos(n\pi t)\mathrm{d}t$ for all $n \in \mathbb{N}$. (*Hint:* use (a) and a change of variable. Be careful when $n = 0$.)

Case 1: If $n = 0$, then

$$\int_0^1 t\cos(n\pi t)\mathsf{d}t = \int_0^1 t\mathsf{d}t = \frac{1}{2}$$

Case 2: If $n \geq 1$, we use the hint and the change of variable $n\pi t \mapsto z$,

$$\int_0^1 t\cos(n\pi t)\mathsf{d}t = \int_0^{n\pi}\frac{z}{n\pi}\cos(z)\frac{\mathsf{d}z}{n\pi} = \frac{1}{(n\pi)^2}\int_0^{n\pi} z\cos(z)\mathsf{d}z = \frac{(-1)^n - 1}{(n\pi)^2}.$$

(c) Compute the cosine series of f.

Using (b) we obtain

$$a_n(f) = \begin{cases} \int_0^1 f(t)\mathrm{d}t = \int_0^1 t\mathrm{d}t = \frac{1}{2} & n = 0, \\ 2\int_0^1 f(t)\cos(n\pi t)\mathrm{d}t = 2\int_0^1 t\cos(n\pi t)\mathrm{d}t = 2\frac{(-1)^n - 1}{(n\pi)^2}, & n \geq 1. \end{cases}$$

Hence, $a_{2n}(f) = 0$ for all $n \geq 0$ and

$$a_{2n+1}(f) = \begin{cases} \frac{1}{2} & n = 0, \\ \frac{-4}{((2n+1)\pi)^2}, & n \geq 1. \end{cases}$$

$$\begin{aligned} \mathsf{CS}(f)(x) &= \sum_{n=0} a_n(f)\cos(n\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) \\ &= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{((2n+1)\pi)^2}\cos((2n+1)\pi x) \end{aligned}$$

Question 3: Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Let $u : [0,1] \to \mathbb{R}$ be the solution to the boundary value problem $-\partial_{xx}u + 7u(x) = f(x)$, $x \in [0,1]$ with $\partial_x u(0) = 0$ and $\partial_x u(1) = 0$. Accept as a fact that all the derivatives of u are continuous and bounded on $[0,1]$. For every integrable function $g : [0,1] \to \mathbb{R}$ we denote by $a_n(g)$ the coefficients of the cosine series of g and $b_n(g)$ the coefficients of the sine series of g.

(a) Compute the coefficients of the sine series of $\partial_x u$ in terms of the coefficients $a_n(u)$.

As u is smooth, there is a theorem (seen in class) that says that we can differentiate the cosine series of u:

$$\partial_x u(x) = \partial_x\left(\sum_{n=0}^{\infty} a_n(u)\cos(n\pi x)\right) = \sum_{n=1}^{\infty} a_n(u)\partial_x\left(\cos(n\pi x)\right) = \sum_{n=1}^{\infty} -n\pi a_n(u)\sin(n\pi x).$$

That is we have $b_n(\partial_x u) = -n\pi a_n(u)$.

(b) Compute the coefficients of the cosine series of $\partial_{xx}u$ in terms of the coefficients $a_n(u)$.

As $\partial_x u$ is smooth AND $\partial_x u(0) = 0$ AND $\partial_x u(1) = 0$, there is a theorem (seen in class) that says that we can differentiate the sine series of $\partial_x u$:

$$\begin{aligned} \partial_{xx}u(x) = \partial_x(\partial_x u(x)) = \partial_x(\mathsf{SS}(\partial_x u(x))) = \partial_x\left(\sum_{n=0}^{\infty} b_n(\partial_x u)\sin(n\pi x)\right) = \sum_{n=0}^{\infty} b_n(\partial_x u)\partial_x(\sin(n\pi x)) \\ = \sum_{n=0}^{\infty} b_n(\partial_x u)n\pi\cos(n\pi x). \end{aligned}$$

Hence, $a_n(\partial_{xx}u(x)) = n\pi b_n(\partial_x u)$, and from (a) we infer that

$$a_n(\partial_{xx}u(x)) = -(n\pi)^2 a_n(u).$$

(c) Compute $\partial_{xx}\left(\mathrm{CS}(u)\right) - \mathrm{CS}(\partial_{xx}u)$. Justify the steps of the argument.

Since u is smooth, we have $\mathsf{CS}(u)(x) = u(x)$ for all $x \in (0,1)$. Hence $\partial_x \mathsf{CS}(u)(x) = \partial_x u(x)$ for all $x \in (0,1)$. Similarly, since $\partial_x u$ is smooth, we have $\mathsf{CS}(\partial_x u)(x)) = \partial_x u(x)$ for all $x \in (0,1)$. Hence

$$\partial_x\left(\mathsf{CS}(u)\right)(x) = \partial_x u(x) = \mathsf{CS}(\partial_x u)(x) \qquad \forall x \in (0,1).$$

The same argument shows that $\partial_x\left(\mathsf{CS}(\partial_x u)\right) = \partial_x(\partial_x u)$. Similarly, since $\partial_{xx}u$ is smooth, we have $\mathsf{CS}(\partial_{xx}u)(x)) = \partial_{xx}u(x)$ for all $x \in (0,1)$. In conclusion,

$$\partial_x\left(\mathsf{CS}(\partial_x u)\right)(x) = \partial_x(\partial_x u)(x) = \partial_{xx}u(x) = \mathsf{CS}(\partial_{xx}u)(x) \qquad \forall x \in (0,1).$$

Hence,

$$\partial_{xx}\left(\mathsf{CS}(u)\right)(x) - \mathsf{CS}(\partial_{xx}u)(x) \qquad \forall x \in (0,1).$$

(d) Compute the coefficients $a_n(u)$ in terms of the coefficients $a_n(f)$. (*Hint*: insert the cosine series of u and f in the equation $-\partial_{xx}u + 7u(x) = f(x)$.)

We have

$$\mathsf{CS}(f) = \mathsf{CS}(-\partial_{xx}u) + 7\mathsf{CS}(u) = -\partial_{xx}\mathsf{CS}(u) + 2\mathsf{CS}(u).$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} a_n(f)\cos(n\pi x) &= \sum_{n=1}^{\infty} n^2\pi^2 a_n(u)\cos(n\pi x) + \sum_{n=1}^{\infty} 7a_n(u)\cos(n\pi x)\\ &= \sum_{n=1}^{\infty}(n^2\pi^2+7)a_n(u)\cos(n\pi x).\end{aligned}$$

This implies that

$$a_n(u) = \frac{a_n(f)}{7+n^2\pi^2}$$

Question 4: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (-\infty,+\infty)$, $t > 0$, with initial data $u(x,0) = \frac{1}{1+x^2}$, $\partial_t u(x,0) = \frac{2x}{(1+x^2)^2}$. Compute the solution $w(x,t)$.

The wave speed is 1. The solution is given by the D'Alembert formula,

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) + \frac{1}{2}\int_{x-t}^{x+t}\frac{2\tau}{(1+\tau^2)^2}\mathsf{d}\tau$$

After integration, we obtain

$$= \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) - \frac{1}{2}\left[\frac{1}{(1+\tau^2)}\right]_{x-t}^{x+t},$$

which finally gives

$$w(x,t) = \frac{1}{1+(x-t)^2}.$$

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Question 5: Let $f \in L^1(\mathbb{R};\mathbb{R})$. Let $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x$ denote the Fourier transform of f.
(a) Compute the Fourier transform of $f(x) + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz$ in terms of $\mathcal{F}(f)$. (*Hint:* $\mathcal{F}(e^{-\alpha|x|})(\omega) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}$ and $\mathcal{F}(h * g)(\omega) = 2\pi\mathcal{F}(h)(\omega)\mathcal{F}(g)(\omega)$ for all $h, g \in L^1(\mathbb{R};\mathbb{R})$.)

Using the convolution theorem, we obtain

$$\mathcal{F}(f) + \frac{1}{2}\mathcal{F}(e^{-|x|} * f) = \mathcal{F}(f) + \frac{1}{2}2\pi\mathcal{F}(e^{-|x|})\mathcal{F}(f) = \mathcal{F}(f) + \pi\mathcal{F}(e^{-|x|})\mathcal{F}(f).$$

Now we use $\mathcal{F}(e^{-|x|}) = \frac{1}{\pi}\frac{1}{\omega^2+1}$. That is

$$\mathcal{F}(f) + \mathcal{F}(e^{-|x|} * f) = \mathcal{F}(f) + \pi\frac{1}{\pi}\frac{1}{\omega^2+1}\mathcal{F}(f) = \mathcal{F}(f)\frac{\omega^2+2}{\omega^2+1}.$$

(b) Find the function y that solves $\partial_{xx}y(x) - 2y(x) = f(x) + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$.

By taking the Fourier transform of the ODE and using the convolution theorem, we obtain

$$-\omega^2\mathcal{F}(y) - 2\mathcal{F}(y) = \mathcal{F}(f) + \frac{1}{2}\mathcal{F}(e^{-|x|} * f) = \mathcal{F}(f)\frac{\omega^2+2}{\omega^2+1}$$

That is

$$-\mathcal{F}(y)(\omega^2+2) = \mathcal{F}(f)\frac{\omega^2+2}{\omega^2+1}.$$

This gives

$$\mathcal{F}(y) = -\mathcal{F}(f)\frac{1}{1+\omega^2}.$$

Using the convolution Theorem, together with $\frac{1}{\omega^2+1} = \pi\mathcal{F}(e^{-|x|})$ gives

$$\mathcal{F}(y) = -\pi\mathcal{F}(f)\mathcal{F}(e^{-|x|}) = -\frac{1}{2}\mathcal{F}(f * e^{-|x|}).$$

Applying $\mathcal{F}^{-1}$ on both sides we obtain

$$y(x) = -\frac{1}{2}f * e^{-|x|} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz$$

That is

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz.$$

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Question 6: (a) Compute the Fourier transform of $g(x) := \frac{1}{x^2+4} + \frac{1}{x^2+1} - \frac{2x}{(1+x^2)^2}$, where $x \in (-\infty, +\infty)$. (Hint: $\mathcal{F}(\frac{2\alpha}{\alpha^2+x^2})(\omega) = e^{-\alpha|\omega|}$ and $\mathcal{F}(\frac{x}{(\alpha^2+x^2)^2})(\omega) = \frac{i\omega}{4\alpha}e^{-\alpha|\omega|}$, for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and recall that $e^{-2|\omega|} = e^{-|\omega|}e^{-|\omega|}$.)

Using the hint we have

$$\mathcal{F}(g)(\omega) = \mathcal{F}(\frac{1}{x^2+4}) + \mathcal{F}(\frac{1}{x^2+1}) - \mathcal{F}(\frac{2x}{(1+x^2)^2}) = \frac{1}{4}e^{-2|\omega|} + \frac{1}{2}e^{-|\omega|} - 2\frac{i\omega}{4}e^{-|\omega|},$$

which gives

$$\mathcal{F}(g)(\omega) = \frac{1}{2}e^{-|\omega|}(\frac{1}{2}e^{-|\omega|} + 1 - i\omega).$$

(b) Let $f \in L^1(\mathbb{R};\mathbb{R})$ with $\partial_x f L^1(\mathbb{R};\mathbb{R})$. Compute the Fourier transform of $\partial_x f(x) + f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1}\mathrm{d}y$ in terms of $\mathcal{F}(f)(\omega)$.

We first observe that

$$\partial_x f(x) + f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1}\mathrm{d}y = \partial_x f(x) + f(x) + \frac{1}{2\pi} f * \frac{1}{1+x^2}.$$

Then

$$\begin{aligned}\mathcal{F}\left(\partial_x f(x) + f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1}\mathrm{d}y\right) &= -i\omega\mathcal{F}(f) + \mathcal{F}(f) + \frac{2\pi}{2\pi}\mathcal{F}(f)\frac{1}{2}e^{-|\omega|} \\ &= \mathcal{F}(f)(-i\omega + 1 + \frac{1}{2}e^{-|\omega|}).\end{aligned}$$

(c) Solve the equation: $\partial_x f(x) + f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1}dy = \frac{1}{x^2+4} + \frac{1}{x^2+1} - \frac{2x}{(1+x^2)^2}$, for all $x \in (-\infty, +\infty)$.

Using (a) and (b) we obtain

$$\mathcal{F}(f)(-i\omega + 1 + \frac{1}{2}e^{-|\omega|}) = \frac{1}{2}e^{-|\omega|}(\frac{1}{2}e^{-|\omega|} + 1 - i\omega).$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{x^2+1}$.

Question 7: Consider the Schrödinger equation $i\partial_t u + (1 - i\epsilon)\partial_{xx} u = 0$, $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$, $t > 0$, where $\epsilon > 0$ and $i^2 = -1$. Note that u is complex-valued.
(a) (i) What is the real part of $(\epsilon + i)s$? (ii) What is the imaginary part of $(\epsilon + i)s$? (iii) What is the modulus of $(\epsilon + i)s$? (iv) What is the sign of the real part of $(\epsilon + i)s$ for $s \in \mathbb{R}$, $s \neq 0$.

(i) We have $\Re((\epsilon + i)s) = \epsilon s$.
(ii) We have $\Im((\epsilon + i)s) = s$.
(iii) We have $|(\epsilon + i)s| = |\epsilon + i||s| = |s|\sqrt{\epsilon^2 + 1}$.
(iv) As we have $\Re((\epsilon + i)s) = \epsilon s$, we conclude that

$$\Re((\epsilon + i)s) \text{ is } \begin{cases} \text{positive if } s > 0 \\ \text{negative if } s < 0. \end{cases}$$

(b) Solve the equation by using the Fourier technique assuming that $u_0 \in L^1(\mathbb{R})$ and decreases fast enough at infinity. (Hint: $\mathcal{F}(\sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}})(\omega) = e^{-\alpha\omega^2}$ for all $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$).)

We take the Fourier transform of the equation.

$$i\partial_t \mathcal{F}(u) + (i\omega)^2(1 - i\epsilon)\mathcal{F}(u) = 0,$$

which gives the ODE

$$\partial_t \mathcal{F}(u) - \omega^2(-\epsilon - i)\mathcal{F}(u) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega) = \mathcal{F}(u)(0)e^{-(\epsilon+i)t\omega^2} = \mathcal{F}(u_0)e^{-(\epsilon+i)t\omega^2}.$$

Observing that $\Re(\epsilon + i)t > \epsilon t > 0$, we can use the hint with $\alpha = (\epsilon + i)t$,

$$\mathcal{F}(u)(\omega) = \mathcal{F}(u_0)\mathcal{F}(\sqrt{\frac{\pi}{(\epsilon + i)t}} e^{-\frac{x^2}{4(\epsilon+i)t}})(\omega) = \frac{1}{2\pi}\mathcal{F}(u_0 * \sqrt{\frac{\pi}{(\epsilon + i)t}} e^{-\frac{x^2}{4(\epsilon+i)t}})(\omega).$$

In conclusion

$$u(x, t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{(\epsilon + i)t}} \int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4(\epsilon+i)t}} \mathrm{d}y = \sqrt{\frac{1}{4\pi(\epsilon + i)t}} \int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4(\epsilon+i)t}} \mathrm{d}y.$$

(c) Let $\Re(z)$, $\overline{z}$, and $|z|$ denote the real part, the conjugate and the modulus of z for all $z \in \mathbb{C}$, respectively. Compute $\Re(\overline{u}\partial_t u) - \frac{1}{2}\partial_t |u|^2$. (*Hint:* $\Re(z) = \frac{1}{2}(z + \overline{z})$, $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, $\overline{\overline{z}} = z$, $\overline{\partial_t u} = \partial_t \overline{u}$.)

Using the rules recalled in the hint, we have

$$\begin{aligned} \Re(\overline{u}\partial_t u) &= \frac{1}{2}(\overline{u}\partial_t u + \overline{\overline{u}\partial_t u}) \\ &= \frac{1}{2}(\overline{u}\partial_t u + \overline{\overline{u}}\,\overline{\partial_t u}) \\ &= \frac{1}{2}(\overline{u}\partial_t u + u\partial_t \overline{u}) \\ &= \frac{1}{2}\partial_t(\overline{u}u) \\ &= \frac{1}{2}\partial_t|u|^2. \end{aligned}$$

Hence $\Re(\overline{u}\partial_t u) - \frac{1}{2}\partial_t |u|^2 = 0$.

(d) Let $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,t)|^2 \mathrm{d}x$ and $P(t) = \int_{-\infty}^{\infty} |\partial_x u(x,t)|^2 \mathrm{d}x$. Prove that $\partial_t E(t) + \epsilon P(t) = 0$ assuming that u dencreases fast enough at infinity. (Hint: Apply the energy method to the Schrödinger equation with $-i\bar{u}$, use $\overline{\partial_x u} = \partial_x \overline{u}$, take the Real part.)

We follow the hint. We test the equation with $\bar{u}$ and integrate over $\mathbb{R}$,

$$\begin{aligned} 0 &= \int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x - i(1-i\epsilon)\int_{\infty}^{\infty} \partial_{xx} u\bar{u}\mathrm{d}x \\ &= \int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + i(1-i\epsilon)\int_{\infty}^{\infty} \partial_x u\partial_x\bar{u}\mathrm{d}x, \qquad \text{we used } \bar{u}(\pm\infty,t)=0 \\ &= \int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (\epsilon+i)\int_{\infty}^{\infty} \partial_x u\overline{\partial_x u}\mathrm{d}x, \\ &= \int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (\epsilon+i)\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x. \end{aligned}$$

(2) Now we take the real part of the equation.

$$0 = \int_{\infty}^{\infty} \Re(\partial_t u\bar{u})\mathrm{d}x + \epsilon\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x.$$

(3) We recall that $\Re(\overline{u}\partial_t u) = \frac{1}{2}\partial_t|u|^2$.

$$0 = \int_{\infty}^{\infty} \frac{1}{2}\partial_t|u|^2\mathrm{d}x + \epsilon\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x.$$

We have obtained the desired result $\partial_t E(t) + \epsilon P(t) = 0$.

(e) Compute $E(t)$ in terms of $E(0)$ for $\epsilon = 0$.

We have $\partial_t E(t) = 0$, which implies that $E(t) = E(0)$,

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} |u_0(x)|^2\mathrm{d}x.$$