M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, March 26, 2012
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{3}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{4}$$

$$\mathcal{F}(f(x-\beta)) = e^{i\omega\beta}\mathcal{F}(f) \tag{5}$$

Question 1: Consider the wave equation $\partial_{tt}w - 4\partial_{xx}w = 0$, $x \in (-\infty, +\infty)$, $t > 0$, with initial data $w(x,0) = \frac{1}{1+x^2}$, $\partial_t w(x,0) = -\frac{4x}{(1+x^2)^2}$. Compute the solution $w(x,t)$.

The wave speed is 2. The solution is given by the D'Alembert formula,

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right) + \frac{1}{4}\int_{x-2t}^{x+2t} -\frac{4\tau}{(1+\tau^2)^2}\mathsf{d}\tau$$

After integration, we obtain

$$= \frac{1}{2}\left(\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right) + \frac{1}{4}\left[\frac{2}{(1+\tau^2)}\right]_{x-2t}^{x+2t},$$

which finally gives

$$w(x,t) = \frac{1}{1+(x+2t)^2}.$$

Question 2: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2}\partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\frac{2t}{1+t^2}\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2}\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega\frac{2t}{1+t^2}\mathcal{F}(u). \end{aligned}$$

This is a first-order linear ODE:

$$\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega\frac{2t}{1+t^2} = i\omega\frac{\mathsf{d}}{\mathsf{d}t}(\log(1+t^2))$$

The solution is

$$\mathcal{F}(u)(\omega,t) = K(\omega)e^{i\omega\log(1+t^2)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega,0) = K(\omega).$$

The shift lemma (see formula (**??**)) implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\log(1+t^2)} = \mathcal{F}(u_0(x-\log(1+t^2))),$$

Applying the inverse Fourier transform finally gives

$$u(x,t) = u_0(x-\log(1+t^2)).$$

Question 3: Consider the wave equation $\partial_{tt} w - \partial_{xx} w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad \partial_x w(0,t) = 0, \quad \partial_x w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f. Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

We know from class that with homogeneous Neumann boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its even extension, say $f_{\mathsf{e,p}}$, where

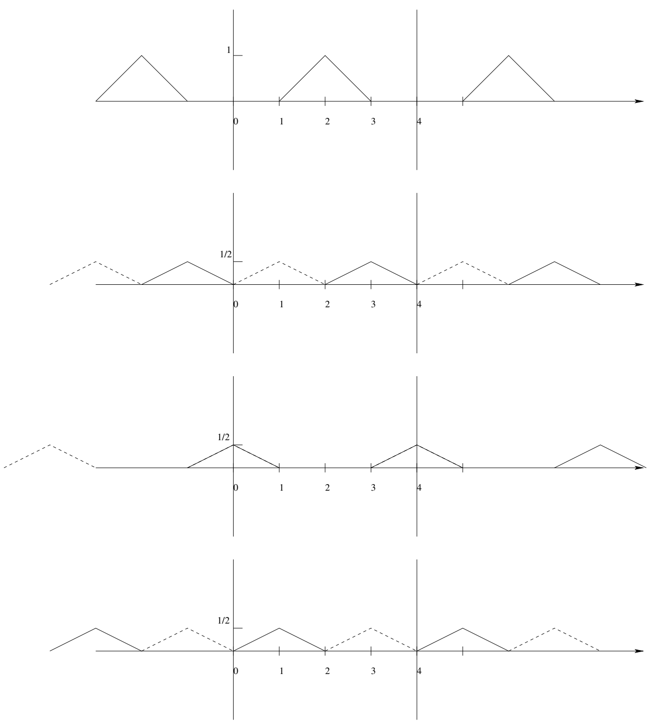
$$f_{\mathsf{e,p}}(x+8) = f_{\mathsf{e,p}}(x), \qquad \forall x \in \mathbb{R}$$

$$f_{\mathsf{e,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ f(-x) & \text{if } x \in [-4,0) \end{cases}$$

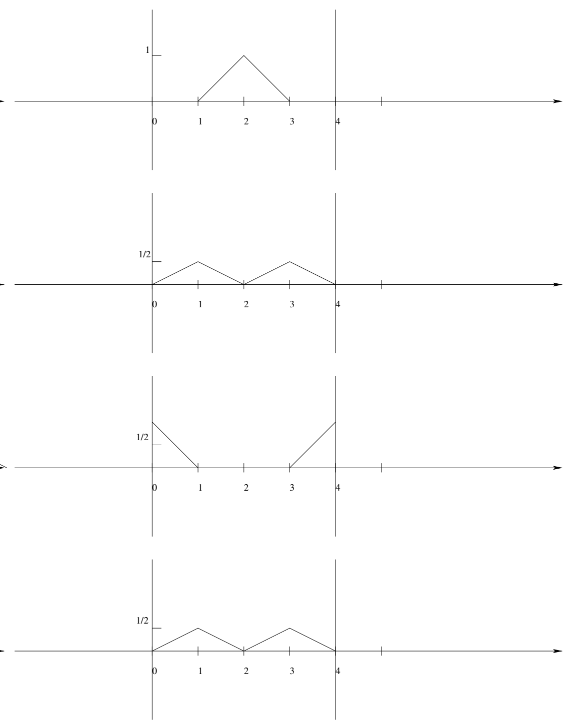
The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{e,p}}(x-t) + f_{\mathsf{e,p}}(x+t)).$$

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.



(a) Initial data + periodic extension of the even extension at $t = 0, 1, 2, 3$. Solid line waves move to the right, dotted line waves move to the left



(b) Solution in domain $(0,4)$ at $t = 0, 1, 2, 3$

Question 4: Solve the integral equation:

$$\int_{-\infty}^{+\infty}\left(f(y)-\sqrt{2}e^{-\frac{y^2}{2\pi}}-\frac{1}{1+y^2}\right)f(x-y)dy=-\int_{-\infty}^{+\infty}\frac{\sqrt{2}}{1+y^2}e^{-\frac{(x-y)^2}{2\pi}}\mathrm{d}y,\qquad \forall x\in(-\infty,+\infty).$$

(Hint: there is an easy factorization after applying the Fourier transform.)

The equation can be re-written

$$f*(f-\sqrt{2}e^{-\frac{x^2}{2\pi}}-\frac{1}{1+x^2})=-\frac{1}{1+x^2}*\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (**??**))

$$2\pi\mathcal{F}(f)\left(\mathcal{F}(f)-\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})-\mathcal{F}(\frac{1}{1+x^2})\right)=-2\pi\mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})$$

Solution 1: Using (**??**), (**??**) we obtain

$$\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})=\sqrt{2}\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\frac{\omega^2}{4\frac{1}{2\pi}}}=e^{-\frac{\pi\omega^2}{2}}$$

$$\mathcal{F}(\frac{1}{1+x^2})=\frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)\left(\mathcal{F}(f)-e^{-\frac{\pi\omega^2}{2}}-\frac{1}{2}e^{-|\omega|}\right)=-\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}.$$

This equation can also be re-written as follows

$$\mathcal{F}(f)^2-\mathcal{F}(f)e^{-\frac{\pi\omega^2}{2}}-\mathcal{F}(f)\frac{1}{2}e^{-|\omega|}+\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}=0,$$

and can be factorized as follows:

$$(\mathcal{F}(f)-e^{-\frac{\pi\omega^2}{2}})(\mathcal{F}(f)-\frac{1}{2}e^{-|\omega|})=0.$$

This means that either $\mathcal{F}(f)=e^{-\frac{\pi\omega^2}{2}}$ or $\mathcal{F}(f)=\frac{1}{2}e^{-|\omega|}$. Taking the inverse Fourier transform, we finally obtain two solutions

$$f(x)=\sqrt{2}e^{-\frac{x^2}{2\pi}},\quad \text{or}\quad f(x)=\frac{1}{1+x^2}.$$

Solution 2: Another solution consists of remarking that the equation with the Fourier transform can be rewritten as follows:

$$\mathcal{F}(f)^2-\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}})\mathcal{F}(f)-\mathcal{F}(\frac{1}{1+x^2})\mathcal{F}(f)+\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}})\mathcal{F}(\frac{1}{1+x^2})=0,$$

which can factorized as follows:

$$(\mathcal{F}(f)-\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}}))(\mathcal{F}(f)-\mathcal{F}(\frac{1}{1+x^2}))=0.$$

The either $\mathcal{F}(f)=\mathcal{F}(\sqrt{2}e^{-\frac{x^2}{2\pi}}))$ or $(\mathcal{F}(f)=\mathcal{F}(\frac{1}{1+x^2})$. The conclusion follows easily.

Question 5: Consider the equation $\partial_{xx}u(x) = f(x)$, $x \in (0, L)$, with $u(0) = a$ and $\partial_x u(L) = b$.
(a) Compute the Green's function of the problem.

Let x_0 be a point in $(0, L)$. The Green's function of the problem is such that

$$\partial_{xx}G(x, x_0) = \delta_{x_0}, \quad G(0, x_0) = 0, \quad \partial_x G(L, x_0) = 0.$$

The following holds for all $x \in (0, x_0)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = ax + b$ in $(0, x_0)$. The boundary condition $G(0, x_0) = 0$ gives $b = 0$. Likewise, the following holds for all $x \in (x_0, L)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = cx + d$ in (x_0, L). The boundary condition $\partial_x G(L, x_0) = 0$ gives $c = 0$. The continuity of $G(x, x_0)$ at x_0 implies that $ax_0 = d$. The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx}G(x, x_0)\mathsf{d}x = 1, \qquad \forall\epsilon > 0,$$

gives the so-called jump condition: $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$. This means that $0 - a = 1$, i.e., $a = -1$ and $d = -x_0$. In conclusion

$$G(x, x_0) = \begin{cases} -x & \text{if} \leq x \leq x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$

(b) Give the integral representation of u using the Green's function.

Let x_0 be a point in $(0, L)$. The definition of the Dirac measure at x_0 is such that

$$\begin{aligned} u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx}G(\cdot, x_0), u \rangle \\ &= -\int_0^L \partial_x G(x, x_0)\partial_x u(x)\mathsf{d}x + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)\partial_{xx}u(x)\mathsf{d}x - [G(x, x_0)\partial_x u(x)]_0^L + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)\partial_x u(L) + G(0, x_0)\partial_x u(0) + \partial_x G(L, x_0)u(L) - \partial_x G(0, x_0)u(0). \end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)b - \partial_x G(0, x_0)a$$

Question 6: Consider the wave equation with variable coefficients

$$m(x)\partial_{tt}u(x,t) - \partial_x(\mu(x)\partial_x u(x,t)) = 0, \qquad x \in (0,L), \quad t > 0,$$

with boundary condition $u(0,t) = 0$, $\partial_x u(L,t) = 0$, where m (density of the material) and μ, (elasticity of the material) and smooth positive functions. Assuming that the solution $u(x,t)$ is smooth, prove that the quantity $\int_0^L \left(\frac{1}{2}m(x)|\partial_t u(x,t)|^2 + \frac{1}{2}\mu(x)|\partial_x u(x,t)|^2\right) \mathrm{d}x$ is independent of t. (Hint: energy argument with $\partial_t u(x,t)$ and use $a(t)\partial_t a(t) = \partial_t(\frac{1}{2}a(t)^2)$.)

Multiply the equation by $\partial_t(ux,t)$ and integrate over $(0,L)$.

$$0 = \int_0^L (m(x)\partial_{tt}u(x,t)\partial_t u(x,t) - \partial_x\mu(x)\partial_x u(x,t)\partial_t u(x,t))\,\mathsf{d}x$$

Integrate by parts, use the boundary conditions, and use twice the formula $a(t)\partial_t a(t) = \partial_t(\frac{1}{2}a(t)^2)$,

$$\begin{aligned}
0 &= \int_0^L \left(m(x)\partial_t\frac{1}{2}|\partial_t u(x,t)|^2 + \mu(x)\partial_x u(x,t)\partial_x\partial_t u(x,t)\right)\mathsf{d}x \\
&= \int_0^L \left(m(x)\partial_t\frac{1}{2}|\partial_t u(x,t)|^2 + \mu(x)\partial_x u(x,t)\partial_t\partial_x u(x,t)\right)\mathsf{d}x \\
&= \int_0^L \left(m(x)\partial_t\frac{1}{2}|\partial_t u(x,t)|^2 + \mu(x)\partial_t\frac{1}{2}|\partial_x u(x,t)|^2\right)\mathsf{d}x.
\end{aligned}$$

Using the fact that m and μ are time-independent, we also have

$$\begin{aligned}
0 &= \int_0^L \left(\partial_t(\frac{1}{2}m(x)|\partial_t u(x,t)|^2) + \partial_t(\frac{1}{2}\mu(x)|\partial_x u(x,t)|^2)\right)\mathsf{d}x, \\
&= \frac{\mathsf{d}}{\mathsf{d}t}\int_0^L \left(\frac{1}{2}m(x)|\partial_t u(x,t)|^2 + \frac{1}{2}\mu(x)|\partial_x u(x,t)|^2\right)\mathsf{d}x,
\end{aligned}$$

which proves the statement.
