Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad \mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{1}$$

Question 1: (a) Prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$ for all $f \in L^1(\mathbb{R})$.

Let $f \in L^1(\mathbb{R})$, then

$$\partial_\omega \mathcal{F}(f)(\omega) = \partial_\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\partial_\omega e^{i\omega x}\mathrm{d}x$$
$$= i\frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)e^{i\omega x}\mathrm{d}x,$$

which prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$.

(b) Let $\alpha \in \mathbb{R}$ with $\alpha > 0$. Prove that $\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = 2\alpha i \partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: use (a).)

We use (a) to deduce that

$$\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = \mathcal{F}(-2\alpha x e^{-\alpha x^2})(\omega) = -2\alpha\mathcal{F}(xe^{-\alpha x^2})(\omega)$$
$$= 2\alpha i\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega).$$

(c) Show that $\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega) = -\frac{\omega}{2\alpha}\mathcal{F}(e^{-\alpha x^2})(\omega)$.

We use the property $\mathcal{F}(\partial_x f(x))(\omega) = -i\omega\mathcal{F}(f(x))(\omega)$ and (b)

$$-i\omega\mathcal{F}(e^{-\alpha x^2})(\omega)) = \mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = 2\alpha i\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega),$$

which implies the desired result.

(d) Given that $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$, compute $\mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: Observe that (c) is an ODE and solve it.)

The solution to the ODE $\partial_\omega g(\omega) = -\frac{\omega}{2\alpha}g(\omega)$ is $g(\omega) = g(0)e^{-\frac{\omega^2}{4\alpha}}$. We apply this formula to $g(\omega) = \mathcal{F}(e^{-\alpha x^2})(\omega)$,

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \mathcal{F}(e^{-\alpha x^2})(0)e^{-\frac{\omega^2}{4\alpha}}.$$

We now need to compute $\mathcal{F}(e^{-\alpha x^2})(0)$,

$$\mathcal{F}(e^{-\alpha x^2})(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha x^2}\mathrm{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-\alpha x^2}\sqrt{\alpha}\mathrm{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{x^2}\mathrm{d}x = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{1}{4\pi\alpha}.$$

Finally

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}.$$

Question 2: Solve the wave equation $\partial_{tt}w - 4\partial_{xx}w = 0$, $x \in (0, +\infty)$, $t > 0$ with initial data $w(x, 0) = (1 + x^3)^{-1}$, $\partial_t w(x, 0) = 0$, $x \in (0, +\infty)$ and boundary condition $\partial_x w(0, t) = 0$, $t > 0$. Give the full expression of the solution in all the cases. (Hint: Consider a particular extension of w over $\mathbb{R}$)

We define $f(x) = (1 + x^3)^{-1}$ and its even extension $f_e(x)$ on $(-\infty, +\infty)$. Let w_e be the solution to the wave equation over the entire real line with f_e as initial data:

$$\begin{aligned} &\partial_{tt}w_e - 4\partial_{xx}w_e = 0, \quad x \in \mathbb{R},\ t > 0 \\ &w_e(x, 0) = f_e(x), \quad x > 0, \qquad \partial_t w_e(x, 0) = 0, \quad x \in \mathbb{R}. \end{aligned}$$

The solution to this problem is given by the D'Alembert formula

$$w_e(x, t) = \frac{1}{2}(f_e(x - 2t) + f_e(x + 2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Let x be positive. Then $w(x, t) = w_e(x, t)$ for all $x \in (0, +\infty)$, since by construction $\partial_x w_e(0, t) = 0$ for all times.

Case 1: If $x - 2t > 0$, $f_e(x - 2t) = f(x - 2t)$; as a result $w(x, t) = \frac{1}{2}(f(x - 2t) + f(x + 2t))$, if $x - 2t > 0$. Hence,

$$w(x, t) = \frac{1}{2}\left(\frac{1}{1 + (x - 2t)^3} + \frac{1}{1 + (x + 2t)^3}\right), \quad \text{If } x - 2t > 0.$$

Case 2: If $x - 2t < 0$, $f_e(x - 2t) = f(-x + 2t)$; as a result $w(x, t) = \frac{1}{2}(f(-x + 2t) + f(x + 2t))$, if $x - 2t < 0$. Hence,

$$w(x, t) = \frac{1}{2}\left(\frac{1}{1 + (-x + 2t)^3} + \frac{1}{1 + (x + 2t)^3}\right), \quad \text{If } x - 2t < 0.$$

Let $H : \mathbb{R} \longrightarrow \mathbb{R}$ be the Heaviside function. (a) Show that the derivative of $xH(x)$ in the distribution sense is equal to $H(x)$. (Hint: Compute $-\int_{-\infty}^{\infty} xH(x)\partial_x\psi(x)\mathrm{d}x$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(xH)$ is the distribution that is such that

$$\langle \partial_x(xH), \psi\rangle = \int \partial_x(xH)\psi := -\int_{-\infty}^{\infty} xH(x)\partial_x\psi(x)\mathsf{d}x$$

for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned}\langle \partial_x(xH), \psi\rangle &= -\int_{-\infty}^{\infty} xH(x)\partial_x\psi(x)\mathsf{d}x \\ &= -\int_0^{\infty} x\partial_x\psi(x)\mathsf{d}x = \int_0^{\infty} \psi(x)\mathsf{d}x \\ &= \int_{-\infty}^{\infty} H(x)\psi(x)\mathsf{d}x.\end{aligned}$$

This means that $\partial_x(xH(x)) = H(x)$.

(b) Show that the derivative of $H(x)e^x$ in the distribution sense is equal to $H(x)e^x + \delta_0$ where δ_0 is the Dirac measure at 0. (Hint: Compute $-\int_{-\infty}^{\infty} H(x)e^x\partial_x\psi(x)\mathrm{d}x$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(e^xH)$ is the distribution that is such that

$$\langle \partial_x(e^xH), \psi\rangle = \int \partial_x(e^xH)\psi := -\int_{-\infty}^{\infty} e^xH(x)\partial_x\psi(x)\mathsf{d}x$$

for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned}\langle \partial_x(e^xH), \psi\rangle &= -\int_{-\infty}^{\infty} e^xH(x)\partial_x\psi(x)\mathsf{d}x \\ &= -\int_0^{\infty} e^x\partial_x\psi(x)\mathsf{d}x = \int_0^{\infty} e^x\psi(x)\mathsf{d}x + \psi(0) \\ &= \langle \delta_0, \psi\rangle + \int_{-\infty}^{\infty} e^xH(x)\psi(x)\mathsf{d}x \\ &= \langle \delta_0 + e^xH, \psi\rangle.\end{aligned}$$

This means that $\partial_x(H(x)\cos(x)) = \delta_0 + e^xH(x)$.

Question 3: (a) Let $s > 0$ and let $Lv = sv - \partial_{xx}v$ for all $v \in D(L) = \{v \in \mathcal{C}^2(\mathbb{R});\, v(\pm\infty) = 0\}$. Define L^{T} and $D(L^{\mathsf{T}})$.

Let $v \in D(L)$ and w be some smooth function, say $w \in \mathcal{C}^2(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} Lvw\mathrm{d}x = \int_{-\infty}^{+\infty} (sv - \partial_{xx}v)w\mathrm{d}x = \int_{-\infty}^{+\infty} (svw + \partial_x v\partial_x w)\mathrm{d}x - \partial_x vw|_{-\infty}^{+\infty}$$

We get rid of the boundary term by imposing $w(\pm\infty) = 0$. Then

$$\int_{-\infty}^{+\infty} Lvw\mathrm{d}x = \int_{-\infty}^{+\infty} v(sw - \partial_{xx}w)\mathrm{d}x + \partial_x wv|_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} v(sw - \partial_{xx}w)\mathrm{d}x.$$

This shows that $L^{\mathsf{T}}w = sw - \partial_{xx}w$ and $D(L^{\mathsf{T}}) = \{w \in \mathcal{C}^2(\mathbb{R});\, w(\pm\infty) = 0\}$.

(b) Let $s > 0$ and consider the equation $Lv = u_0(x)$, $v \in D(L)$. Compute Green's function, $G(x, x_0)$, $x, x_0 \in \mathbb{R}$.

Let $G(x, x_0)$ be Green's function. Since the operator is self-adjoint (shown in class many times), G satisfies

$$sG(x, x_0) - \partial_{xx}G(x, x_0) = \delta_{x_0}, \qquad G(\pm\infty, x_0) = 0.$$

Case 1: Assume $x < x_0$, then $G(x, x_0) = ae^{\sqrt{s}x} + be^{-\sqrt{s}x}$. The condition $G(-\infty, x_0) = 0$ implies that $b = 0$. Hence $G(x, x_0) = ae^{\sqrt{s}x}$ when $x < x_0$.
Case 2: Assume $x > x_0$, then $G(x, x_0) = ce^{\sqrt{s}x} + de^{-\sqrt{s}x}$. The condition $G(+\infty, x_0) = 0$ implies that $c = 0$. Hence $G(x, x_0) = de^{-\sqrt{s}x}$ when $x < x_0$.
Now we impose the continuity at x_0: $ae^{\sqrt{s}x_0} = de^{-\sqrt{s}x_0}$. We conclude with the jump condition,

$$\int_{x_0-\epsilon}^{x_0+\epsilon} (sG(x, x_0) - \partial_{xx}G(x, x_0))\mathrm{d}x = 1,$$

implying that $-\partial_x G(x_0^+, x_0) + \partial_x G(x_0^-, x_0) = 1$ when passing to the limit $\epsilon \to 0$. Hence $d\sqrt{s}e^{-\sqrt{s}x_0} + a\sqrt{s}e^{\sqrt{s}x_0} = 1$. Then using $ae^{\sqrt{s}x_0} = de^{-\sqrt{s}x_0}$, we infer that $d\sqrt{s}e^{-\sqrt{s}x_0} + d\sqrt{s}e^{-\sqrt{s}x_0} = 1$, i.e., $d = \frac{1}{2\sqrt{s}}e^{\sqrt{s}x_0}$ and $a = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}x_0}$. In conclusion

$$G(x, x_0) = \begin{cases} \frac{1}{2\sqrt{s}}e^{\sqrt{s}(x-x_0)} & \text{if } x < x_0, \\ \frac{1}{2\sqrt{s}}e^{\sqrt{s}(x_0-x)} & \text{otherwise,} \end{cases}$$

which can also be re-written $G(x, x_0) = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x-x_0|}$.

Question 4: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid x > 0,\ x+3t > 0\}$. Use the method of characteristics to solve the equation $\partial_t u + 4\partial_x u + 2u = 0$ for $(x,t) \in \Omega$ and $u(x,0) = x+4$, for $x > 0$, $u(-3t,t) = t+4$, for $t > 0$.

(i) We first parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 3s & s < 0 \\ s & s > 0, \end{cases} \qquad t_\Gamma(s) = \begin{cases} -s & s < 0 \\ 0 & s > 0. \end{cases}$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 4, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s))$.
(iii) Set $\Phi(t,s) = u(X(t,s),t)$. Then

$$\begin{aligned} \partial_t \Phi(t,s) &= \partial_x u(X(t,s),t)\partial_t X(t,s) + \partial_t u(X(t,s),t)\partial_t t \\ &= 4\partial_x u(X(t,s),t) + \partial_t u(X(t,s),t) = -2u(X(t,s),t) = -2\Phi(s,t) \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{-2(t-t_\Gamma(s))}$, i.e., $u(X(t,s)) = u(X(t_\Gamma(s),s), t_\Gamma(s))e^{-2(t-t_\Gamma(s))} = u(x_\Gamma(s), t_\Gamma(s))e^{-2(t-t_\Gamma(s))}$.
(iv) The implicit representation of the solution is

$$X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s)), \quad u(X(t,s)) = u(x_\Gamma(s), t_\Gamma(s))e^{-2(t-t_\Gamma(s))}.$$

(v) The explicit representation is obtained by replacing the parameterization (t,s) by (X,t). Using the definitions of $x_\Gamma(s)$ and $t_\Gamma(s)$, we have two cases:
Case 1: $s < 0$. The definition of $X(t,s)$ gives $X(s,t) = 3s + 4(t+s)$, i.e., $s = (X-4t)/7$. Then

$$\begin{aligned} u(X,t) = (t_\Gamma(s)+4)e^{-2(t-t_\Gamma(s))} &= (-s+4)e^{-2(t+s)} = (4-(X-4t)/7)e^{-2(t+(X-4t)/7)} \\ &= (4 + \frac{4t-X}{7})e^{-\frac{2}{7}(3t+X)} \end{aligned}$$

i.e., $\boxed{u(X,t) = (4 + \frac{4t-X}{7})e^{-\frac{2}{7}(3t+X)} \text{ if } X < 4t}$.
Case 2: $s > 0$. The definition of $X(t,s)$ gives $X(s,t) = s + 4t$, i.e., $s = X - 4t$. Then

$$u(X,t) = (x_\Gamma(s)+4)e^{-2(t-t_\Gamma(s))} = (s+4)e^{-2t} = (4+X-4t)e^{-2t}.$$

i.e., $\boxed{u(X,t) = (4+X-4t)e^{-2t} \text{ if } X > 4t}$.

Question 5: Let $\alpha := e^{\frac{\pi}{2}}$. Consider the operator $L : \phi \longmapsto \partial_{xx}\phi(x)) + \frac{1}{x}\partial_x\phi(x) + \frac{1}{x^2}\phi(x)$, with domain $D = \{v \in \mathcal{C}^2(1,2);\ v(1) = 0,\ v(\alpha) = 0\}$.
(i) What is the Null space of L? (Hint: The general solution to $L\phi = 0$ is $\phi(x) = c_1 \cos(\log(x)) + c_2 \sin(\log(x))$.)

Let ϕ be a member of the null space of L, say $\mathsf{N}(L)$. Then

$$\partial_{xx}\phi(x) + \frac{1}{x}\partial_x\phi(x) + \frac{1}{x^2}\phi(x) = 0$$

In other words, using the hint, $\phi(x) = c_1 \cos(\log(x)) + c_2 \sin(\log(x))$. The boundary conditions imply that

$$\phi(1) = 0 = c_1,$$

In conclusion $\mathsf{N}(L) = \mathsf{span}\{\sin(\log(x))\}$.

(iii) Give the formal adjoint of L and its domain.

Let $u \in D$ and $v \in D^*$, then

$$\int_1^2 (Lu(x))v(x)\mathsf{d}x = \int_1^\alpha (\partial_{xx}u(x) + \frac{1}{x}\partial_x u(x) + \frac{1}{x^2}u(x))v(x)\mathsf{d}x$$
$$= \int_1^\alpha (u(x)\partial_{xx}v(x) - u(x)\partial_x(\frac{1}{x}v(x)) + \frac{1}{x^2}u(x)v(x))\mathsf{d}x + \frac{1}{x}u(x)v(x)|_1^\alpha + \partial_x u(x)v(x)|_1^\alpha - \partial_x v(x)u(x)|_1^\alpha$$

We enforce $v(1) = v(\alpha) = 0$ to get rid of the boundary terms. Then

$$\int_1^2 (Lu(x))v(x)\mathsf{d}x = \int_1^\alpha u(x)(\partial_{xx}v(x) - \partial_x(\frac{1}{x}v(x)) + \frac{1}{x^2}v(x))\mathsf{d}x$$

This means that $D* = \{v \in \mathcal{C}^2(1,2);\ v(1) = 0,\ v(\alpha) = 0\} = D$ and $L^*v = \partial_{xx}v(x) - \partial_x(\frac{1}{x}v(x)) + \frac{1}{x^2}v(x)$.

(iii) The general solution of $\partial_{xx}v(x) - \partial_x(\frac{1}{x}v(x)) + \frac{1}{x^2}v(x) = 0$, is $\phi(x) = c_1 x\cos(\log(x)) + c_2 x\sin(\log(x))$. Under which condition does the problem $Lu = f(x)$, $x \in (1,\alpha)$, with $u(1) = 0$, $u(\alpha) = 0$ has a solution?

We are in the second case of the Fredholm alternative. We must compute $\mathsf{Null}(L^\mathsf{T})$. Let $v \in \mathsf{Null}(L^\mathsf{T})$, i.e., $L^\mathsf{T}v = 0$ and $v \in D(L^\mathsf{T})$, then $v(x) = c_1 x\cos(\log(x)) + c_2 x\sin(\log(x))$. The boundary conditions imply that

$$c_1 = 0,$$

meaning that $v(x) = c_2 x\sin(\log(x))$. In conclusion, $\mathsf{Null}(L^\mathsf{T}) = \mathsf{span}\{x\sin(\log(x))\}$. There is a unique solution to the above problem if and only if

$$\int_1^\alpha f(x)x\sin(\log(x))\mathsf{d}x = 0.$$