M602: Methods and Applications of of Partial Differential Equations
Mid-Term TEST, September 19, 2008
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1

Let u solve $\partial_t u - \partial_x((2x+1)\partial_x u) = 3$, $x \in (0, L)$, with $\partial_x u(0,t) = \alpha$, $\partial_x u(L,t) = -1$, $u(x,0) = f(x)$.
(a) Let $\alpha = 1$. Compute $\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over the domain $(0, L)$:

$$\begin{aligned} d_t \int_0^L u(x,t)dx &= \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_x((2x+1)\partial_x u)dx + 3L \\ &= (2L+1)\partial_x u(L,t) - \partial_x u(0,t) + 3L = -(2L+1) - \alpha + 3L \\ &= L - 1 - \alpha \end{aligned}$$

That is $d_t \int_0^L u(x,t)dx = L - 2$. This implies $\int_0^L u(x,t)dx = (L-2)t + \int_0^L f(x)dx$.

(b) Let α be an arbitrary number. For which value of α, $\int_0^L u(x,t)dx$ does not depend on t?

The above computation yields $\int_0^L u(x,t)dx = (L-1-\alpha)t + \int_0^L f(x)dx$. This is independent of t if $L - 1 - \alpha = 0$, meaning $\alpha = L - 1$.

Question 2

Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi$, $t \in (0,\pi)$, supplemented with the boundary conditions $2\phi(0) = \phi'(0)$, $\phi(\pi) = 0$.
(a) What should be the sign of λ for a non-zero solution to exist? Prove your answer.

Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain.

$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) = \lambda \int_0^\pi \phi^2(t)dt.$$

Using the BCs, we infer

$$\int_0^\pi (\phi'(t))^2 dt + 2\phi(0)^2 = \lambda \int_0^\pi \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(b) Assume that $2\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) \neq 0$. Prove that $\phi = 0$ is the only possible solution.

Observe first that λ cannot be zero, otherwise we would have $2\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$. As a result, (a) implies that λ is positive. Then ϕ is of the following form

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi'(0) = 2\phi(0)$ implies $2c_1 - \sqrt{\lambda}c_2 = 0$. The other BC implies $c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi) = 0$. The constants c_1 and c_2 solve the following linear system

$$\begin{cases} 2c_1 - \sqrt{\lambda}c_2 = 0 \\ c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi) = 0 \end{cases}$$

The determinant is equal to $2\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$ and is non-zero by hypothesis. Then the only solution is $c_1 = c_2 = 0$, which again gives $\phi = 0$.

In conclusion, the only possible solution to the problem is $\phi = 0$ if $2\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) \neq 0$.

Question 3

Consider the Laplace equation $\Delta u = 0$ in the rectangle $x \in [0, L]$, $y \in [0, H]$ with the boundary conditions $u(0, y) = 0$, $\partial_x u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = \sin(\frac{3}{2}\pi x/L)$. Solve the Equation using the method of separation of variables. (Give all the details.)

Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda$. Observe that $\phi(0) = 0$ and $\phi'(L) = 0$. The usual energy technique implies that λ is negative. That is to say $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The Dirichlet condition at $x = 0$ implies $a = 0$. The Neumann condition at L implies $\cos(\sqrt{\lambda}L) = 0$, which implies $\sqrt{\lambda}L = (n + \frac{1}{2})\pi$, where n is an integer. This means that $\phi(x) = b\sin((n + \frac{1}{2})\pi x/L)$. The fact that λ is negative implies $\psi(y) = c\cosh(\sqrt{\lambda}y) + d\sinh(\sqrt{\lambda}y)$. The boundary condition at $y = 0$ implies $c = 0$. Then

$$u(x, y) = A\sin((n + \tfrac{1}{2})\pi\frac{x}{L})\sinh((n + \tfrac{1}{2})\pi\frac{y}{L}).$$

The boundary condition at $y = H$ gives

$$\sin(\frac{3}{2}\pi\frac{x}{L}) = A\sin((n + \tfrac{1}{2})\pi\frac{x}{L})\sinh((n + \tfrac{1}{2})\pi\frac{H}{L}),$$

which implies $n = 1$ and $A = \sinh^{-1}\left(\frac{\frac{3}{2}\pi H}{L}\right)$, i.e.,

$$u(x, y) = \frac{\sinh\left(\frac{\frac{3}{2}\pi y}{L}\right)}{\sinh\left(\frac{\frac{3}{2}\pi H}{L}\right)}\sin(\frac{3}{2}\pi x/L)$$

Question 4

Let $f : [-1, +1] \longrightarrow \mathbb{R}$ be such that $f(x) = x$, if $x \in [0, 1]$ and $f(x) = 0$ if $x \in [-1, 0]$.
(a) Sketch the graph of the Fourier series of f on $(-\infty, \infty)$.

The Fourier series is equal to the periodic extension of f, except at the points $(2n + 1)L$, $n \in \mathbb{Z}$, where it is equal to $\frac{1}{2} = \frac{1}{2}(1 - 0)$.

(b) Compute the Fourier coefficients of f (recall $a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx$, and for $n \geq 1$, $a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx$, $b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx$. Hint: integrate by parts).

Clearly $a_0 = \frac{1}{4}$. For $n \geq 1$ we have

$$a_n = \int_0^1 x\cos(n\pi x)dx = -\frac{1}{n\pi}\int_0^1 \sin(n\pi x) + \frac{1}{n\pi}x\sin(n\pi x)|_0^1 = \frac{1}{(n\pi)^2}\cos(n\pi x)|_0^1 = \frac{(-1)^n - 1}{(n\pi)^2}$$

and

$$b_n = \int_0^1 x\sin(n\pi x)dx = \frac{1}{n\pi}\int_0^1 \cos(n\pi x) - \frac{1}{n\pi}x\cos(n\pi x)|_0^1 = \frac{(-1)^{n+1}}{n\pi}.$$

Question 5

Using cylindrical coordinates and the method of separation of variables, solve the Laplace equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{\pi}{2}],\ r \in [0,1]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(1,\theta) = \sin(2\theta)$. (Give all the details.)

We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$, and $rd_r(rd_r g(r)) = \lambda g(r)$. The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\frac{\pi}{2}) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(1,\theta) = \sin(2\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{\pi}{2}) = 0$ implies $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = 2n$. From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $rd_r(rd_r r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $2n = \alpha$. The boundary condition at $r = 1$ gives $\sin(2\theta) = 1^{2n}\sin(2n\theta)$ for all $\theta \in [0, \frac{\pi}{2}]$. This implies $n = 1$. The solution to the problem is

$$u(r,\theta) = r^2\sin(2\theta).$$
