M602: Methods and Applications of Partial Differential Equations
Second Mid-Term TEST, March 25th
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\text{If } f \text{ is piecewise } \mathcal{C}^1, \qquad \frac{1}{2}(f(x^-)+f(x^+)) = \mathcal{F}^{-1}(\mathcal{F}(f))(x),\ \forall x \in \mathbb{R}, \tag{2}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{3}$$

$$\mathcal{F}(e^{-|x|})(\omega) = \frac{1}{\pi}\frac{1}{\omega^2+1}, \tag{4}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha} \tag{5}$$

Question 1

Solve the PDE

$$\begin{aligned} &u_{tt} - a^2u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t, \\ &u(x,0) = \sin(x), \quad u_t(x,0) = a\cos(x), && -\infty < x < +\infty. \end{aligned}$$

Apply D'Alembert's Formula.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(x+at)+\sin(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} a\cos(\xi)\xi \\ &= \frac{1}{2}(\sin(x+at)+\sin(x-at)) + \frac{1}{2}(\sin(x+at)-\sin(x-at)) \\ &= \sin(x+at). \end{aligned}$$

Question 2

Solve the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 \le x \le 1, \quad 0 < t, \\ &u(0,t) = 0, \quad u(1,t) = 0 && 0 < t, \\ &u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, && 0 < x < +\infty. \end{aligned}$$

(Hint: Consider the periodic extension over $\mathbb{R}$ of the odd extension of u over $[-1+1]$).

The odd extension of u over $[-1+1]$, say u_o, satisfies the PDE and the initial conditions, and always satisfies $u_o(0,t) = 0$, $u_o(1,t) = 0$, $u_o(-1,t) = 0$. Since $u_o(1,t) = u_o(-1,t) = 0$, the periodic extension, says u_p, is smooth and also satisfies the PDE plus the initial conditions. As a result, we can obtain u by computing the solution of the wave equation on $\mathbb{R}$ using the periodic extension over $\mathbb{R}$ of the odd extension of the initial data over $[-1+1]$, i.e., $u = u_p|_{[0,1]}$

We have to define the odd extension of $\sin(\pi x)$ on $(-1,+1)$. Clearly $\sin(\pi x)$ is the odd extension. Now we define the periodic extension of $\sin(\pi x)$ over the entire real line. Clearly $\sin(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\sin(\pi(x-t)) + \sin(\pi(x+t)) \\ &= \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) - \sin(\pi t)\cos(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) + \sin(\pi t)\cos(\pi x)) \\ &= \cos(\pi t)\sin(\pi x). \end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\sin(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 3

Use the Fourier transform method to compute the solution of $u_{tt} - a^2 u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0, +\infty)$, with $u(0, x) = f(x) := \sin^2(x)$ and $u_t(0, x) = 0$ for all $x \in \mathbb{R}$.

Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2 a^2 \mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t, \omega) = c_1(\omega)\cos(\omega a t) + c_2(\omega)\sin(\omega a t).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0, \xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t, \omega) = \mathcal{F}(f)(\omega)\cos(\omega a t) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Taking the inverse transform gives

$$\begin{aligned} u(t, x) &= \frac{1}{2}\int_{-\infty}^{+\infty} \mathcal{F}(f)(\omega)(e^{ia\omega t - i\omega x} + e^{-ia\omega t - i\omega x})d\omega \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \mathcal{F}(f)(\omega)(e^{-i\omega(x-at)} + e^{-i\omega(x+at)})d\omega \\ &= \frac{1}{2}(f(x-at) + f(x+at)) = \frac{1}{2}(\sin^2(x+at) + \sin^2(x-at)). \end{aligned}$$

Note that this is the D'Alembert formula.

Question 4

(a) Compute the Fourier transform of the function $f(x)$ defined by $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$

By definition

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-1}^{1} e^{i\xi\omega} = = \frac{1}{2\pi} \frac{1}{i\omega} (e^{i\omega} - e^{-i\omega})$$
$$= \frac{1}{2\pi} \frac{2\sin(\omega)}{\omega}.$$

Hence

$$\mathcal{F}(f)(\omega) = \frac{1}{\pi} \frac{\sin(\omega)}{\omega}.$$

(b) Find the inverse Fourier transform of $g(\omega) = \frac{\sin(\omega)}{\omega}$. (Hint: use (a)+(2))

Using (a) we deduce that $g(\omega) = \pi \mathcal{F}(f)(\omega)$, that is to say, $\mathcal{F}^{-1}(g)(x) = \pi \mathcal{F}^{-1}(\mathcal{F}(f))(x)$. Now, using (2), we deduce that $\mathcal{F}^{-1}(g)(x) = \pi f(x)$ at every point x where $f(x)$ is of class C^1 and $\mathcal{F}^{-1}(g)(x) = \frac{\pi}{2}(f(x^-) + f(x^+))$ at discontinuity points of f. As a result:

$$\mathcal{F}^{-1}(g)(x) = \begin{cases} \pi & \text{if } |x| < 1 \\ \frac{\pi}{2} & \text{at } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Question 5

Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$, where f is a function such that $|f|$ is integrable over $\mathbb{R}$.

By taking the Fourier transform of the ODE, one obtains

$$-\omega^2 \mathcal{F}(y) - \mathcal{F}(y) = \mathcal{F}(f).$$

That is

$$\mathcal{F}(y) = -\mathcal{F}(f)\frac{1}{1+\omega^2}.$$

and the convolution Theorem, see (3), together with (4) gives

$$\mathcal{F}(y) = -\pi\mathcal{F}(f)\mathcal{F}(e^{-|x|}) = -\frac{1}{2}\mathcal{F}(f * e^{-|x|}).$$

Applying $\mathcal{F}^{-1}$ on both sides we obtain

$$y(x) = -\frac{1}{2} f * e^{-|x|} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|} f(z) dz$$

That is

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|} f(z) dz.$$

Question 6

Solve the following integral equation $\int_{-\infty}^{+\infty} e^{-(x-y)^2} g(y)dy = e^{-\frac{1}{2}x^2}$ for all $x \in (-\infty, +\infty)$, i.e. find the function g that solves the above equation.

The left-hand side of the equation is a convolution; hence,

$$e^{-x^2} * g(x) = e^{-\frac{1}{2}x^2}.$$

By taking the Fourier transform, we obtain

$$2\pi \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}\omega^2} \mathcal{F}g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2}.$$

That gives

$$\mathcal{F}(g)(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4}\omega^2}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \frac{\sqrt{4\pi}}{\sqrt{2\pi}} e^{-x^2} = \sqrt{\frac{2}{\pi}} e^{-x^2}.$$