Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\Phi \in C^1(\mathbb{R}^3;\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^3;\mathbb{R})$ be defined by $\Phi(x_1,x_2,x_3) := x_1^3 + \cos(x_2+x_3)$.
(a) Compute $\nabla\Phi$ and $\nabla\Psi$.

Applying the chain rule we obtain

$$\nabla\Phi(x_1,x_2,x_3) = \left(3x_1^2, -\sin(x_2+x_3), -\sin(x_2+x_3)\right),$$

Question 2: Let $\nabla\times$ denote the curl operator acting on vector fields: i.e., let $\mathbf{A} = (A_1,A_2,A_3) \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ be a three-dimensional vector field over $\mathbb{R}^3$, then $\nabla\times\mathbf{A} := (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$.
(a) Let $\varphi \in C^2(\mathbb{R}^3;\mathbb{R})$. Compute $\nabla\times(\nabla\varphi)$. (*Hint:* Recall that $\partial_{ij}\varphi = \partial_{ji}\varphi$, for all $i,j \in \{1,2,3\}$.)

The definitions imply that $\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$. Hence,

$$\nabla\times(\nabla\varphi) = (\partial_2\partial_3\varphi - \partial_3\partial_2\varphi, \partial_3\partial_1\varphi - \partial_1\partial_3\varphi, \partial_1\partial_2\varphi - \partial_2\partial_1\varphi) = \partial_{31}\varphi - \partial_{13}\varphi, \partial_{12}\varphi - \partial_{21}\varphi) = \mathbf{0}.$$

(b) Let $\mathbf{v}(x_1,x_2,x_3) := (-x_1, x_2, x_1x_3)$. Compute $\nabla\times(\mathbf{v})$.

By definition,

$$\nabla\times\mathbf{v} = (0, -x_3, 0).$$

(c) Let $\mathbf{v}(x_1,x_2,x_3) := (x_1^2, x_2x_3, x_1x_3)$. Compute $\nabla\cdot(\mathbf{v})$.

By definition,

$$\nabla\cdot\mathbf{v} = 2x_1 + x_3 + x_1 = 3x_1 + x_3.$$

Question 3: Let $k(x,y) = 1 + \sin(x\pi)^2 \sin(\pi y)^2$. Let $D = (-1,1)\times(-1,1)$ be the square centered at $(0,0)$ and of side 2.
(a) Compute the value of k over the boundary of D (i.e., $k|_{\partial D}$)

Since the boundary of D is the set $\{(x,y)\in\mathbb{R}^2;\ |x|=1, -1\le y\le 1\}\cup\{(x,y)\in\mathbb{R}^2;\ |y|=1,-1\le x\le 1\}$, we infer that

$$k|_{\partial D} = 1.$$

(b) Let $f(\mathbf{X}) = \frac{1}{4}$ for all $\mathbf{X}\in D$. Let $\Phi \in C^2(D)$ solve $-\nabla\cdot(k\nabla\Phi(\mathbf{X})) = f(\mathbf{X})$ for all $\mathbf{X}\in D$ and $\Phi_{|\partial D} = 0$. Compute $\int_{\partial D} k\partial_n\Phi \mathrm{d}s$.

Applying the divergence theorem gives

$$\int_{\partial D} k\partial_n\Phi \mathsf{d}s = \int_D \nabla\cdot(k\nabla\Phi(\mathbf{X}))\mathsf{d}\mathbf{X} = -\int_D f(\mathbf{X})\mathsf{d}\mathbf{X} = -\int_D \frac{1}{4}\mathsf{d}\mathbf{X} = -\frac{1}{4}|D|$$

where $|D| = 4$ is the surface of D. As a result,

$$\int_{\partial D} k\partial_n\Phi \mathsf{d}s = -1.$$

Question 4: Let $L>0$ and $D := (0,L)$. Let u solve $\partial_t u - \partial_x\big(xu + \partial_x u\big) = f(x)e^{-2t}$, $x\in D$, with $\partial_x u(0,t) = 1$, $Lu(L,t) + \partial_x u(L,t) = 1$, $u(x,0) = u_0(x)$, where f and u_0 are two smooth functions.
(a) Compute $\partial_t \int_0^L u(x,t)\mathrm{d}x$ as a function of t.

Integrate the equation over the domain $(0,L)$ and apply the fundamental theorem of calculus:

$$\begin{aligned}\partial_t \int_0^L u(x,t)\mathsf{d}x = \int_0^L \partial_t u(x,t)\mathsf{d}x &= \int_0^L \partial_x\big(xu+\partial_x u\big)\mathsf{d}x + e^{-2t}\int_0^L f(x)\mathsf{d}x \\ &= Lu(L,t) + \partial_x u(L,t) - \partial_x u(0,t) + e^{-2t}\int_0^L f(x)\mathsf{d}x \\ &= 1 - 1 + e^{-2t}\int_0^L f(x)\mathsf{d}x = e^{-2t}\int_0^L f(x)\mathsf{d}x.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)dx = e^{-2t}\int_0^L f(x)dx.$$