HW 1

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: (a) Let $\Phi \in C^1(\mathbb{R}^3;\mathbb{R})$ be defined by $\Phi(x_1,x_2,x_3) := x_2x_3^2 + \sin(x_1+x_3)$. Compute $\nabla\Phi$.

Applying the chain rule we obtain

$$\nabla\Phi(x_1,x_2,x_3) = \left(\cos(x_1+x_3), x_3^2, 2x_2x_3 + \cos(x_2+x_3)\right),$$

(b) Let $\Psi \in C^1(\mathbb{R}^3;\mathbb{R})$ be defined by $\Psi(x_1,x_2,x_3) := \frac{x_2^3}{2+\sin(x_1+2x_3)}$. Compute $\nabla\Psi$.

Applying the chain rule we obtain

$$\nabla\Psi(x_1,x_2,x_3) = \left(-\frac{x_2^3\cos(x_1+2x_3)}{(2+\sin(x_1+2x_3))^2}, \frac{3x_2^2}{2+\sin(x_1+2x_3)}, -\frac{2x_2^3\cos(x_1+2x_3)}{(2+\sin(x_1+2x_3))^2}\right)$$

Question 2: Let $\nabla\times$ denote the curl operator acting on vector fields: i.e., let $\mathbf{A} = (A_1,A_2,A_3) \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ be a three-dimensional vector field over $\mathbb{R}^3$, then $\nabla\times\mathbf{A} := (\partial_2A_3 - \partial_3A_2, \partial_3A_1 - \partial_1A_3, \partial_1A_2 - \partial_2A_1)$.
(a) Let $\varphi \in C^2(\mathbb{R}^3;\mathbb{R})$. Compute $\nabla\times(\nabla\varphi)$. (*Hint:* Recall that $\partial_{ij}\varphi = \partial_{ji}\varphi$, for all $i,j \in \{1,2,3\}$.)

The definitions imply that $\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$. Hence,

$$\nabla\times(\nabla\varphi) = (\partial_2\partial_3\varphi - \partial_3\partial_2\varphi, \partial_3\partial_1\varphi - \partial_1\partial_3\varphi, \partial_1\partial_2\varphi - \partial_2\partial_1\varphi) = \partial_{31}\varphi - \partial_{13}\varphi, \partial_{12}\varphi - \partial_{21}\varphi) = \mathbf{0}.$$

(b) Let $\mathbf{v}(x_1,x_2,x_3) := (-x_1, x_2, x_1x_3)$. Compute $\nabla\times\mathbf{v}$.

By definition,

$$\nabla\times\mathbf{v} = (0, -x_3, 0).$$

Question 3: (a) Recal the definition of the divergence operator acting on vector fields: i.e., let $\mathbf{A} = (A_1, \ldots, A_d) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ be a d-dimensional vector field over $\mathbb{R}^d$, then $\nabla\cdot\mathbf{A} :=?$.

$$\nabla\cdot\mathbf{A} = \partial_1 A_1 + \ldots + \partial_d A_d.$$

(b) Let $\mathbf{v}(x_1, x_2, x_3) := (x_1^2, x_2 x_3, x_1 x_3)$. Compute $\nabla\cdot\mathbf{v}$.

By definition,

$$\nabla\cdot\mathbf{v} = 2x_1 + x_3 + x_1 = 3x_1 + x_3.$$

(c) Let $\mathbf{w} = (w_1, w_2, w_3) \in C^2(\mathbb{R}^3, \mathbb{R}^3)$. Compute $\nabla\cdot(\nabla\times\mathbf{w})$.

(Recall that $\nabla\times\mathbf{w} = \partial_2 w_3 - \partial_3 w_2, \partial_3 w_1 - \partial_1 w_3, \partial_1 w_2 - \partial_2 w_1)$. Then, by definition,

$$\begin{aligned}
\nabla\cdot(\nabla\times\mathbf{w}) &= \partial_1(\partial_2 w_3 - \partial_3 w_2) + \partial_2(\partial_3 w_1 - \partial_1 w_3) + \partial_3(\partial_1 w_2 - \partial_2 w_1) \\
&= \partial_{12} w_3 - \partial_{13} w_2 + \partial_{23} w_1 - \partial_{12} w_3 + \partial_{13} w_2 - \partial_{23} w_1 \\
&= 0.
\end{aligned}$$

Hence $\nabla\cdot(\nabla\times\mathbf{w}) = 0$ for all $\mathbf{w} = (w_1, w_2, w_3) \in C^2(\mathbb{R}^3, \mathbb{R}^3)$.

Question 4: Let $D = (-1,1)\times(-1,1)$ be the square centered at $(0,0)$ and of side 2.

(a) Let $h(x,y) = 3 + 5\sin(2\pi x)^2 \sin(3\pi y)^2$. Compute the value of h over the boundary of D (i.e., $h|_{\partial D}$)

Since the boundary of D is the set $\{(x,y)\in\mathbb{R}^2;\, |x| = 1, -1 \le y \le 1\} \cup \{(x,y)\in\mathbb{R}^2;\, |y| = 1, -1 \le x \le 1\}$, we infer that

$$h|_{\partial D} = 3.$$

(b) Let $f(\mathbf{X}) = \frac{\pi}{7}$ for all $\mathbf{X}\in D$. Let $k(x,y) = 1 + x^2|y|$. Let $\Phi \in C^2(D)$ solve $-\nabla\cdot(k\nabla\Phi(\mathbf{X})) = f(\mathbf{X})$ for all $\mathbf{X}\in D$ and $\Phi_{|\partial D} = 0$. We denote $\partial_n\Phi := \mathbf{n}\cdot\nabla\Phi$. Compute $\int_{\partial D} k\partial_n\Phi \mathrm{d}s$. (*Hint:* Use the divergence theorem.)

Applying the divergence theorem gives

$$\int_{\partial D} k\partial_n\Phi \mathsf{d}s = \int_D \nabla\cdot(k\nabla\Phi(\mathbf{X}))\mathsf{d}\mathbf{X} = -\int_D f(\mathbf{X})\mathsf{d}\mathbf{X} = -\int_D \frac{1}{4}\mathsf{d}\mathbf{X} = -\frac{\pi}{7}|D|$$

where $|D| = 4$ is the surface of D. As a result,

$$\int_{\partial D} k\partial_n\Phi \mathsf{d}s = -\frac{4\pi}{7}.$$

Question 5: Let $L > 0$ and $D := (0, L)$. Let u solve $\partial_t u - \partial_x\big(xu + \partial_x u\big) = f(x)e^{-2t}$, $x \in D$, with $\partial_x u(0,t) = 1$, $Lu(L,t) + \partial_x u(L,t) = 3$, $u(x,0) = u_0(x)$, where f and u_0 are two smooth functions.
(a) Compute $\partial_t \int_0^L u(x,t)\mathrm{d}x$ as a function of t.

Integrate the equation over the domain $(0, L)$ and apply the fundamental theorem of calculus:

$$\begin{aligned}\partial_t \int_0^L u(x,t)\mathsf{d}x &= \int_0^L \partial_t u(x,t)\mathsf{d}x = \int_0^L \partial_x\big(xu + \partial_x u\big)\mathsf{d}x + e^{-2t}\int_0^L f(x)\mathsf{d}x \\ &= Lu(L,t) + \partial_x u(L,t) - \partial_x u(0,t) + e^{-2t}\int_0^L f(x)\mathsf{d}x \\ &= 3 - 1 + e^{-2t}\int_0^L f(x)\mathsf{d}xan = 2 + e^{-2t}\int_0^L f(x)\mathsf{d}x.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)\mathsf{d}x = 2 + e^{-2t}\int_0^L f(x)\mathsf{d}x.$$

(b) Assume that that $\int_0^L u_0(x)dx = 0$. Compute $\int_0^L u(x,t)dx$ for all $t > 0$.

Using the fundamental theorem of calculus we obtain

$$\begin{aligned}\int_0^L u(x,t)\mathsf{d}x &= \int_0^L u(x,0)\mathsf{d}x + 2t - \frac{1}{2}(e^{-2t} - 1)\int_0^L f(x)\mathsf{d}x \\ &= 2t - \frac{1}{2}(e^{-2t} - 1)\int_0^L f(x)\mathsf{d}x\end{aligned}$$