Quiz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $k : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, if $x \in [-1,0]$ and $k(x) = 1$ if $x \in (0,1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $-\partial_x T(-1) + T(-1) = -1$ and $T(1) = 3$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$.

(ii) Solve the problem, i.e., find $T(x)$ for all $x \in [-1,+1]$.

On $[-1,0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $-\partial_x T^-(-1) + T^-(-1) = -1 = -2b + a$. This gives $a = 2b - 1$ and $T^-(x) = 2b - 1 + bx$.

We proceed similarly on $[0,+1]$ and we obtain $T^+(x) = c + dx$. The Dirichlet boundary condition at $x = +1$ gives $T^+(1) = 3 = c + d$. This implies $c = 3 - d$ and $T^+(x) = 3 - d + dx$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b - 1 = 3 - d, \qquad \text{and} \qquad 2b = d.$$

This implies $d = 2$ and $b = 1$. In conclusion

$$T(x) = \begin{cases} x + 1 & \text{if } x \in [-1,0], \\ 2x + 1 & \text{if } x \in [0,+1]. \end{cases}$$

Question 2: Let $k, f : [-1, +1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, $f(x) = 0$ if $x \in [-1, 0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0, 1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = -2$ and $T(1) = 2$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find $T(x)$ for all $x \in [-1, +1]$.

On $[-1, 0]$ we have $k^-(x) = 2$ and $f^-(x) = 0$ which implies $-\partial_{xx}T^-(x) = 0$. This in turn implies $T^-(x) = ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = -2 = -a + b$. This gives $a = b + 2$ and $T^-(x) = (b + 2)x + b$.

We proceed similarly on $[0, +1]$ and we obtain $-\partial_{xx}T^-(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Dirichlet condition at $x = 1$ implies $T^+(1) = 2 = -1 + c + d$. This gives $c = 3 - d$ and $T^-(x) = -x^2 + (3 - d)x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $2(b + 2) = 3 - d$, respectively. In conclusion $b = -\frac{1}{3}$, $d = -\frac{1}{3}$ and

$$T(x) = \begin{cases} \frac{5}{3}x - \frac{1}{3} & \text{if } x \in [-1, 0], \\ -x^2 + \frac{10}{3}x - \frac{1}{3} & \text{if } x \in [0, 1]. \end{cases}$$

Question 3: Assume that the following equation has a smooth solution: $-\partial_x((2+\sin(x))\partial_x T(x)) + \partial_x T(x) + T(x) = x^3$, $T(a) = \sqrt{5}$, $T(b) = \pi$, $x \in [a, b]$, $t > 0$, where $k > 0$, $b > a$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify the expression $-\partial_x((2 + \sin(x))\partial_x T)$.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((2 + \sin(x))\partial_x\phi(x)) + \partial_x\phi(x) + \phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a, b), and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b \big(-\partial_x((2 + \sin(x))\partial_x\phi(x))\phi(x) + (\partial_x\phi(x))\phi(x) + (\phi(x))^2\big)\mathrm{d}x \\ &= \int_a^b \big(-\partial_x(\phi(x)(2 + \sin(x))\partial_x\phi(x)) + (1 + |x|)(\partial_x\phi(x))^2 + \partial_x(\frac{1}{2}\phi(x)^2) + (\phi(x))^2\big)\mathrm{d}x \\ &= \int_a^b \big((2 + \sin(x))(\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathrm{d}x \geq \int_a^b \big((\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathrm{d}x \end{aligned}$$

because $2 + \sin(x) > 0$ for all $x \in [a, b]$. This implies $\int_a^b(\phi(x))^2\mathrm{d}x = 0$, i.e. $\phi = 0$, meaning that $T_2 = T_1$.

Question 4: Assume that the following equation has a smooth solution: $-\partial_x((1+\sin(x)^2)\partial_x T(x)) - \pi\partial_x T(x) + (1+b-x)T(x) = \cos(x)$, $T(a) = 1$, $T(b) = \pi$, $x \in [a, b]$, $t > 0$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify $-\partial_x((1+x^2)\partial_x T)$.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1+\sin(x)^2)\partial_x\phi(x)) - \pi\partial_x\phi(x) + (1+b-x)\phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a, b), and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b \big(-\partial_x((1+\sin(x)^2)\partial_x\phi(x))\phi(x) - \pi(\partial_x\phi(x))\phi(x) + (1+b-x)(\phi(x))^2\big)\mathrm{d}x \\ &= -[\phi(x)(1+\sin(x)^2)\partial_x\phi(x)]_a^b + \int_a^b \big((1+\sin(x)^2)(\partial_x\phi(x))^2 - \pi\partial_x\Big(\frac{1}{2}\phi(x)^2\Big) + (1+b-x)(\phi(x))^2\big)\mathrm{d}x. \end{aligned}$$

But $\phi(a) = T_2(a) - T_1(a) = 0$ and $\phi(b) = T_2(b) - T_1(b) = 0$. Hence,

$$0 = \int_a^b \big((1+\sin(x)^2)(\partial_x\phi(x))^2 + (1+b-x)(\phi(x))^2\big)\mathrm{d}x \geq \int_a^b \phi^2(x)\mathrm{d}x,$$

since $1 + b - x \geq 1$ for all $x \in [a, b]$. This implies $\int_a^b(\phi(x))^2\mathrm{d}x = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.

Question 5: Let ϕ be a non-zero solution to the eigenvalue problem $\phi(x) - \partial_x\left((1+\sin(x)^2)\partial_x\phi(x)\right) = \lambda\phi(x)$, $x \in (0,\pi)$, $\phi(\pi) = 0$, $-\partial_x\phi(0) + \phi(0) = 0$. Determine the sign of λ using the energy method.

Multiply the equation by ϕ, integrate over $(0,\pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$\begin{aligned}
\lambda \int_0^\pi (\phi(x))^2 \mathsf{d}x &= \int_0^\pi \phi^2(x)\mathsf{d}x - \int_0^\pi \phi(x)\partial_x\left((1+\sin(x)^2)\partial_x\phi(x)\right)\mathsf{d}x \\
&= \int_0^\pi \phi^2(x)\mathsf{d}x - \int_0^\pi (\partial_x(\phi(x)(1+\sin(x)^2)\partial_x\phi(x)) - (1+\sin(x)^2)(\partial_x\phi(x))^2)\mathsf{d}x \\
&= -\phi(\pi)(1+\sin(\pi)^2)\partial_x\phi(\pi) + \phi(0)\partial_x\phi(0) + \int_0^\pi \left(\phi^2(x) + (1+\sin(x)^2)(\partial_x\phi(x))^2\right)\mathsf{d}x \\
&= (\phi(0))^2 + \int_0^\pi \left(\phi^2(x) + (1+\sin(x)^2)(\partial_x\phi(x))^2\right)\mathsf{d}x.
\end{aligned}$$

In conclusion

$$(\phi(0))^2 + \int_0^\pi \left(\phi^2(x) + (1+\sin(x)^2)(\partial_x\phi(x))^2\right)\mathsf{d}x = \lambda \int_0^\pi (\phi(x))^2\mathsf{d}x.$$

As ϕ is nonzero, we obtain that $\lambda = ((\phi(0))^2 + \int_0^\pi ((1+\sin(x)^2)\phi^2(x) + (\partial_x\phi(x))^2)\mathsf{d}x) / \int_0^\pi (\phi(x))^2\mathsf{d}x \geq 0$, i.e. λ is non-negative.

Question 6: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t \in (0,1)$, supplemented with the boundary condition $\phi(0) = 0$, $\partial_t\phi(1) = 0$.
(a) Prove that it is necessary that λ be positive for a non-zero smooth solution to exist.

(i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the Fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t = 0$, which implies that $\phi'(t) = 0$ for all $t \in (0,1]$. This implies that $\phi(t)$ is constant, and this constant is zero since $\phi(0) = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.

(b) The general solution to $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$ is $\phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda})$ for $\lambda \geq 0$. Find all the eigenvalues $\lambda > 0$ and the associated nonzero eigenfunctions.

Since $\lambda \geq 0$ by hypothesis, ϕ is of the following form

$$\phi(t) = \phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda}).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\partial_x\phi(1) = 0 = c_2\sqrt{\lambda}\cos(2\sqrt{\lambda})$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $2\sqrt{\lambda} = (n+\frac{1}{2})\pi$, $n = 0, 2, \ldots$. In conclusion

$$\lambda = ((2n+1)\pi)^2/16, \quad n = 1, 2, \ldots, \qquad \phi(t) = c\sin((n+\frac{1}{2})\pi\sqrt{t}).$$