HW 3

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = \cos(x - y + z^2)\sin(x + \cos(y - z)).$$

(a) Give the expression of $f(y, z, x)$ for any real numbers x, y, z (i.e., I want the value of f at (y, z, x)).

We just have to replace x by y, y by z, and z by x in the definition of f.

$$f(y, z, x) = \cos(y - z + x^2)\sin(y + \cos(z - x)).$$

(b) Compute $(\partial_y f)(y, z, x)$ (here $(\partial_y f)(y, z, x)$ means "value at (y, z, x) of the partial derivative of f with respect to the second variable of f". You may also write it $(\partial_2 f)(y, z, x)$ if you want).

We start by computing the partial derivative of f with respect to the second variable of f:

$$\partial_y f(x, y, z) = \sin(x - y + z^2)\sin(x + \cos(y - z)) - \cos(x - y + z^2)\cos(x + \cos(y - z))\sin(z - x).$$

Then

$$\partial_y f(y, z, x) = \sin(y - z + x^2)\sin(y + \cos(z - x)) - \cos(y - z + x^2)\cos(y + \cos(z - x))\sin(x - y).$$

(c) Compute $(\partial_y f)(y, z^2, x)$ (here $(\partial_y f)(y, z^2, x)$ means "value at (y, z^2, x) of the partial derivative of f with respect to the second variable of f". You may also write it $(\partial_2 f)(y, z^2, x)$ if you want).

We have established above that

$$\partial_y f(x, y, z) = \sin(x - y + z^2)\sin(x + \cos(y - z)) - \cos(x - y + z^2)\cos(x + \cos(y - z))\sin(z - x).$$

Then

$$\partial_y f(y, z^2, x) = \sin(y - z^2 + x^2)\sin(y + \cos(z^2 - x)) - \cos(y - z^2 + x^2)\cos(y + \cos(z^2 - x))\sin(x - y).$$

(d) Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\Phi(x, y, z) := f(y, z, x)$. Compute $(\partial_y \Phi)(x, y, z)$.

We have

$$\Phi(x, y, z) = \cos(y - z + x^2)\sin(y + \cos(z - x)).$$

Then

$$(\partial_y \Phi)(x, y, z) = -\sin(y - z + x^2)\sin(y + \cos(z - x)) + \cos(y - z + x^2)\cos(y + \cos(z - x)).$$

(e) Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\Phi(x, y, z) := f(y, z, x)$. Compute $(\partial_y \Phi)(y, z, x)$.

We have established above that

$$\Phi(x, y, z) = \cos(y - z + x^2) \sin(y + \cos(z - x)).$$

Then

$$(\partial_y \Phi)(y, z, x) = -\sin(z - x + y^2) \sin(z + \cos(x - y)) + \cos(z - x + y^2) \cos(z + \cos(x - y)).$$

(f) Give the expression of $f(z^2, y, \sin(x + y))$ for any any real numbers x, y, z.

We just have to replace x by z^2, y by y, and z by $\sin(x + y)$ in the definition of f.

$$f(z^2, y, \sin(x + y)) = \cos(z^2 - y + (\sin(x + y)^2) \sin(z^2 + \cos(y - \sin(x + y))).$$

(g) Compute $(\partial_x f)(z^2, y, \sin(x + y))$.

We have

$$(\partial_x f)(x, y, z) = -\sin(x - y + z^2) \sin(x + \cos(y - z)) + \cos(x - y + z^2) \cos(x + \cos(y - z)).$$

Hence,

$$(\partial_x f)(x, y, z) = -\sin(z^2 - y + \sin(x + y)^2) \sin(z^2 + \cos(y - \sin(x + y))) + \cos(z^2 - y + \sin(x + y)^2) \cos(z^2 + \cos(y - \sin(x + y))).$$

(h) Give the expression of $f(\sin(u), \cos(v), \sinh(u + w))$ for any any real numbers u, v, w.

We have

$$f(\sin(u), \cos(v), \sinh(u + w)) = \cos(\sin(u) - \cos(v) + \sinh(u + w)^2) \sin(\sin(u) + \cos(\cos(v) - \sinh(u + w))).$$

Question 2: All the following expressions solve the Laplace equation inside the rectangular domain $D := [0, L]\times[0, H]$ (do not check it).
(a) Show that none of these solutions satisfies the following boundary conditions $\partial_x u(0, y) = \frac{20\pi}{H}\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi L}{H})$, $\partial_x u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = 0$? (justify clearly your answer):

$$u_1(x,y) = 5\cos(\frac{4\pi y}{H})\cosh(\frac{4\pi(x-L)}{H}), \quad u_2(x,y) = 5\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi(x-L)}{H}),$$
$$u_3(x,y) = 5\cos(\frac{4\pi y}{H})\sinh(\frac{4\pi(x-L)}{H}), \quad u_4(x,y) = 5\sin(\frac{4\pi y}{H})\sinh(\frac{4\pi(x-L)}{H}).$$

From class, we know that all the above expressions solve the Laplace equation, hence we just need to verify the boundary conditions. We observe that u_1 and u_3 do not satisfy the Dirichlet boundary conditions $u(x, 0) = 0$, $u(x, H) = 0$; therefore u_1 and u_3 must be discarded.

Both u_2 and u_4 satify that Dirichlet conditions: $u_2(x, 0) = 0$, $u_2(x, H) = 0$, and $u_4(x, 0) = 0$, $u_4(x, H) = 0$. Now we need to check the Neumann conditions.

Note that u_4 is such that $\partial_x u_4(L, y) = 5\frac{4\pi}{H}\sin(\frac{4\pi y}{H})\cosh(0) \neq 0$; a result u_4 must be discarded as well.

Finally u_2 is such that $\partial_x u_2(L, y) = 5\frac{4\pi}{H}\sin(\frac{4\pi y}{H})\sinh(0) = 0$, but $\partial_x u_2(0, y) = 3\frac{4\pi}{H}\sin(\frac{4\pi y}{H})\sinh(-\frac{4\pi L}{H})$, which shows that u_2 is not the solution to our problem either.

(b) Give the expression of the solution that satisfies the boundary conditions $\partial_x u(0, y) = \frac{20\pi}{H}\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi L}{H})$, $\partial_x u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = 0$.

The correct solution is of the form

$$u(x,y) = a\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi(x-L)}{H}).$$

We know from class that $\Delta u(x, y) = 0$ (you calso verify it). We also have

$$u(x,0) = a\sin(\frac{4\pi 0}{H})\cosh(\frac{4\pi(x-L)}{H}) = 0.$$

$$u(x,H) = a\sin(\frac{4\pi H}{H})\cosh(\frac{4\pi(x-L)}{H}) = 0.$$

$$\partial_x u(L,y) = a\frac{4\pi}{H}\sin(\frac{4\pi y}{H})\sinh(\frac{4\pi(L-L}{H}) = 0.$$

But to enforce

$$\partial_x u(0,y) = a\frac{4\pi}{H}\sin(\frac{4\pi y}{H})\sinh(-\frac{4\pi L}{H}) = \frac{20\pi}{H}\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi L}{H}).$$

we need to set

$$a := 5\frac{\cosh(\frac{4\pi L}{H})}{\sinh(-\frac{4\pi L}{H})}.$$

Hence, the solution is

$$u(x,y) = 5\frac{\cosh(\frac{4\pi L}{H})}{\sinh(\frac{-4\pi L}{H})}\sin(\frac{4\pi y}{H})\cosh(\frac{4\pi(x-L)}{H}).$$

Question 3: The solution of the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{1}{2}\pi],\ r \in [0,2]\}$, subject to the boundary conditions $u(r,0) = 0$, $\partial_\theta u(r, \frac{1}{2}\pi) = 0$, $u(2,\theta) = g(\theta)$ is $u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n+1}\sin((2n+1)\theta)$. What is the solution corresponding to $g(\theta) = 5\sin(3\theta) + 2\sin(7\theta)$? (Give all the details.)

One must have

$$g(\theta) = 5\sin(3\theta) + 2\sin(7\theta) = \sum_{n=1}^{\infty} b_n r^{2n+1}\sin((2n+1)\theta).$$

The only non-zero terms in the expansion are $b_1 r^3\sin(3\theta) + b_3 r^7 \sin(7\theta)$, corresponding to $n = 1$ and $n = 3$. Hence, one must have

$$5 = b_1 2^3, \quad \text{and} \quad 2 = b_3 2^7.$$

This means $b_1 = \frac{5}{2^3}$ and $b_3 = \frac{2}{2^7}$ and the other coefficients are zero. In conclusion

$$u(r,\theta) = 5\frac{r^3}{2^3}\sin(3\theta) + 2\frac{r^7}{2^7}\sin(7\theta).$$

Question 4: Let $u \in C^2(\mathbb{R}^2;\mathbb{R})$. Using the cylindrical coordinates, assume that $\Delta u(r,\theta) = 0$ for all $r \le 1$ with boundary condition $u(1,\theta) = \sin(\theta)^3$.

(a) Compute u at the point $\mathbf{0}$ (*Hint:* Use the mean value theorem).

Using the mean value theorem, we infer that for all $\theta \in [0, 2\pi)$

$$u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\phi)\mathsf{d}\phi = \frac{1}{2\pi}\int_0^{2\pi} \sin(\theta)^3 \mathsf{d}\phi = 0.$$

Hence

$$u(0,\theta) = 0.$$

Question 5: Assume that the following equation has a nonzero solutions $-\partial_{\theta\theta}u = \lambda u$, $\theta \in (0, \frac{1}{2})$ with the boundary conditions $-\partial_\theta u(0) + u(0) = 0$ and $u(\frac{\pi}{2}) = 0$.

(a) What is the sign of λ?

The energy method gives

$$\lambda\int_0^{\frac{\pi}{2}} u^2(\theta)\mathsf{d}\theta = -\int_0^{\frac{\pi}{2}} (\partial_{\theta\theta}u)u\mathsf{d}\theta = \int_0^{\frac{\pi}{2}} (\partial_\theta u)^2\mathsf{d}\theta - \partial_\theta u(\frac{\pi}{2})u(\frac{\pi}{2}) + \partial_\theta u(0)u(0)$$

This gives

$$\lambda\int_0^{\frac{\pi}{2}} u^2(\theta)\mathsf{d}\theta = \int_0^{\frac{\pi}{2}} (\partial_\theta u)^2\mathsf{d}\theta + u(0)^2.$$

Since $u \neq 0$, we infer that $\int_0^{\frac{\pi}{2}} u^2(\theta)\mathsf{d}\theta \neq 0$, hence $\lambda = (u(0)^2 + \int_0^{\frac{\pi}{2}}(\partial_\theta u)^2\mathsf{d}\theta)/\int_0^{\frac{\pi}{2}} u^2(\theta)\mathsf{d}\theta \ge 0$, i.e., $\lambda \ge 0$.

(b) Show that λ cannot be zero.

If $\lambda = 0$, then $u(0)^2 = 0$ and $\int_0^{\frac{\pi}{2}}(\partial_\theta u)^2\mathsf{d}\theta = 0$. The second conditions means that u is contant. But $u(0) = 0$ implies that $u(\theta) = 0$ for all $\theta \in (0, \frac{1}{2})$. Which is a contradiction since u is nonzero by assumption. In conclusion $\lambda > 0$.