HW 4

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $f : [-L, L] \to \mathbb{R}$ and assume that $|f|$ is integrable over $[-L, L]$. Recall the definition of the Fourier series of f.

Recall that $\mathsf{FS}(f) : \mathbb{R} \to \mathbb{R}$ is defined by $\mathsf{FS}(f)(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi \frac{x}{L}) + \sum_{n=1}^{\infty} b_n \sin(n\pi \frac{x}{L})$ with

$$a_n := \begin{cases} \frac{1}{2L} \int_{-L}^{L} f(x) \cos(\pi n \frac{x}{L}) \mathrm{d}x & \text{if } n = 0 \\ \frac{1}{L} \int_{-L}^{L} f(x) \cos(\pi n \frac{x}{L}) \mathrm{d}x & \text{if } n \neq 0 \end{cases} \qquad b_n := \frac{1}{L} \int_{-L}^{L} f(x) \sin(2\pi n \frac{x}{L}) \mathrm{d}x.$$

Question 2: (a) Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) = \sin(2x)$. Compute the Fourier series of f.

Recall that $\mathsf{FS}(f)(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi \frac{x}{\pi}) + \sum_{n=1}^{\infty} b_n \sin(n\pi \frac{x}{\pi}) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ with

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(2x) \mathrm{d}x, \quad a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2x) \cos(nx) \mathrm{d}x, \ \forall n \geq 1.$$

We obtain $a_n = 0$ for all $n \geq 0$. And

$$b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2x) \sin(nx) \mathrm{d}x, \ \forall n \geq 1.$$

We obtain $b_2 = 1$ and $b_n = 0$ for all $n \neq 2$, $n \geq 1$. Hence $\mathsf{FS}(f)(x) = \sin(2x)$ for all $x \in \mathbb{R}$.

(b) For which values of $x \in [-\pi, \pi]$ do $\mathrm{FS}(f)(x)$ and $f(x)$ coincide?

f is a smooth function in $(-\pi, \pi)$; hence, $\mathsf{FS}(f)(x)$ and $f(x)$ coincide for all $x \in (-\pi, \pi)$. But also, $f(-\pi) = f(\pi)$; hence, $\mathsf{FS}(f)$ and f also coincide at $-\pi$ and π.

(c) Let $f : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by $f(x) = \sin(2x)$. Compute the Fourier series of f.

Recall that $\mathsf{FS}(f)(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi \frac{x}{2\pi}) + \sum_{n=1}^{\infty} b_n \sin(n\pi \frac{x}{2\pi}) = \sum_{n=0}^{\infty} a_n \cos(n\frac{x}{2}) + \sum_{n=1}^{\infty} b_n \sin(n\frac{x}{2})$ with

$$a_0 := \frac{1}{4\pi} \int_{-2\pi}^{2\pi} \sin(2x) \mathrm{d}x, \quad a_n := \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sin(4\frac{x}{2}) \cos(n\frac{x}{2}) \mathrm{d}x, \ \forall n \geq 1.$$

We obtain $a_n = 0$ for all $n \geq 0$. And

$$b_n := \frac{1}{2\pi} \int_{-1}^{1} \sin(2x) \sin(n\frac{x}{2}) \mathrm{d}x = \frac{1}{2\pi} \int_{-1}^{1} \sin(4\frac{x}{2}) \sin(n\frac{x}{2}) \mathrm{d}x, \ \forall n \geq 1.$$

We obtain $b_4 = 1$ and $b_n = 0$ for all $n \neq 4$, $n \geq 1$. Hence $\mathsf{FS}(f)(x) = \sin(2x)$ for all $x \in \mathbb{R}$.

(d) For which values of $x \in [-2\pi, 2\pi]$ do $\mathrm{FS}(f)(x)$ and $f(x)$ coincide?

f is a smooth function in $(-2\pi, 2\pi)$; hence, $\mathsf{FS}(f)(x)$ and $f(x)$ coincide for all $x \in (-2\pi, 2\pi)$. But also, $f(-2\pi) = f(2\pi)$; hence, $\mathsf{FS}(f)$ and f also coincide at -2π and 2π.

Question 3: Let $f : (-\pi, \pi) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x \leq 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

(a) Compute the Fourier series of f.

As f is odd, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathsf{d}x = 0, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathsf{d}x = 0.$$

Moreover,

$$\begin{aligned}\pi b_n &= \int_{-\pi}^{\pi} f(x)\sin(nx)\mathsf{d}x = -\int_{-\pi}^{0}\sin(nx)\mathsf{d}x + \int_{0}^{\pi}\sin(nx)\mathsf{d}x \\ &= -\frac{1}{n}(-\cos(0) + \cos(n\pi)) + \frac{1}{n\pi}(-\cos(n\pi) + \cos(0)) \\ &= \frac{2}{n}(1 - (-1)^n).\end{aligned}$$

Hence $b_{2n} = 0$ and $b_{2n+1} = \frac{4}{(2n+1)\pi}$. Hence

$$\mathsf{FS}(f)(x) == \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\sin((2n+1)x)$$

(b) For which values of x in $(-\pi, \pi)$ do $\mathrm{FS}(f)(x)$ and $f(x)$ coincide?

As f is a smooth function in $(-\pi, 0)$ and $(0, \pi)$, $\mathsf{FS}(f)(x)$ and $f(x)$ coincide for all $x \in (-\pi, 0)$ and $(0, \pi)$. But $f(0) = -1$ whereas $\mathsf{FS}(f)(0) = 0$.

(c) Let $g : [-\pi, \pi] \to \mathbb{R}$ be defined by

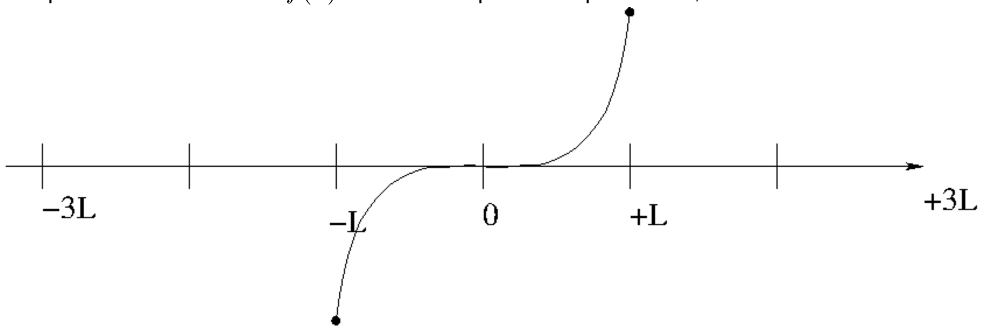
$$g(x) = \begin{cases} a & \text{if } -\pi = x \\ -1 & \text{if } -\pi < x < 0 \\ b & \text{if } 0 = x \\ 1 & \text{if } 0 < x < \pi \\ c & \text{if } \pi = x \end{cases}$$

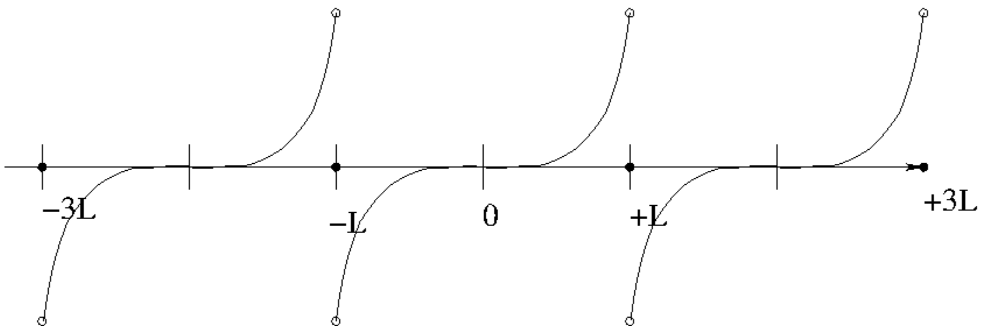
What should be the values of a, b, c so that $\mathrm{FS}(g)(x)$ and $g(x)$ coincide over the entire domain $[-\pi, \pi]$?

By arguing as above we conclude that $\mathsf{FS}(g)(x)$ and $g(x)$ coincide in $(-\pi, 0)$ and $(0, \pi)$. We also have $\mathsf{FS}(g)(-\pi) = 0$, $\mathsf{FS}(g)(0) = 0$, and $\mathsf{FS}(g)(\pi) = 0$. Hence we must set $a = b = c$ so that $\mathsf{FS}(g)$ and g coincide over the entire domain $[-\pi, \pi]$.

Question 4: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = |x|x$. (a) Sketch the graph of the Fourier series of f and the graph of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$ except at the points kL, $k \in \mathbb{Z}$.



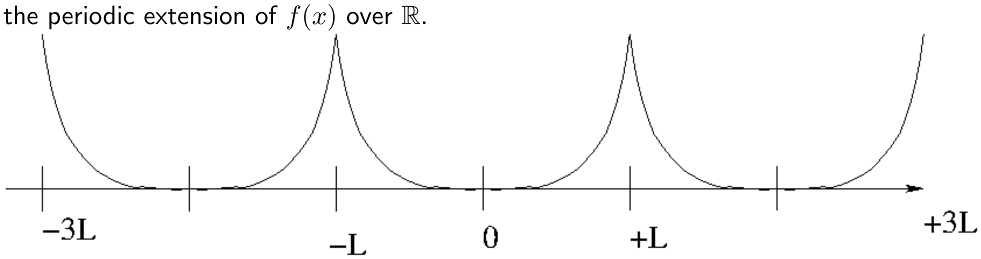


(b) For which values of $x \in \mathbb{R}$ is $FS(f)(x)$ equal to $x|x|$? (Explain)

The periodic extension of $f(x) = x|x|$ over $\mathbb{R}$ is smooth over each interval $[(2k-1)L, (2k+1)L]$, $k \in \mathbb{Z}$, but discontinuous at all the points $(2k+1)L$, $k \in \mathbb{Z}$. This means that the Fourier series is equal to $x|x|$ over the interval $(-L, L)$. Since $f(-L) + f(+L) = 0$, the Fourier series is equal to 0 at all the points $-L$ and L, i.e., $FS(f)(\pm L) \neq \pm L|L|$.

Question 5: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f and the graph of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.



(b) For which values of $x \in \mathbb{R}$ is $FS(f)(x)$ equal to x^4? (Explain)

The periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is piecewise smooth and globally continuous since $f(L) = f(-L)$. This means that the Fourier series is equal to x^4 over the entire interval $[-L, +L]$.

Question 6: Let $h : [0, \pi] \to \mathbb{R}$ be defined by $h(x) := x(\pi - x)$.
(a) Compute the sine series of h.

Using the definition we have $\mathsf{SS}(f)(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi\frac{x}{\pi})$ with

$$b_n := \frac{2}{\pi}\int_0^{\pi} h(x)\sin(n\pi\frac{x}{\pi})\mathsf{d}x = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\mathsf{d}x = \frac{4}{n^3\pi}(1 + (-1)^{n+1}).$$

Hence

$$\mathsf{SS}(h)(x) = \sum_{m=1}^{\infty} \frac{4}{m^3\pi}(1 + (-1)^{m+1})\sin(mx) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)x).$$

(a) Where do h and SS(h) coincide?

h is smooth over $[0, \pi]$ and $h(0) = 0 = h(\pi)$; hence, h and $\mathsf{SS}(h)$ coincide over the entire interval $[0, \pi]$.

(b) Let $g : [0, \pi] \to \mathbb{R}$ be defined by $g(x) := \pi - 2x$. Compute the cosine series of g (*Hint:* $\partial_x h = g$.)

Observe that $h(0) = h(\pi) = 0$; as a result the sine series of h is continuous at 0 and $+\pi$. This in turn implies that it is legitimate to differentiate the sine series of h term by term to obtain the cosine series of $h'(x) = g(x)$. In other words,

$$\mathsf{CS}(g)(x) = \partial_x \mathsf{SS}(h)(x) = \sum_{m=1}^{\infty} \frac{4}{m^3\pi}(1 + (-1)^{m+1})\partial_x \sin(mx) = \sum_{m=1}^{\infty} \frac{4}{m^2\pi}(1 + (-1)^{m+1})\cos(mx).$$

Notice that the equality holds true for all $x \in [0, \pi]$ since g is smooth and the above is a cosine series.

Question 7: Compute the sine series of $h(x) = \sin(x)$ for $x \in [0, +\pi]$.

By definition

$$\mathsf{SS}(h)(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\pi\frac{x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\mathsf{d}x = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously

$$\mathsf{SS}(h)(x) = \sin(x), \quad \forall x \in \mathbb{R}.$$