HW 5

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$. Let $u : [0,1] \to \mathbb{R}$ be the solution to the boundary value problem $-\partial_{xx}u + 2u(x) = f(x)$, $x \in [0,1]$ with $u(0) = 0$ and $u(1) = 0$. Accept as a fact that all the derivatives of u are continuous and bounded on $[0,1]$.

(a) Compute the sine series of f.

We have

$$b_n(f) = 2\int_0^1 t\sin(n\pi t)\mathsf{d}t = \frac{2(-1)^{n+1}}{n\pi}$$

Hence

$$\mathsf{SS}(f)(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x).$$

(b) Let $b_n(u)$ be the coefficients of the sine series of u. Give the definition of $b_n(u)$.

The definition of $b_n(u)$ is

$$b_n(u) := 2\int_0^1 u(t)\sin(n\pi t)\mathsf{d}t.$$

(c) Compute the cosine series of $\partial_x u$ in terms of the coefficients $b_n(u)$.

As u is smooth and $u(0) = u(1) = 0$, we can differentiate the sine series of u:

$$\partial_x u(x) = \partial_x\left(\sum_{n=1}^{\infty} b_n(u)\sin(n\pi x)\right) = \sum_{n=1}^{\infty} n\pi b_n(u)\cos(n\pi x).$$

That is we have $a_n(\partial_x u) = n\pi b_n(u)$.

(d) Compute the sine series of $\partial_{xx}u$ in terms of the coefficients $b_n(u)$.

As $\partial_x u$ is smooth, we can differentiate the cosine series of $\partial_x u$:

$$\partial_x(\partial_x u(x)) = \partial_x\left(\sum_{n=1}^{\infty} n\pi b_n(u)\cos(n\pi x)\right) = \sum_{n=1}^{\infty} -n^2\pi^2 b_n(u)\sin(n\pi x).$$

Hence $b_n(\partial_{xx}u) = -n^2\pi^2 b_n(u)$.

(e) Compute $\partial_{xx}\left(\mathrm{SS}(u)\right)(x) - \mathrm{SS}(\partial_{xx}u)(x)$ for all $x \in (0,1)$.

Since u is smooth, we have $\mathsf{SS}(u)(x) = u(x)$ for all $x \in (0,1)$. Hence $\partial_x\mathsf{SS}(u)(x) = \partial_x u(x)$ for all $x \in (0,1)$. Similarly, since $\partial_x u$ is smooth, we have $\mathsf{SS}(\partial_x u)(x)) = \partial_x u(x)$ for all $x \in (0,1)$. Hence

$$\partial_x\left(\mathsf{SS}(u)\right)(x) = \partial_x u(x) = \mathsf{SS}(\partial_x u)(x) \qquad \forall x \in (0,1).$$

The same argument shows that $\partial_x\left(\mathsf{SS}(\partial_x u)\right) = \partial_x(\partial_x u)$. Similarly, since $\partial_{xx}u$ is smooth, we have $\mathsf{SS}(\partial_{xx}u)(x)) = \partial_{xx}u(x)$ for all $x \in (0,1)$. In conclusion,

$$\partial_x\left(\mathsf{SS}(\partial_x u)\right)(x) = \partial_x(\partial_x u)(x) = \partial_{xx}u(x) = \mathsf{SS}(\partial_{xx}u)(x) \qquad \forall x \in (0,1).$$

Hence,

$$\partial_{xx}\left(\mathsf{SS}(u)\right)(x) - \mathsf{SS}(\partial_{xx}u)(x) \qquad \forall x \in (0,1). = 0$$

(f) Compute the coefficients $b_n(u)$. (*Hint*: insert the sine series of u and f in the equation $-\partial_{xx}u + 2u(x) = f(x)$.)

We have

$$\mathsf{SS}(f) = \mathsf{SS}(-\partial_{xx}u) + 2\mathsf{SS}(u) = -\partial_{xx}\mathsf{SS}(u) + 2\mathsf{SS}(u).$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} b_n(f)\sin(n\pi x) &= \sum_{n=1}^{\infty} n^2\pi^2 b_n(u)\sin(n\pi x) + \sum_{n=1}^{\infty} 2b_n(u)\sin(n\pi x) \\ &= \sum_{n=1}^{\infty}(n^2\pi^2+2)b_n(u)\sin(n\pi x).\end{aligned}$$

This implies that

$$b_n(u) = \frac{b_n(f)}{2+n^2\pi^2}$$

Question 2: Let $f : [0,\pi] \to \mathbb{R}$ be defined by $f(x) = \cos(x)$.

(a) Compute the sine series of f.

We have

$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} \cos(t)\sin(n\pi\frac{t}{\pi})\mathrm{d}t$$

Recall that $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$. Then

$$\begin{aligned}b_n(f) &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin((1+n)t) + \sin((1-n)t)\right)\mathrm{d}t \\ &= -\frac{1}{\pi}\left(\frac{1}{n+1}(\cos((1+n)\pi) - 1) + \frac{1}{n-1}(\cos(1-n)\pi - 1)\right) \\ &= \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\end{aligned}$$

Hence,

$$\mathsf{SS}(f)(x) = \sum_{n=1}^{\infty}\frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx) = \sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin(2nx)$$

(b) Compute the cosine series of f.

Solution 1: We have

$$a_n(f) = \frac{2}{\pi}\int_0^{\pi}\cos(t)\cos(n\pi\frac{t}{\pi})\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(nt)\mathrm{d}t$$

Hence

$$a_n(f) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.}\end{cases}$$

This means

$$\mathsf{CS}(f)(x) = \cos(x).$$

Solution 2: The cosine serie of f coincides with the Fourier series of the even extension of f, say $f_{\mathsf{even}} : [-\pi,\pi] \to \mathbb{R}$. We have $f_{\mathsf{even}}(x) = \cos(x)$ and $\mathsf{FS}(f_{\mathsf{even}})(x) = \cos(x)$. Hence $\mathsf{CS}(f)(x) = \cos(x)$ for all $x \in \mathbb{R}$.

Question 3:

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$. The cosine serie series of $f(x)$ is

$$\mathrm{CS}(f)(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\pi x).$$

(a) Let $g : [0,1] \to \mathbb{R}$ be defined by $g(x) = x$. Compute the sine series of g.

As f is smooth, we can differentiate the cosine series of f and we obtain

$$2\mathsf{SS}(g)(x) = \mathsf{SS}(2g)(x) = \mathsf{SS}(\partial_x f)(x) = \partial_x(\mathsf{CS}(f))(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\pi}{n}\sin(n\pi x).$$

Hence

$$\mathsf{SS}(g)(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\pi}{n}\sin(n\pi x).$$

(Note in passing: $\mathsf{SS}(g)(x) = g(x)$ for all $x \in [0,1)$, but $\mathsf{SS}(g)(1) = 0 \neq 1 = g(1)$.)

(b) Is it posible to obtain the cosine series of the function $h : [0,1] \to \mathbb{R}$ defined by $h(x) = 1$ by differentiating the sine series of g.

No it is not legitimate to differentiate the sine series of g because $g(1) \neq 0$.

Question 4: Let N be a positive integer and let $\mathbb{P}_N$ be the set of trigonometric polynomials of degree at most N; that is, $\mathbb{P}_N = \mathrm{span}\{1, \cos(x), \sin(x), \ldots, \cos(Nx), \sin(Nx)\}$.

(i) Consider the function $f : (-\pi,\pi) \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{1000}\frac{1}{n^5}\sin(7n)\cos(2nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_7$ over $(-\pi,\pi)$.

The best L^2-approximation of f in $\mathbb{P}_7$ over $(-\pi,\pi)$ is the truncated Fourier series $\mathsf{FS}_7(f)$. Clearly

$$\mathsf{FS}_7(f)(x) = \sum_{n=1}^{3}\frac{1}{n^5}\sin(7n)\cos(2nx)$$

(ii) Consider the function $f : (-\pi,\pi) \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{11}\frac{1}{n^3+1}\cos(35n^2)\sin(3nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_{35}$ over $(-\pi,\pi)$.

The best L^2-approximation of f in $\mathbb{P}_{35}$ over $(0,2\pi)$ is the truncated Fourier series $FS_{35}(f)$. Observing that $f \in \mathbb{P}_{11} \subset \mathbb{P}_{35}$ it is clear that $\mathsf{FS}_{35}(f) = f$.

Question 5: Let $L > 0$. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2 \text{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $2 - 3t$ in $\mathbb{P}_1$ with respect to the L^2-norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\text{d}t = 2L^2/\pi$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L}(2-3t)\text{d}t = 2, \\
a_1 &= \frac{1}{L}\int_{-L}^{L}(2-3t)\cos(\pi t/L)\text{d}t = 0 \\
b_1 &= \frac{1}{L}\int_{-L}^{L}(2-3t)\sin(\pi t/L)\text{d}t = \frac{1}{L}\int_{-L}^{L} -3t\sin(\pi t/L)\text{d}t = -6\cos(\pi)\frac{L}{\pi} = -\frac{6L}{\pi}.
\end{aligned}$$

As a result

$$FS_1(t) = 2 - \frac{6L}{\pi}\sin(\pi t/L)$$

(b) Compute the best approximation of $h(t) = 2\cos(\pi t/L) + 7\sin(3\pi t/L)$ in $\mathbb{P}_1$.

The function $h(t) - 2\cos(\pi t/L) = 7\sin(3\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $2\cos(\pi t/L)$. (Recall that the best approximation of h in $\mathbb{P}_1$ is such that $\int_{-L}^{L}(h(t) - FS_1(h))p(t)\text{d}t = 0$ for all $p \in \mathbb{P}_1$.) In conclusion

$$FS_1(h) = 2\cos(\pi t/L).$$
