HW 6

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

We recall the following definitions and results:

$$\mathcal{F}(f)(\omega) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \quad (1)$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}, \qquad \forall\alpha > 0 \quad (2)$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \quad (3)$$

Question 1: (a) State the shift Lemma (use your class notes).

For every $\beta \in \mathbb{R}$, every $\omega \in \mathbb{R}$ and every function f integrable over $\mathbb{R}$, we have

$$\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}.$$

(b) Prove the shift Lemma (use your class notes).

Let f be an integrable function over $\mathbb{R}$ (in $L^1(\mathbb{R})$), and let $\beta \in \mathbb{R}$. Using the definitions above, the following holds:

$$\mathcal{F}(f(x-\beta))(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\beta)e^{i\omega x}\mathrm{d}x.$$

Upon making the change of variable $z = x - \beta$, we obtain

$$\begin{aligned}\mathcal{F}(f(x-\beta))(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)e^{i\omega(z+\beta)}\mathrm{d}z = e^{i\omega\beta}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)e^{i\omega z}\mathrm{d}z \\ &= e^{i\omega\beta}\mathcal{F}(f)(\omega),\end{aligned}$$

which proves the lemma.

Question 2: (a) Let f be an integrable function on $(-\infty, +\infty)$. Prove that for all $a, b \in \mathbb{R}$, and for all $\xi \in \mathbb{R}$, $\mathcal{F}(e^{ibx}f(ax))(\xi) = \frac{1}{a}\mathcal{F}(f)(\frac{\xi+b}{a})$.

The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}\mathcal{F}(e^{ibx}f(ax))(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{ibx}e^{i\xi x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{i(b+\xi)x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{a}f(x')e^{i\frac{(\xi+b)}{a}x'}dx' \\ &= \tfrac{1}{a}\mathcal{F}(f)(\tfrac{\xi+b}{a}).\end{aligned}$$

(b) Let c be a positive real number. Compute the Fourier transform of $f(x) = e^{-cx^2}\sin(bx)$. (*Hint:* use that $\sin(bx) = -i\frac{1}{2}(e^{ibx} - e^{-ibx})$.)

Using the fact that $\sin(bx) = -i\frac{1}{2}(e^{ibx} - e^{-ibx})$, and setting $a = \sqrt{c}$ we infer that

$$\begin{aligned} f(x) &= \tfrac{1}{2i}e^{-(ax)^2}(e^{ibx} - e^{-ibx}) \\ &= \tfrac{1}{2i}e^{-(ax)^2}e^{ibx} - \tfrac{1}{2i}e^{-(ax)^2}e^{i(-b)x} \end{aligned}$$

and using step (a) and formula (2) we deduce

$$\mathcal{F}(f)(\xi) = \tfrac{1}{2i}\tfrac{1}{a}\tfrac{1}{\sqrt{4\pi}}(e^{-\frac{1}{4}(\frac{\xi+b}{a})^2} - e^{-\frac{1}{4}(\frac{\xi-b}{a})^2}).$$

In conclusion

$$\mathcal{F}(f)(\xi) = \tfrac{1}{2i}\tfrac{1}{\sqrt{4\pi c}}(e^{-\frac{1}{4c}(\xi+b)^2} - e^{-\frac{1}{4c}(\xi-b)^2}).$$

(c) Compute the Fourier transform of $e^{-a|x|}\cos(bx)$ (Hint: use $\cos(bx) = \frac{1}{2}(e^{ibx} + e^{-ibx})$), where a and b are real numbers and $a > 0$ (computation should be short and simple).

$$\begin{aligned} e^{-a|x|}\cos(bx) &= e^{-a|x|}\frac{1}{2}(e^{ibx} + e^{-ibx}) \\ &= \frac{1}{2}\left(e^{-a|x|}e^{ibx} + e^{-a|x|}e^{-ibx}\right) \end{aligned}$$

Hence,

$$\mathcal{F}(e^{-a|x|}\cos(bx))(\omega) = \frac{1}{2}\frac{1}{a}\left(\mathcal{F}(e^{-|x|})(\frac{b+\omega}{a}) + \mathcal{F}(e^{-a|x|})(\frac{\omega-b}{a})\right)$$

Now we use formula (3), i.e., $\mathcal{F}(e^{-|x|}) = \frac{1}{\pi}\frac{1}{\omega^2+1}$, and obatin

$$\begin{aligned} \mathcal{F}(e^{-a|x|}\cos(bx))(\omega) &= \frac{1}{2\pi}\frac{1}{a}\left(\frac{1}{\frac{(b+\omega)^2}{a^2}+1} + \frac{1}{\frac{(b-\omega)^2}{a^2}+1}\right), \\ &= \frac{1}{2\pi}\left(\frac{a}{(b+\omega)^2+a^2} + \frac{a}{(b-\omega)^2+a^2}\right). \end{aligned}$$

Question 3: (a) Let M be a positive real number and let $\omega \in \mathbb{R}$. Compute $\lim_{M\to\infty} e^{(-1+i\omega)M}$. (Justify all the steps.)

We have

$$|e^{(-1+i\omega)M}| = |e^{-M}||e^{i\omega)M}| = e^{-M}.$$

Hence $\lim_{M\to\infty} |e^{(-1+i\omega)M}| = 0$, which in turn implies that $\lim_{M\to\infty} e^{(-1+i\omega)M} = 0$ since the function $\mathbb{C} \ni z \longmapsto |z| \in \mathbb{R}$ is continuous.

(b) Compute the Fourier transform of $f(x) = H(x)e^{-x}$ where H is the Heaviside function: $H(x) = 0$ if $x < 0$ and $H(x) = 1$ otherwise. (*Hint:* Compute $\lim_{M\to\infty} \frac{1}{2\pi}\int_{-M}^{M} H(x)e^{-x}e^{i\omega x}\mathrm{d}x$.)

Using the definitions we have

$$\mathcal{F}(f)(\omega) = \lim_{M\to\infty} \frac{1}{2\pi}\int_{-M}^{M} H(x)e^{-x}e^{i\omega x}\mathrm{d}x = \lim_{M\to\infty} \frac{1}{2\pi}\int_{0}^{M} e^{(-1+i\omega)x}\mathrm{d}x$$
$$= \frac{1}{2\pi}\frac{1}{1-i\omega}.$$

Note that here we used that $\lim_{M\to\infty} e^{(-1+i\omega)M} = 0$

Question 4: Use the Fourier transform technique to solve $\partial_t u(x,t) + t\partial_x u(x,t) + 2u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$.

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + t(-i\omega)\mathcal{F}(u)(\omega,t) + 2\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega t - 2.$$

Then applying the fundamental theorem of calculus we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = i\omega\frac{1}{2}t^2 - 2t.$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\frac{1}{2}t^2}e^{-2t}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x - \frac{1}{2}t^2)(\omega)e^{-2t}.$$

This finally gives

$$u(x,t) = u_0(x - \frac{1}{2}t^2)e^{-2t}.$$

Question 5: Solve the following integral equation : $\int_{-\infty}^{+\infty}(f(y)-3\sqrt{2}e^{-\frac{y^2}{2\pi}})f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}$, for all $x\in\mathbb{R}$. (Hint: $z^2-3za+2a^2=(z-a)(z-2a)$)

This equation can be re-written using the convolution operator:

$$f*f-3\sqrt{2}e^{-\frac{x^2}{2\pi}}*f=-4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use $\mathcal{F}(e^{-\alpha x^2})=\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$ with $\alpha=\frac{1}{2\pi}$ to obtain

$$\begin{aligned}2\pi\mathcal{F}(f)^2-2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}}\\ \mathcal{F}(f)^2-3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}}+2e^{-\omega^2\pi} &= 0\\ (\mathcal{F}(f)-e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f)-2e^{-\omega^2\frac{\pi}{2}}) &= 0.\end{aligned}$$

This implies

$$\text{either}\quad \mathcal{F}(f)=e^{-\omega^2\frac{\pi}{2}},\quad\text{or}\quad \mathcal{F}(f)=2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either}\quad f(x)=\sqrt{2}e^{-\frac{x^2}{2\pi}},\quad\text{or}\quad f(x)=2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 6: Let $\lambda>0$ and $S_\lambda(x)=\begin{cases}1 & \text{if } |x|\le\lambda\\ 0 & \text{otherwise}\end{cases}$. Prove that $\mathcal{F}(S_\lambda)(\omega)=\frac{1}{\pi}\frac{\sin(\omega\lambda)}{\omega}$.

By definition

$$\begin{aligned}\mathcal{F}(S_\lambda)(\omega)=\frac{1}{2\pi}\int_{-\lambda}^{\lambda}e^{i\xi\omega}\mathrm{d}\xi &= \frac{1}{2\pi}\frac{1}{i\omega}(e^{i\omega\lambda}-e^{-i\omega\lambda})\\ &= \frac{1}{2\pi}\frac{2\sin(\omega\lambda)}{\omega}.\end{aligned}$$

Hence

$$\mathcal{F}(S_\lambda)(\omega)=\frac{1}{\pi}\frac{\sin(\omega\lambda)}{\omega}.$$

Question 7: Consider the telegraph equation $\partial_{tt}u + 2\alpha\partial_t u + \alpha^2 u - c^2\partial_{xx}u = 0$ with $\alpha \geq 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x)$, $x \in \mathbb{R}$, $t > 0$ and boundary condition at infinity $u(\pm\infty, t) = 0$, $\partial_x u(\pm\infty, t) = 0$. Solve the equation by the Fourier transform technique. (Hint: Use that $\mathcal{F}(S_\lambda)(\omega) = \frac{1}{\pi}\frac{\sin(\omega\lambda)}{\omega}$ +the solution to the ODE $\phi''(t) + 2\alpha\phi'(t) + (\alpha^2 + \lambda^2)\phi(t) = 0$ is $\phi(t) = \mathrm{e}^{-\alpha t}(a\cos(\lambda t) + b\sin(\lambda t))$)

Applying the Fourier transform with respect to x to the equation, we infer that

$$\begin{aligned} 0 &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + \alpha^2\mathcal{F}(u)(\omega,t) - c^2(-i\omega)^2\mathcal{F}(u)(\omega,t) \\ &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + (\alpha^2 + c^2\omega^2)\mathcal{F}(u)(\omega,t) \end{aligned}$$

Using the hint, we deduce that

$$\mathcal{F}(u)(\omega,t) = \mathrm{e}^{-\alpha t}(a(\omega)\cos(\omega ct) + b(\omega)\sin(\omega ct)).$$

The initial condition $u(x,0) = 0$ gives

$$0 = \mathcal{F}(u)(\omega,0) = a(\omega).$$

Hence $a(\omega) = 0$. The other initial condition $\partial_t u(x,0) = g(x)$ gives

$$\mathcal{F}(g)(\omega) = \mathcal{F}(\partial_t u(\cdot,0))(\omega) = \partial_t\mathcal{F}(u)(\omega,0) = \omega c b(\omega).$$

As a result, $b(\omega) = \mathcal{F}(g)(\omega)/(\omega c)$. In conclusion,

$$\mathcal{F}(u)(\omega,t) = \mathrm{e}^{-\alpha t}\mathcal{F}(g)\frac{\sin(\omega ct)}{\omega c}.$$

Then using the identity $\mathcal{F}(S_\lambda)(\omega) = \frac{1}{\pi}\frac{\sin(\omega\lambda)}{\omega}$ proved in Question 2, we have

$$\mathcal{F}(u)(\omega,t) = \frac{\pi}{c}\mathrm{e}^{-\alpha t}\mathcal{F}(g)\mathcal{F}(S_{ct}).$$

The convolution theorem implies that

$$u(x,t) = \mathrm{e}^{-\alpha t}\frac{1}{2c}g * S_{ct} = \mathrm{e}^{-\alpha t}\frac{1}{2c}\int_{-\infty}^{\infty} g(y)S_{ct}(x-y)\mathrm{d}y.$$

Finally the definition of S_{ct} implies that $S_{ct}(x-y)$ is equal to 1 if $-ct < x - y < ct$ and is equal zero otherwise, which finally means that

$$u(x,t) = \mathrm{e}^{-\alpha t}\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathrm{d}y.$$
