HW 7

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: We want to solve the following PDE: $\partial_t w + 3\partial_x w = 0$, for $x > -t$, $t > 0$, with $w(x,t) = w_\Gamma(x,t)$, for all $(x,t) \in \Gamma$ where $\Gamma = \{(x,t) \in \mathbb{R}^2 \text{ s.t. } x = -t,\ x < 0\} \cup \{(x,t) \in \mathbb{R}^2 \text{ s.t. } t = 0,\ x \geq 0\}$ and w_Γ is a given function.

(a) Draw a picture of the domain Ω where the PDE must be solved and properly identify the boundary Γ.

(b) Define a one-to-one parametric representation of the boundary Γ.

For negative s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$; clearly we have $x_\Gamma(s) = -t_\Gamma(s)$ for all $s < 0$. For positive s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. The map $\mathbb{R} \ni s \mapsto (x_\Gamma(s), t_\Gamma(s)) \in \Gamma$ is one-t-one.

(c) Give a parametric representation of the characteristics associated with the PDE.

(i) We use t and s to parameterize the characteristics. The characteristics are defined by

$$\partial_t X(t,s) = 3, \quad \text{with} \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

This yields the following parametric representation of the characteristics

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),$$

where $t \geq 0$ and $s \in (-\infty, +\infty)$.

(d) Give an implicit parametric representation of the solution to the PDE.

(i) Now we set $\phi(t,s) = w(X(t,s),t)$ and we insert this ansatz in the equation. This gives $\partial_t \phi(t,s) = 0$, i.e., $\phi(t,s)$ does not depend on t. In other words

$$w(X(t,s),t) = \phi(t,s) = \phi(0,s) = w(X(0,s), t(0,s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s))$$

A parametric representation of the solution is given by

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),$$
$$w(X(t,s),t) = w_\Gamma(x_\Gamma(s), t_\Gamma(s)).$$

(e) Give an explicit representation of the solution.

(i) We have to find the inverse map $(x,t) \mapsto (s,t)$, where $x - 3t = x_\Gamma(s) - 3t_\Gamma(s)$. Then, there are two cases depending on the sign of s.
case 1: If $s < 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$. That means $x - 3t = 4s$, which in turns implies $s = \frac{1}{4}(x - 3t)$. Then

$$w(x,t) = w_\Gamma(\tfrac{1}{4}(x-3t), -\tfrac{1}{4}(x-3t)), \quad \text{if } x - 3t < 0.$$

case 2: If $s \geq 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. That means $x - 3t = s$. Then

$$w(x,t) = w_\Gamma(x - 3t, 0), \quad \text{if } x - 3t \geq 0.$$

Note that the explicit representation of the solution does not depend on the choice of the parameterization.

Question 2: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t > 0, x \geq \frac{1}{t}\}$. Solve the following PDE in explicit form with the method of characteristics: (Solution: $u(x,t) = (2+\cos(s))e^{\frac{1}{s}-t}$ with $s = \frac{1}{2}[(x-2t) + \sqrt{(x-2t)^2+8}])$

$$\partial_t u(x,t) + 2\partial_x u(x,t) = -u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 2+\cos(x), \text{ if } x = 1/t.$$

(i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = \frac{1}{s}$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 2 + \cos(s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 2, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t\Phi(t,s)$. This gives

$$\begin{aligned}\partial_t\Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s)\\ &= \partial_t u(X(t,s),t) + 2\partial_x u(X(t,s),t) = u(X(t,s),t) = -\Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{-t+t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s), \qquad u(X(t,s)) = u_\Gamma(s)e^{-t+t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 2(t - \frac{1}{s}) + s = 2t - \frac{2}{s} + s$$

which gives the equation

$$s^2 - s(X-2t) - 2 = 0$$

The solutions are $s_\pm = \frac{1}{2}\left((X-2t) \pm \sqrt{(X-2t)^2+8}\right)$. The only legitimate solution is the positive one:

$$s = \frac{1}{2}\left((X-2t) + \sqrt{(X-2t)^2+8}\right)$$

The solution is

$$\begin{aligned}&u(x,t) = (2+\cos(s))e^{\frac{1}{s}-t}\\ &\text{with } s = \frac{1}{2}((x-2t) + \sqrt{(x-2t)^2+8})\end{aligned}$$

Question 3: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t > 0,\ x+3t > 0\}$. Use the method of characteristics to solve the equation $\partial_t u + 4\partial_x u + 2u = 0$ for $(x,t) \in \Omega$ and $u(x,0) = x+4$, for $x > 0$, $u(-3t,t) = t+4$, for $t > 0$.

(i) We first parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 3s & s < 0 \\ s & s > 0, \end{cases} \qquad t_\Gamma(s) = \begin{cases} -s & s < 0 \\ 0 & s > 0. \end{cases}$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 4, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s))$.
(iii) Set $\Phi(t,s) = u(X(t,s),t)$. Then

$$\begin{aligned} \partial_t \Phi(t,s) &= \partial_x u(X(t,s),t)\partial_t X(t,s) + \partial_t u(X(t,s),t)\partial_t t \\ &= 4\partial_x u(X(t,s),t) + \partial_t u(X(t,s),t) = -2u(X(t,s),t) = -2\Phi(s,t) \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{-2(t-t_\Gamma(s))}$, i.e., $u(X(t,s)) = u(X(t_\Gamma(s),s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))} = u(x_\Gamma(s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))}$.
(iv) The implicit representation of the solution is

$$X(t,s) = x_\Gamma(s) + 4(t - t_\Gamma(s)), \quad u(X(t,s)) = u(x_\Gamma(s),t_\Gamma(s))e^{-2(t-t_\Gamma(s))}.$$

(v) The explicit representation is obtained by replacing the parameterization (t,s) by (X,t). Using the definitions of $x_\Gamma(s)$ and $t_\Gamma(s)$, we have two cases:
Case 1: $s < 0$. The definition of $X(t,s)$ gives $X(s,t) = 3s + 4(t+s)$, i.e., $s = (X-4t)/7$. Then

$$\begin{aligned} u(X,t) = (t_\Gamma(s)+4)e^{-2(t-t_\Gamma(s))} &= (-s+4)e^{-2(t+s)} = (4-(X-4t)/7)e^{-2(t+(X-4t)/7)} \\ &= (4 + \frac{4t-X}{7})e^{-\frac{2}{7}(3t+X)} \end{aligned}$$

i.e., $\boxed{u(X,t) = (4+\frac{4t-X}{7})e^{-\frac{2}{7}(3t+X)} \text{ if } X < 4t}$.
Case 2: $s > 0$. The definition of $X(t,s)$ gives $X(s,t) = s+4t$, i.e., $s = X - 4t$. Then

$$u(X,t) = (x_\Gamma(s)+4)e^{-2(t-t_\Gamma(s))} = (s+4)e^{-2t} = (4+X-4t)e^{-2t}.$$

i.e., $\boxed{u(X,t) = (4+X-4t)e^{-2t} \text{ if } X > 4t}$.

Question 4: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x \geq 0,\ t \geq 0\}$. Solve the following PDE in explicit form

$$\partial_t u(x,t) + t\partial_x u(x,t) = 2u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(0,t) = t,\ u(x,0) = x.$$

(i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s > 0$ and $x_\Gamma(s) = 0$ and $t_\Gamma(s) = -s$ if $s \leq 0$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = t, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned} \partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + t\partial_x u(X(t,s),t) = 2u(X(t,s),t) = 2\Phi(t,s). \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{2(t-t_\Gamma(s))}$.

(iv) The implicit representation of the solution is

$$X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s), \quad u(X(t,s)) = u_\Gamma(s)e^{2(t-t_\Gamma(s))}, \quad u_\Gamma(s) = \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(v) We distinguish two cases to get the explicit form of the solution:
Case 1: Assume $s > 0$, then $t_\Gamma(s) = 0$ and $x_\Gamma(s) = s$. This implies $X(t,s) = \frac{1}{2}t^2 + s$, meaning $s = X - \frac{1}{2}t^2$. The solution is

$$u(x,t) = (x - \frac{1}{2}t^2)e^{2t}, \quad \text{if} \quad x > \frac{1}{2}t^2.$$

Case 2: Assume $s \leq 0$, then $t_\Gamma(s) = -s$ and $x_\Gamma(s) = 0$. This implies $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}s^2$, meaning $s = -\sqrt{t^2 - 2X}$. The solution is

$$u(x,t) = \sqrt{t^2 - 2x}\, e^{2(t - \sqrt{t^2-2x})}, \quad \text{if} \quad x \leq \frac{1}{2}t^2.$$
