HW 8

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the following conservation equation

$$\partial_t \rho + \partial_x (q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

(i) What is the wave speed for this problem when $q(\rho) = \rho(2 - \sin(\rho))$ (and $\rho(x,t)$ is the conserved quantity).?

The wave speed is the quantity $q'(\rho) = 2 - \sin(\rho) - \rho\cos(\rho)$.

(ii) What is the wave speed for this problem when $q(\rho) = 2\rho + \cos(\rho^2)$ (and $\rho(x,t)$ is the conserved quantity).

The wave speed is the quantity $q'(\rho) = 2 - 2\rho\sin(\rho^2)$.

Question 2: Consider the equation $\partial_t u + \partial_x (u^4) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$.

(a) Give the explicit representation of the solution when the initial data is $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x$.

Case $s < 0$: $u_0(s) = 0$, $f'(u_0(s) = 0$. Then

$$X(s,t) = s$$

Hence

$$u(x,t) = 0 \quad \text{if } x < 0.$$

Case 2 $0 < s$: The $u_0(s) = s^{\frac{1}{3}}$ and $f'(u_0(s) = 4s$. Then

$$X(s,t) = s + 4st.$$

This means $s = \frac{X}{1+4t}$. In conclusion

$$u(x,t) = \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} \quad \text{if } 0 < x.$$

(b) Draw the characteristics

TODO

(c) Give the explicit representation of the solution when the initial data is $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x < 1$, and $u_0(x) = 0$ if $1 < x$.

There is a shock moving to the right. From (a) we know that the solution is $u(x,t) = (\frac{x}{1+4t})^{\frac{1}{3}}$ on the left of the shock and $u(x,t) = 0$ on the right. (this is visible when one draws the characteristics).

Solution 1: The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{u_+^4 - u_-^4}{u_+ - u_-}, \qquad \text{and } x_s(0) = 1,$$

where $u_+(t) = 0$ and $u_-(t) = \left(\frac{x_s(t)}{1+4t}\right)^{\frac{1}{3}}$. This gives

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = u_-(t)^3 = \frac{x_s(t)}{1+4t},$$

which we re-write as follows:

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{1+4t} = \frac{1}{4}\frac{\mathrm{d}\log(1+4t)}{\mathrm{d}t}.$$

Applying the fundamental of calculus between 0 and t gives

$$\log(x_s(t)) - \log(1) = \frac{1}{4}(\log(1+4t) - \log(1)).$$

This give

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Solution 2: Another (equivalent) way of solving this problem, that does not require to solve the Rankin-Hugoniot relation, consists of writing that the value of u_- is such that the total mass is conserved:

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = \int_0^{x_s(t)} u(x,t)\mathrm{d}x = \int_0^{x_s(0)} u_0(x)\mathrm{d}x = \int_0^1 x^{\frac{1}{3}}\mathrm{d}x = \frac{3}{4}$$

i.e., using the fact that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$, we have

$$\frac{3}{4} = (1+4t)^{-\frac{1}{3}} \int_0^{x_s(t)} x^{\frac{1}{3}}\mathrm{d}x = (1+4t)^{-\frac{1}{3}}\frac{3}{4}x_s(t)^{\frac{4}{3}}.$$

This again gives

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Conclusion: The solution is finally expressed as follows:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} & \text{if } 0 < x < (1+4t)^{\frac{1}{4}} \\ 0 & \text{if } (1+4t)^{\frac{1}{4}} < x \end{cases}$$

Question 3: Consider the conservation equation with flux $q(\rho) = \rho^4$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$. (i) Draw the characteristics

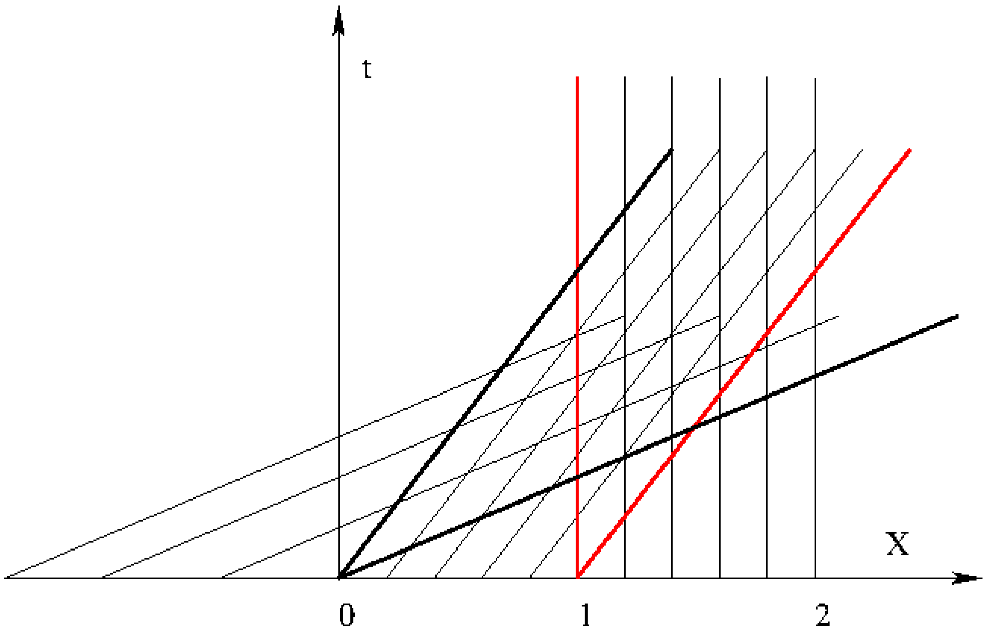
There are three families of characteristics.

Case 1: $s < 0$, $X(s,t) = 32t + s$. In the x-t plane, these are lines with slope $\frac{1}{32}$.

Case 2: $0 < s < 1$, $X(s,t) = 4t + s$. In the x-t plane, these are lines with slope $\frac{1}{4}$.

Case 3: $1 < s$, $X(s,t) = s$. In the x-t plane, these are verical lines.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).



(ii) Describe qualitatively the nature of the solution.

We have two shocks moving to the right. One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).

(iii) When does the left shock catch up with the right one?

The speeds of the shocks are

$$\frac{dx_1(t)}{dt} = \frac{2^4 - 1}{2 - 1} = 15, \quad \text{and} \quad \frac{dx_2(t)}{dt} = \frac{1 - 0}{1 - 0} = 1.$$

The location of the left shock at time t is $x_1(t) = 15t$ and that of the right shock is $x_2(t) = t + 1$. The two shocks are at the same location when $15t = t + 1$, i.e., $t = \frac{1}{14}$.

(iv) What is the speed of the shock when the two shocks have merged and what is the position of the shock as function of time?

When the shocks have merged the left state is $\rho = 2$ and the right state is $\rho = 0$; as a result the speed of the shock is

$$\frac{dx_3(t)}{dt} = \frac{2^4 - 0}{2 - 0} = 8,$$

and the shock trajectory is $x_3(t) = 8(t - \frac{1}{14}) + \frac{15}{14}$.

Question 4: Consider the conservation equation $\partial_t \rho + \partial_x(\sin(\frac{\pi}{2}\rho)) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 1$ if $x < 0$ and $\rho_0(x) = 0$ if $x > 0$. Draw the characteristics and give the explicit representation of the solution.

The implicit representation of the solution to the equation $\partial_t \rho + \partial_x q(\rho) = 0$, $\rho(x, 0) = \rho_0(x)$, is

$$X(s,t) = q'(\rho_0(s))t + s; \quad \rho(X(s,t), t) = \rho_0(s). \tag{1}$$

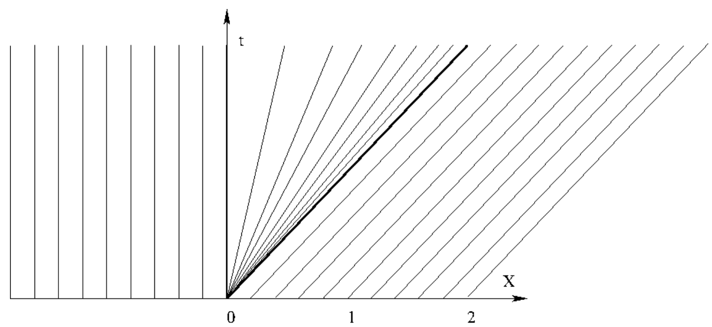
The explicit representation is obtained by expressing s in terms of X and t.
Case 1: $s < 0$, we have $\rho_0(s) = 1$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(\frac{\pi}{2}) = 0$, which implies $X = s$. Then

$$\rho(x,t) = 1 \text{ if } x < 0.$$

Case 2: $0 < s$, we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$, $X = \frac{\pi}{2}t + s$, which means $s = X - \frac{\pi}{2}t$. Then

$$\rho(x,t) = 0 \text{ if } \frac{\pi}{2}t < x.$$

Case 3: $s = 0$. Note that there is no characteristics in the region $0 < x < \frac{\pi}{2}t$; this means that there is an expansion wave in this region.



The solution is given by setting $s = 0$ in the expression defining te characteristics: $X = q'(\rho_0)t + 0 = \frac{\pi}{2}\cos(\frac{\pi}{2}\rho_0)t$ with the constraint that $0 < \rho_0(s) < 1$. This means that

$$\rho_0 = \frac{2}{\pi}\arccos(\frac{2X}{\pi t}), \quad \text{for} \quad 0 \le X \le \frac{\pi}{2}t.$$

Finally

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{2}{\pi}\arccos(\frac{2x}{\pi t}), & 0 \le x \le \frac{\pi}{2}t, \\ 0 & \text{if } \frac{\pi}{2}t < x. \end{cases} \tag{2}$$

Question 5: Consider the following conservation equation

$$\partial_t \rho + \partial_x (q(\rho)) = 0, \quad x \in (-\infty, +\infty), \ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{6} & \text{if } x < 0, \\ \frac{1}{3} & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

The characteristics are defined by

$$\frac{dX(s,t)}{dt} = q'(\rho) = 2(1 - 3\rho(X(s,t),t)), \quad X(s,t) = s.$$

Set $\phi(s,t) = \rho(X(s,t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(s,t),t) = \rho(s,0) = \rho_0(s)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1-3\rho_0(s))t + s$. We then have two cases depending whether s is positive or negative.

1. $s < 0$, then $\rho_0(s) = \frac{1}{6}$ and $X(s,t) = t + s$. This means

$$\rho(x,t) = \frac{1}{6} \quad \text{if} \quad x < t.$$

2. $s > 0$, then $\rho_0(s) = \frac{1}{3}$ and $X(s,t) = s$. This means

$$\rho(x,t) = \frac{1}{3} \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{1}{6}\frac{3}{2} - \frac{1}{3}}{\frac{1}{6} - \frac{1}{3}} = \frac{1}{12}6 = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{6}, \quad x < \frac{t}{2},$$
$$\rho = \frac{1}{3}, \quad x > \frac{t}{2}.$$
