Last name: ***name:***

HW 9

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: (a) Consider the following flux $f(v) = \sin(v)$ and the entropy $\eta(v) = \frac{1}{2}v^2$. Find the associated entropy flux $G(v)$ (*Hint:* Recall that $G'(v) := \eta'(v)f'(v)$.)

By definition

$$G(v) = \int_0^v \eta'(z)f'(z)\mathsf{d}z.$$

Integrating by parts gives

$$G(v) = \int_0^v z\cos(z)\mathsf{d}z = -\int_0^v \sin(z)\mathsf{d}z + v\sin(v) = \cos(v) - 1 + v\sin(v).$$

Hence, up to a nonessential constant, the entropy flux associated with the entropy $\eta(v) = \frac{1}{2}v^2$ is

$$G(v) = \cos(v) + v\sin(v).$$

(b) Let $\eta(v) = \frac{1}{3}|v|v^2$. Compute $\eta'(v) - |v|v$ for all $v \in \mathbb{R}$. (*Hint:* Consider the two cases $v \leq 0$ and $0 \leq v$.)

If $v \geq 0$, we have

$$\eta(v) = \frac{1}{3}v^3, \quad \text{and} \quad \eta'(v) = v^2 = |v|v.$$

If $v \leq 0$, we have

$$\eta(v) = -\frac{1}{3}v^3, \quad \text{and} \quad \eta'(v) = -v^2 = |v|v.$$

Hence

$$\eta'(v) = |v|v.$$

(c) Show that the function $\eta(v) = \frac{1}{3}|v|v^2$ is convex. (*Hint:* Consider the two cases $v \leq 0$ and $0 \leq v$.)

If $v \geq 0$, we have

$$\eta(v) = \frac{1}{3}v^3, \quad \text{and} \quad \eta''(v) = 2v = 2|v| \geq 0.$$

If $v \leq 0$, we have

$$\eta(v) = -\frac{1}{3}v^3, \quad \text{and} \quad \eta''(v) = -2v = 2|v| \geq 0.$$

Hence $\eta''(v) \geq 0$ for all $v \in \mathbb{R}$. This proves that η is convex.

(d) Consider the following flux $f(v) = \cos(v)$ and the entropy $\eta(v) = \frac{1}{3}|v|v^2$. Find the associated entropy flux $G(v)$ (*Hint:* Recall that $G'(v) := \eta'(v)f'(v)$.)

From the previous question we know that $\eta'(v) = |v|v$. If $v \geq 0$, we have

$$G(v) = \int_0^v \eta'(z)f'(z)\mathsf{d}z = \int_0^v |z|zf'(z)\mathsf{d}z = \int_0^v z^2 f'(z)\mathsf{d}z = -\int_0^v z^2\sin(z)\mathsf{d}z.$$

Integrating by parts gives

$$\begin{aligned}G(v) &= -\int_0^v 2z\cos(z)\mathsf{d}z + z^2\cos(z)|_v^0 = \int_0^v 2\sin(z)\mathsf{d}z - 2z\sin(z)|_0^v + z^2\cos(z)|_0^v \\ &= -2\cos(0) + 2\cos(v) + 2v\sin(v) - v^2\cos(v) \\ &= \cos(0) + 2\cos(v) + 2|v|\sin(v) - |v|v\cos(v).\end{aligned}$$

If $v \leq 0$, we have

$$G(v) = \int_0^v \eta'(z)f'(z)\mathsf{d}z = \int_0^v |z|zf'(z)\mathsf{d}z = \int_0^v -z^2 f'(z)\mathsf{d}z = \int_0^v z^2\sin(z)\mathsf{d}z.$$

Hence

$$\begin{aligned}G(v) &= 2\cos(0) - 2\cos(v) - 2v\sin(v) + v^2\cos(v) \\ &= 2\cos(0) - 2\cos(v) + 2|v|\sin(v) - |v|v\cos(v)\end{aligned}$$

In conclusion the entropy flux is

$$G(v) = 2\cos(0) - 2\cos(v) + 2|v|\sin(v) - |v|v\cos(v).$$

Question 2: Let $k \in \mathbb{R}$ and $\eta(v) := |v-k|$. (a) Show that for all $v, w \in \mathbb{R}$, all $k \in \mathbb{R}$, all $\theta \in [0,1]$, $\eta(\theta v + (1-\theta)w) \leq \theta\eta(v) + (1-\theta)\eta(w)$, i.e., η is convex. (*Hint:* recall that $|a+b| \leq |a| + |b|$ and $k = \theta k + (1-\theta)k$.)

Using the hint, we have

$$\begin{aligned}\eta(\theta v + (1-\theta)w) &= |\theta v + (1-\theta)w - k| = |\theta v + (1-\theta)w - \theta k - (1-\theta)k| = |\theta(v-k) + (1-\theta)(w-k)| \\ &\leq |\theta(v-k)| + |(1-\theta)(w-k)| = \theta|(v-k)| + (1-\theta)|(w-k)| \\ &= \theta\eta(v) + (1-\theta)\eta(w).\end{aligned}$$

Hence

$$\eta(\theta v + (1-\theta)w) \leq \theta\eta(v) + (1-\theta)\eta(w).$$

(b) Let $f \in C^1(\mathbb{R};\mathbb{R})$. Show that the entropy flux associated with the flux $f(v)$ and the entropy $\eta(v) := |v-k|$ is $G(v) := \text{sgn}(v-k)(f(v) - f(k))$ for all $v \neq k$, where $\text{sgn}(v-k)$ is the sign of $v-k$.

If $v > k$, then

$$G(v) = f(v) - f(k), \quad \text{and} \quad \eta(v) = v - k.$$

Hence

$$G'(v) = f'(v), \quad \text{and} \quad \eta'(v) = 1.$$

Which shows that

$$G'(v) = \eta'(v)f'(v).$$

If $v < k$, then

$$G(v) = -f(v) + f(k), \quad \text{and} \quad \eta(v) = k - v.$$

Hence

$$G'(v) = -f'(v), \quad \text{and} \quad \eta'(v) = -1.$$

Which shows again that

$$G'(v) = \eta'(v)f'(v).$$

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Question 3: Consider the equation $\partial_t u + \partial_x(-u^6) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$, with initial data $u_0(x) = 1$ if $x < 0$, $u_0(x) = 0$ otherwise.

(a) Show that $u(x,t) = 1 - H(x+t)$ is a solution in the distribution sense.

Using the chain rule and denoting by δ_0 the Dirac measure at $\{x = 0\}$, we have

$$\partial_t u = -\delta_0(x+t).$$

Moreover, using that $(1 - H(x+t))^6 = 1 - H(x+t)$, we also have

$$\partial_x(-u^6) = +\delta_0(x+t).$$

This implies that $\partial_t u + \partial_x(-u^6) = 0$, i.e., $u(x,t) = H(x+t)$ is a solution in the distribution sense.

(b) What is the entropy flux associated with the entropy $\eta(v) = v^2$?

By definition, the entropy flux is

$$F(v) = \int_0^v f'(z)\eta'(z)\mathsf{d}z = \int_0^v -6z^5 2z\mathsf{d}z = -\frac{12}{7}v^7,$$

i.e., $F(v) = -\frac{12}{7}v^7$.

(c) Is it the entropy solution? Clearly justify your answer either by invoking the characteristics or invoking an entropy inequality (say using $\eta(v) = v^2$).

Solution 1: By looking at the characteristics $\big(X(t,s) = s - 6t$ for $s < 0$ and $X(s,t) = s$ for $s > 0\big)$ we observe that the correct solution should be an expansion wave.

Solution 2: Consider the entropy $\eta(v) = v^2$, then the entropy flux is $F(v) = \int_0^v -6z^5 2z\mathsf{d}z = -\frac{12}{7}v^7$. Then upon observing that $\eta(u) = u^2 = 1 - H(x+t)$ and $F(u) = -\frac{12}{7}u^7 = -\frac{12}{7}(1 - H(x+t))$, we infer that

$$\partial_t \eta(u) = -\partial_t H(x+t) = -\delta_0(x+t)$$

and

$$\partial_t F(u) = \frac{12}{7}\partial_t H(x+t) = \frac{12}{7}\delta_0(x+t).$$

In conclusion

$$\partial_t \eta(u) + \partial_t F(u) = (-1 + \frac{12}{7})\delta_0(x+t) = \frac{5}{7}\delta_0(x+t) > 0.$$

The entropy residual $\partial_t \eta(u) + \partial_t F(u)$ is a positive measure, which implies that the $1 - H(x+t)$ is not the entropy solution.

Question 4: Consider the conservation equation $\partial_t u + \partial_x f(u) = 0$ with $f(v) = v^4$ and initial data $u_0(x) = 2H(x) - 1$.
(a) Compute $f(u_0(x))$ (*Hint:* consider the cases $x < 0$ and $0 < x$).

If $x < 0$, we have $u_0(x) = -1$, whence

$$f(u_0(x)) = 1.$$

If $0 < x$, then $u_0(x) = 1$, whence

$$f(u_0(x)) = 1.$$

(b) Show that $u(x,t) = u_0(x)$ is a weak solution. (*Hint:* compute $\partial_t u + \partial_x f(u)$)

Using the hint we have

$$\partial_t u(x,t) = \partial_t u_0(x) = 0,$$

and

$$\partial_x f(u_0) = \partial_x(1) = 0.$$

In conclusion, $\partial_t u + \partial_x f(u) = 0$.

(c) Consider the entropy $\eta(v) = v^2$. Compute the associated entropy flux $q(v)$.

We have

$$q(v) = \int_0^v 2z4z^3 \mathsf{d}z = \frac{8}{5}v^5.$$

(d) Using the entropy $\eta(v) = v^2$ and the associated entropy flux, compute $\partial_t \eta(u(x,t)) + \partial_x q(u(x,t)) = 0$ where $u(x,t) = u_0(x)$ (*Hint:* Use the distribution theory/theory of weak derivatives.)

We have

$$\eta(u(x,t)) = (u_0(x))^2 = 1,$$

and

$$g(u(x,t)) = \frac{8}{5}(u(x,t))^5 = \frac{8}{5}(u_0(x))^5 = \frac{8}{5}u_0(x).$$

This show that

$$\partial_t \eta(u(x,t)) + \partial_x q(u(x,t)) = \partial_t(1) + \partial_x \frac{8}{5}u_0(x) = \frac{8}{5}2\delta_0 = \frac{16}{5}\delta_0.$$

(e) What do you conclude from (d)? (Based on the observation that $\int \delta_0 \psi = \psi(0) \geq 0$ for all nonnegative functions $\psi \in C^0(\mathbb{R}; [0,\infty))$, we say that $\delta_0 \geq 0$, i.e., δ_0 is a positive measure)

We conclude that

$$\partial_t \eta(u(x,t)) + \partial_x q(u(x,t)) = \frac{16}{5}\delta_0 \geq 0.$$

Hence, the entropy inequality is violated. The proposed solution is not the entropy solution.

(f) Compute the entropy solution to this problem.

By looking at the characteristics we conclude that the entropy solution is an expansion wave.

Case 1: $s < 0$, $u_0(x) = -1$, $f'(u_0(x)) = -4$. Hence

$$X(x,t) = s - 4t.$$

This gives $s(X,t) = X + 4t$. Hence,

$$u(X,t) = -1, \quad \text{if } X \leq -4t.$$

Case 2: $s = 0$ and we use u_0 as parameter with $-1 \leq u_0 \leq 1$. Then

$$X(s,t) = 0 + 4u_0^3 t.$$

This implies that $u_0 = \frac{X}{4t}$.

$$u(X,t) = u_0 = \frac{X}{4t}, \quad \text{if } -1 \leq \frac{X}{4t} \leq 1.$$

Case 3: $0 < s$, $u_0(x) = 1$, $f'(u_0(x)) = 4$. Hence

$$X(x,t) = s + 4t.$$

This gives $s(X,t) = X - 4t$. Hence,

$$u(X,t) = 1, \quad \text{if } 4t \leq X.$$