1 Theory

Definition 1.1. For space $E$ (total space), $B$ (base), $F$ (fiber) and surjection
\[ \pi : E \rightarrow B \] (1)
if there is an open neighborhood $U$ of $\pi(x)$ ($x \in E$) such that $\pi^{-1}(U)$ and $U \times F$ are homeomorphic, $(E, B, F, \pi)$ is a fiber bundle. And $E$ has the form of product space $B \times F$.

Figure 1: Base and Fiber

Here are two kinds of fiber bundle on a torus.

Figure 2: Trivial Fibers
Figure 3: Nontrivial Fibers
Definition 1.2. \( \{U_i\} \) is an atlas of an \( n \)-dimensional manifold \( U \). For each chart \( U_i \), define

\[
\phi_i : U_i \rightarrow \mathbb{R}^n
\]  

(2)

If \( U_i \cap U_j \neq \emptyset \), define

\[
\varphi_{ij} = \phi_j \circ \phi_i^{-1}
\]  

(3)

And if \( \varphi_{ij} \) has the form of

\[
\varphi_{ij} = (\varphi_{ij}^1(x), \varphi_{ij}^2(x, y))
\]  

(4)

\[
\varphi_{ij}^1 : \mathbb{R}^{n-p} \rightarrow \mathbb{R}^{n-p}
\]  

(5)

\[
\varphi_{ij}^2 : \mathbb{R}^n \rightarrow \mathbb{R}^p
\]  

(6)

then for any stripe \( s_i : x = c_i \) in \( U_i \) there must be a stripe in \( U_j \) \( s_j : x = c_j \) such that \( s_i \) and \( s_j \) are the same curve in \( U_i \cap U_j \) and can be connected. The maximal connection is a leaf. This structure is called a \( p \)-dimensional foliation \( F \) of an \( n \)-dimensional manifold \( U \).

Property 1.1. All the fiber bundles compose a group \( G_F = \{ F_i \} \).

If we regard the fiber bundle(1-manifold) as a parameterization on a surface(2-manifold), we can prove it easily. As an example, we use the parameterization on a torus.

\[
\{u, v\} \rightarrow \{(R + r \cos v) \cos u, (R + r \cos v) \sin u, r \sin v\}
\]  

(7)

Then we can get the trivial fibers in Fig. 2.

\[
\{u, v\} = \{t, \text{const}\}
\]  

(8)

To generalize the creating method of fiber bundle, it’s necessary to define the transformation functions.

Definition 1.3. A transformation function \( F : [0, 1]^2 \rightarrow [0, 1]^2 \) for fiber bundle must be:

- \( F \) is \( C^1([0, 1]^2) \)
- \( F \) is bijection
  - \( F(u, 0) = F(u, 1) \), \( F(0, v) = F(1, v) \)
  - \( F_u(u, 0) = F_u(u, 1) \), \( F_u(0, v) = F_u(1, v) \)
  - \( F_v(u, 0) = F_v(u, 1) \), \( F_v(0, v) = F_v(1, v) \)

An example is \( F(u, v) = \{u, \{v + u/2\}\} \). The \( \{\} \) inside means the fractional part. Generally, transformation functions always use \( \{\} \) to make sure that \( F \) have the properties listed above. In the following parts, we will omit it.

With transformation functions, we can easily get any fiber in any fiber bundle:

\[
F : \{u, v\} = \{t, \text{const}\} \rightarrow \{u', v'\} = F(u, v) \rightarrow \{(R + r \cos v') \cos u', (R + r \cos v') \sin u', r \sin v'\}
\]  

(9)

And

Property 1.2. For \( G_F \), we have following properties:

- \( \forall F_i, F_j, F_i \circ F_j \) is defined as
  \[
  F_j \circ F_i : \{u', v'\} = F_j(F_i(u, v)) \rightarrow \{(R + r \cos v') \cos u', (R + r \cos v') \sin u', r \sin v'\}
  \]  

(10)

- Identity element is
  \[
  I : I(u, v) = \{u, v\}
  \]  

(11)

- \( \forall F_i \),
  \[
  F_i^{-1} : \{u', v'\} = F_i^{-1}(u, v) \rightarrow \{(R + r \cos v') \cos u', (R + r \cos v') \sin u', r \sin v'\}
  \]  

(12)
2 Result

As an example, assume

\begin{align*}
F_1(u, v) &= \{u, v + \frac{u}{2}\} \\
F_2(u, v) &= \{u, v + \sin 2\pi u\} \\
F_3(u, v) &= \{u + \frac{1}{2} \sin 2\pi v, v\} \\
F &= F_3 \circ F_2 \circ F_1
\end{align*}

Figure 4: A fiber of $I$

Figure 5: A fiber of $I$

Figure 6: Five fibers of $I$

Figure 7: Five fibers of $I$
Figure 8: A fiber of $F_1$

Figure 9: A fiber of $F_1$

Figure 10: Five fibers of $F_1$

Figure 11: Five fibers of $F_1$
Figure 12: A fiber of $F_2$

Figure 13: A fiber of $F_2$

Figure 14: Five fibers of $F_2$

Figure 15: Five fibers of $F_2$
Figure 16: A fiber of $F_3$

Figure 17: A fiber of $F_3$

Figure 18: Five fibers of $F_3$

Figure 19: Five fibers of $F_3$
Figure 20: A fiber of $F_2 \circ F_1$

Figure 21: A fiber of $F_2 \circ F_1$

Figure 22: Five fibers of $F_2 \circ F_1$

Figure 23: Five fibers of $F_2 \circ F_1$
Figure 24: A fiber of $F_3 \circ F_2 \circ F_1$

Figure 25: A fiber of $F_3 \circ F_2 \circ F_1$

Figure 26: Five fibers of $F_3 \circ F_2 \circ F_1$

Figure 27: Five fibers of $F_3 \circ F_2 \circ F_1$
Figure 28: Base and Fiber