

Physics 408 Problem Set 11 Due Tues, Dec 7 at beginning of class

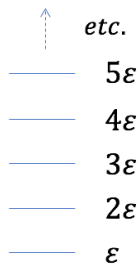
1) Callen problem 9.7.1.

Added part (b): Following the description given in class for such systems, consider the temperature $(T_1 + T_0)/2$, appearing in the middle of the temperature range of the vaporization-condensation curves. Sketch the Gibbs free energy curves vs. x_A at this temperature, and in the Gibbs plot show specifically the coexistence line joining the liquid and gas phases.

2) Callen problem 18.2.1

3) Consider a system with equally spaced single-particle energy levels (see diagram), with values $\varepsilon, 2\varepsilon, \text{etc.}$, as shown. Suppose that 100 identical, non-interacting spin-1/2 Fermions occupy these levels, with each energy level able to hold two of the particles due to spin.

(a) Find the ground state configuration and total energy, and the degeneracy of the ground state.
 (b) At low energies where the states can be counted, this situation can be readily analyzed using the microcanonical ensemble, as opposed to the grand canonical ensemble which we used for a



more general analysis. Noting that excitations above the ground state include sets of excited eigenstates with relative energy $\varepsilon, 2\varepsilon, \text{etc.}$, consider the lowest 3 excited energy macrostates, and find the multiplicity for each of the three cases. (Some microstates differ from each other by spin flips; be sure to include the spin degeneracy when counting if needed.) As a result find the Boltzmann statistical entropy for each macrostate.

(c) The temperature can be obtained by the derivative of energy vs entropy; as an approximation to the derivative find the temperature for each excited macrostate by considering the discrete differences between each state and the one that is immediately lower in energy.

(d) $D(\varepsilon)$ can also be obtained here by assuming ε is very small, in which case dN/dE is a constant. Find $D(\varepsilon)$ in this way (including the spin degeneracy of 2 as I did in class).

(e) Based on the above results, it is possible to compare the statistical entropy to the thermodynamic entropy, using the specific heat result obtained in class for a continuum independent-Fermi particle system ($C = k^2 T \frac{\pi^2}{3} D(\varepsilon_F)$). Since there is no work in this constant- V system, all energy goes in as heat, and the entropy can be obtained as an integral. Do this and find S for the 3ε excited state considered above. The result should be within a factor of 3 of the statistical result obtained above; the Boltzmann and thermodynamic entropy are comparable but only identically equal for the large-number limit.

4) For this problem consider the situation of problem 3, but with 100 zero-spin noninteracting Bosons occupying these energy levels.

(a) Find the ground state energy and its degeneracy.

(b) Find the energy and degeneracy for the first three excited energy-macrostates.

(c) Find the temperature of the 3 excited states, in the same microcanonical approximate treatment as utilized in the last problem.

(d) For each macrostate, find the average number of excited particles (e.g. average of the different number of excitations in each microstate). This is the analog of N_e (text equation 10.53), the number of excited particles within the Bose-condensed configuration.

5) The entropy at the bottom of table 18.2 in the text is shown but not evaluated there. For the $T < T_c$ case, show that an expression for S in terms of (T/T_c) can be obtained starting from the heat capacity, also shown in table 18.2 (you can use the numerical value which is a bit easier to handle). Do this by integration from $T = 0$. As a result, and based on the values for U below T_c also shown in the table, show that ST/U is equal to $5/3$. This can be obtained numerically from what is shown in the table, but the $5/3$ value is exact. The entropy obtained this way depends only on the excited particles; the condensate entropy is zero and it does not contribute to the heat capacity.

6) In treating spin-1/2 Fermi gases, we assumed the number of up and down spins to be identical, at least in absence of a magnetic field. (Hence the factor 2 multiplying the density of states to convert from spin 0 to spin 1/2.) However, when Coulomb interaction between electrons is taken into account, it can be shown that there is an extra energy $\Delta E = -e^2 \left(\frac{\pi^2}{18}\right)^{1/3} \frac{\Delta N^2}{N^{2/3} V^{1/3}}$.

This is the Hartree-Fock model for the exchange energy; e is the electron charge and ΔN is the difference between up and down spin numbers: $\Delta N = N_+ - N_-$. This energy can cause the spins to become unbalanced, and note that the result comes from a purely electrical repulsion of electrons, rather than an interaction between magnetic dipoles, however this type of exchange is what typically causes ferromagnets to align magnetically. For those taking the QM 414 course, the Griffiths text has a simplified expression for an exchange energy; essentially it comes about because Pauli exclusion and the anti-symmetry of Fermion wavefunctions tend to keep like-spin electrons further apart, and this minimizes the Coulomb interaction when the spins are balanced.

(a) Without the exchange term identified above, if ΔN becomes nonzero, with N remaining unchanged, the total energy of a spin-1/2 Fermi gas increases. Show that this is the case by treating the spins as separate Fermi gases, and find the change in total energy to leading order in ΔN . (Your result should be proportional to ΔN^2 .)

(b) By balancing the energy increase of part (a) with the exchange term, show that there is a minimum electron density below which the spins will become spontaneously polarized. This corresponds to a general result, which is that Fermi gases are nonmagnetic at large density but become unstable to becoming magnetic for low densities. Re-expressing your result, you should find that it simplifies to $n < \frac{2}{3\pi^2} \left(\frac{1}{a_0}\right)^3$, where a_0 is the Bohr radius.