Physics 408 Problem Set 8 Due Thurs, Nov 4 at beginning of class

1) [corrected Nov. 2] Consider an N-atom paramagnet consisting of independent magnetic spins with angular momentum quantum number J, and individual spin energy $\mu m_J B$ in a magnetic field B, and with μJ corresponding to the atomic magnetic moment (for an electron μ would be the g-factor times the Bohr magneton). This is the generalization of the spin-1/2 case considered in class.

a) Show that the single-atom partition function can be expressed, $Z = \frac{sinh\left\{(J+\frac{1}{2})\mu B/kT\right\}}{sinh\{(\mu B/2kT\}}$. Note that the finite sum $1 + x + x^2 + \cdots x^n$ is equivalent to $\frac{1-x^{n+1}}{1-x}$, as can be easily established by considering the difference of the sums for n and for n+1.

b) Find a general expression for the magnetization of such a system, demonstrating that it is proportional to $\frac{\partial Z}{\partial B}$.

c) As a result, show that the N-atom magnetization can be written as a sum of two terms, proportional to $coth\left\{(J+\frac{1}{2})\mu B/kT\right\}$ and $coth\{\mu B/2kT\}$. You should obtain the complete resulting expression, which gives the magnetization in terms of the *Brillouin function*.

d) Show that when $kT \gg \mu JB$, the magnetization becomes proportional to (1/T), and find the proportionality constant. This is the generalization of the *Curie law*.

2) The "Magnetic free energy", $F_M = U - TS - IB$ is sometimes used to describe magnetic systems. F_M is closely related to the alternative magnetic thermodynamic potentials discussed in section 7-5 of the text; we won't discuss those. In this case, as used in the text, I is the notation for the total moment, which is equivalent to MV, with M the magnetization (which by standard units is a per-volume quantity). Note that for this problem we don't consider any volume changes, so the implicit volume-dependence of I is not important. For magnetic systems, U includes the BdI term described in appendix B of the text. As a result, the differential form of F_M becomes, $dF_M = -SdT - IdB$. This is appropriate for systems at constant-T and constant-B, with the subtraction of the IB term playing a similar role as the PV term added to F to get the Gibbs free energy.

a) Based on the differential form given above, identify a Maxwell relation connecting derivatives of S and I.

b) Find a relationship for $\frac{\partial T}{\partial B}_S$ (the adiabatic B-induced temperature change), in terms of $\frac{\partial I}{\partial T}_B$ and C_B, where the latter is the constant-B heat capacity, $C_B = T\frac{\partial S}{\partial T}_B$. (Hint, it is helpful to use a cyclic relation with S-B-T derivatives.)

c) Based on the properties of paramagnets seen in class and in the problem above, show from your result that an isolated paramagnet is always heated by an increase in field, and cooled by a field decrease.

3) Show that in general, for N held constant, $\frac{\partial U}{\partial V}_T = T\frac{\partial P}{\partial T}_V - P$. (This will require a Maxwell relation plus another general result for partial derivatives.) Then, show that this relation yields the expected behavior for an ideal gas based on its familiar equation of state.