Notes:

Homework : Due next Tuesday. (I am not sure yet about presentations, depends upon timing, I will send email.)

Last class day: Weds. Dec 8.

Final Exam: Friday Dec. 10, 12:30 PM, in room 203. Exam will be comprehensive, with no particular focus on new material. A formula sheet will be allowed, similar to the previous exam.

I have a **sample exam** I will post – the format will be similar to exam 1, but with more problems & fewer parts per problem (tentatively 8 problems).

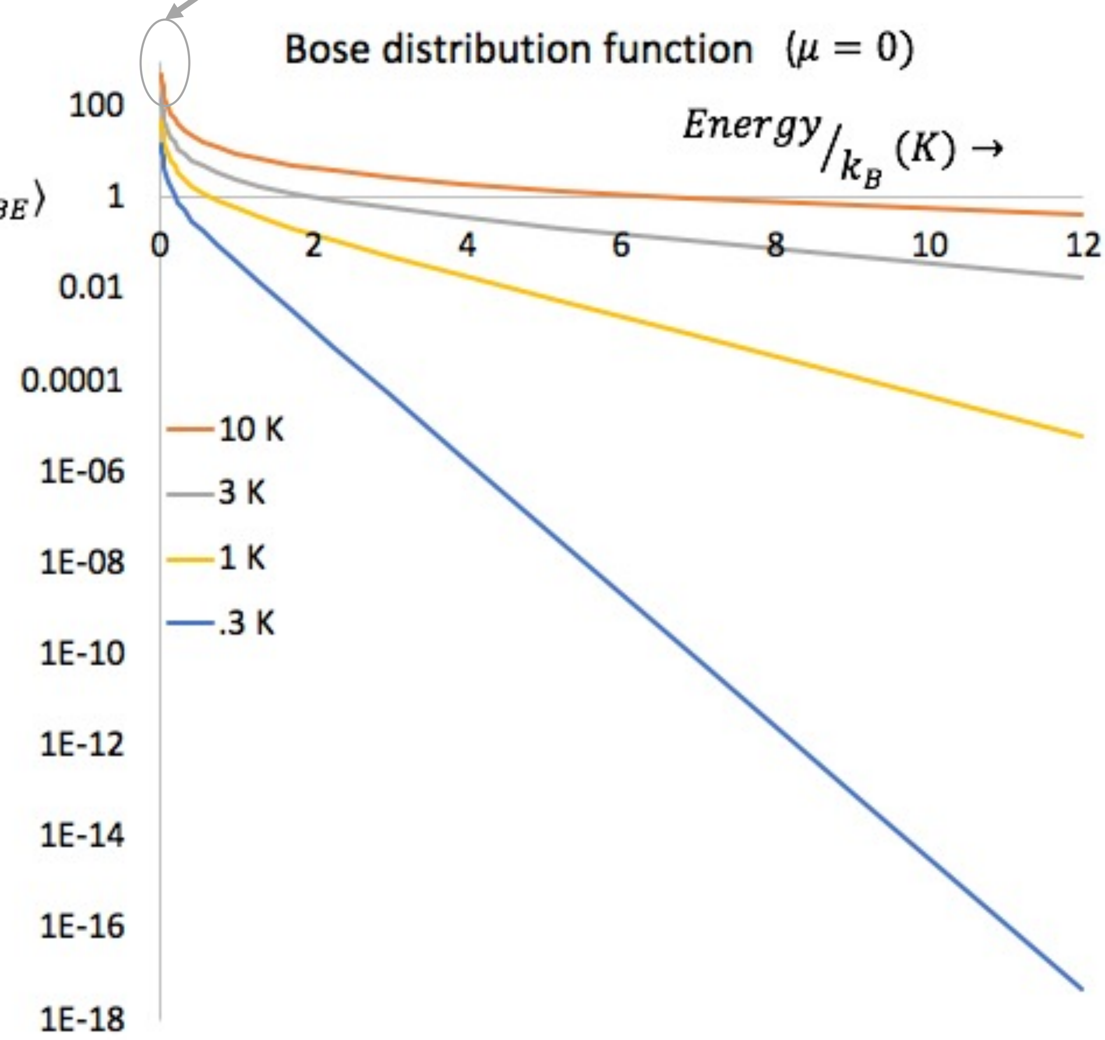
Bose gases; Bose-condensation

$$N = \int_0^\infty D(\varepsilon) \boxed{\frac{1}{e^{\beta(\varepsilon-\mu)} - 1}} d\varepsilon$$

n_{BE}

Plotted, with $\mu = 0$

n undefined when $\varepsilon = \mu$



High T, requires increasingly more negative μ to conserve N.

Also note, $e^{\beta\mu} \equiv \xi$ is fugacity (must be < 1) :

$$n_{BE} = \frac{1}{e^{\beta(\varepsilon)}/\xi - 1}$$

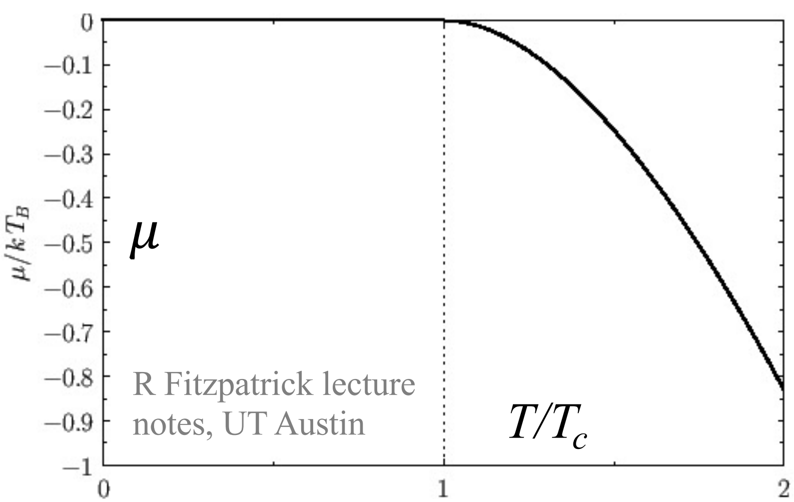
see text, expanding in powers of ξ gives formal high-T solution for fixed N.

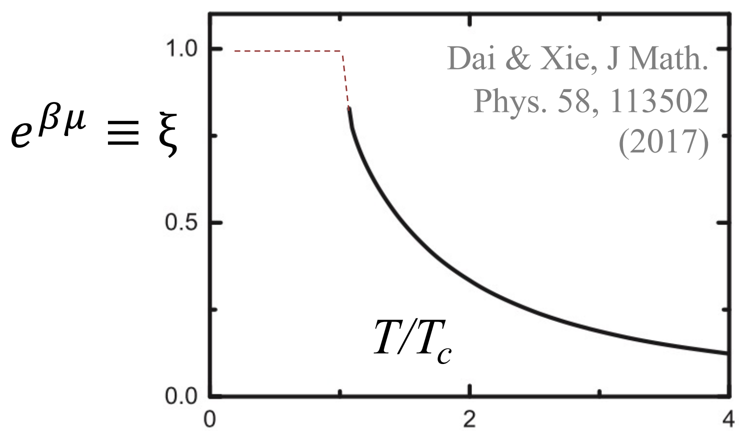
Bose gases; Bose-condensation

$$D(\varepsilon) = \frac{V}{(4\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

$$N = \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}-1}d\varepsilon = \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta(\varepsilon)}/\xi - 1}d\varepsilon$$

Solve for fixed N, can show





- chemical potential & fugacity determined by # particles N.
- Upper plot generated from the series-solution shown in text (p. 414), lower plot based on similar method.

$$\tilde{N}_e = \left[\frac{g_0 V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\right]\frac{\sqrt{\pi}}{2}(k_B T)^{3/2}F_{3/2}(\xi) = \frac{g_0 V}{\lambda_T^3}F_{3/2}(\xi)$$

where λ_T is the "thermal wavelength" (equation 18.26) and

$$F_{3/2}(\xi) = \sum_{r=1}^{\infty}\frac{\xi^r}{r^{3/2}} = \xi + \frac{\xi^2}{2\sqrt{2}} + \frac{\xi^3}{3\sqrt{3}} + \cdots$$

- I will focus on analytical results for <u>low-T regime</u>.
- Zero slope in μ vs T at T_c: 2nd order phase transition.

Bose gases; Bose-condensation

$$D(\varepsilon) = \frac{V}{(4\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon} \equiv \alpha V\sqrt{\varepsilon}$$

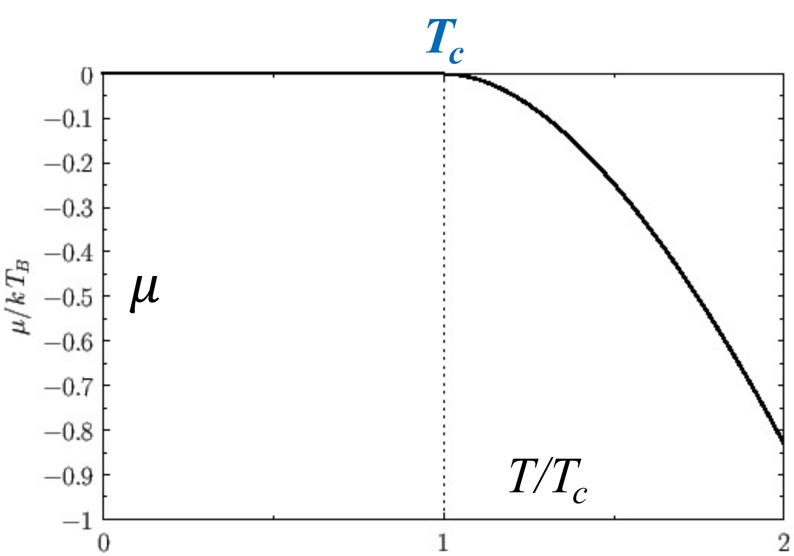
$$N = \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}-1}d\varepsilon = \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta(\varepsilon)}/\xi - 1}d\varepsilon$$

Limiting case: $\mu = 0$ ($\xi = 1$) can solve analytically:

$$N = \int_0^\infty \frac{\alpha V\sqrt{\varepsilon}d\varepsilon}{e^{\beta\varepsilon}-1} = \alpha V(kT)^{3/2}\int_0^\infty \frac{\sqrt{u}du}{e^u - 1}$$

$$\frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right) \cong 2.612\frac{\sqrt{\pi}}{2}$$

Riemann zeta fn.



- determines T_c: $kT_c \cong 6.626\frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}$
- $T < T_c$, μ can't further increase, so appears that N should decrease.
- Actually, zero-level condensate is *not included in integration* assuming continuum of levels $D(\varepsilon)$; setting $\mu = 0$ means infinite n in ground level.

Some numerics; position chemical potential just below zero:

```
Integrate[Sqrt[x] / (Exp[x] - 1), {x, 0, Infinity}]
```

$\frac{1}{2}\sqrt{\pi}\,\text{Zeta}\left[\frac{3}{2}\right]$

```
Integrate[Sqrt[x] / (Exp[x] - 1), {x, 0, Infinity}] // N
```

2.31516

Integration for $N_{excited}$ with zero chemical potential

```
Exp[1 / 1000] // N
```

1.001 < corresponds to $e^{-\beta\mu}$

```
Integrate[x ^ .5 / (1.001 * Exp[x] - 1), {x, 0, Infinity}]
```

2.21713

$n = 1000$ in ground state.

```
N[Exp[1 / 1 000 000], 10]
```

1.000001000

```
Integrate[x ^ .5 / (1.000001 * Exp[x] - 1), {x, 0, Infinity}]
```

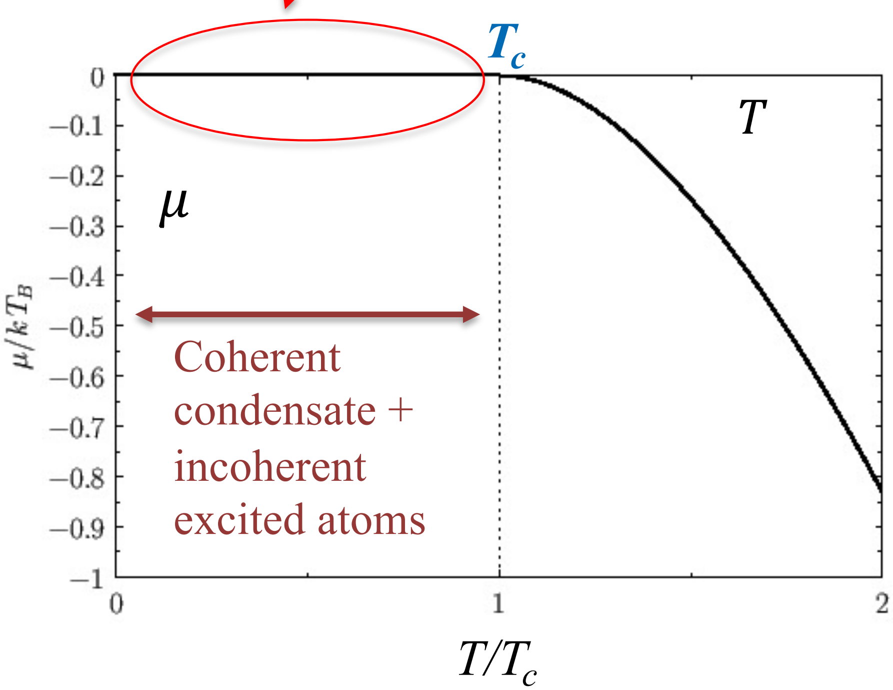
2.31202

$n = 10^6$
Excited population essentially same as for $\mu = 0$

Result: zero level treat separately.

Bose condensate region: states "pile on" to ground state.
μ just less than zero. (approaches zero in large-N limit, where transition becomes sharp.)

$$kT_c \cong 6.626\,\frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}$$



Modified summation for N below T_c:

$$N \equiv N_{excited} + N_o$$

$$N = \int_0^\infty \frac{D(\varepsilon)d\varepsilon}{e^{\beta\varepsilon}-1} + \frac{1}{(e^{-\beta\mu}-1)}$$

excited states $\propto T^{3/2}$

ground state $n_{BE} = N_o$

(& note, limiting consideration to cases with non-degenerate ground state.)

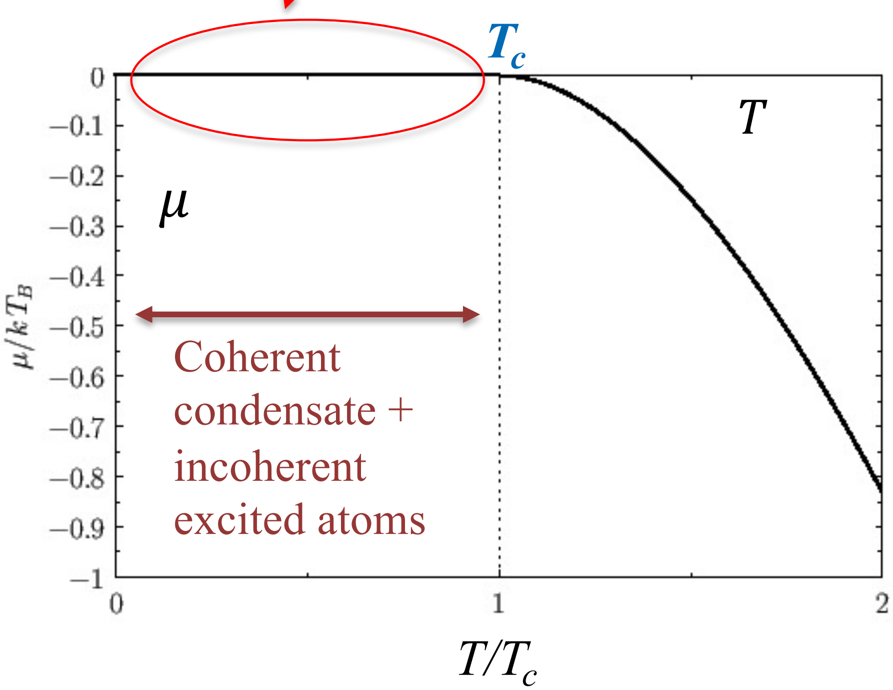
& note $N_{excited}$ which I refer to in HW problem with small # of particles

Result: zero level treat separately.

$$\left(\frac{h^2}{2\pi mkT}\right)^{1/2} = \lambda_{th}$$

Bose condensate region: states "pile on" to ground state.
μ just less than zero. (approaches zero in large-N limit, where transition becomes sharp.)

$$kT_c \cong 6.626 \frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}$$



Also note, transition occurs when thermal wavelengths (roughly) overlap:

$$\left(\frac{V}{N}\right)^{2/3} \cong 6.626 \frac{\hbar^2}{2mkT_c}$$

so $\frac{V}{N} \cong \lambda_{th}^3$

Solutions below T_c:

$$\alpha V (kT)^{3/2} \int_0^\infty \frac{\sqrt{u}du}{e^u - 1} \cong 2.31\alpha V(kT)^{3/2}$$

$$\alpha = \frac{1}{(4\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}$$

$$N = Nf(T) + \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta\varepsilon} - 1}d\varepsilon$$

ground state = N_o

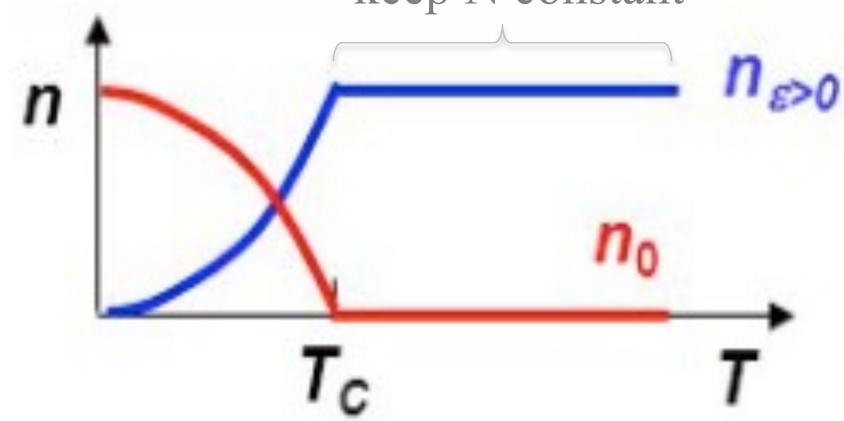
All other states: normal Bose gas (large N limit)

solution:

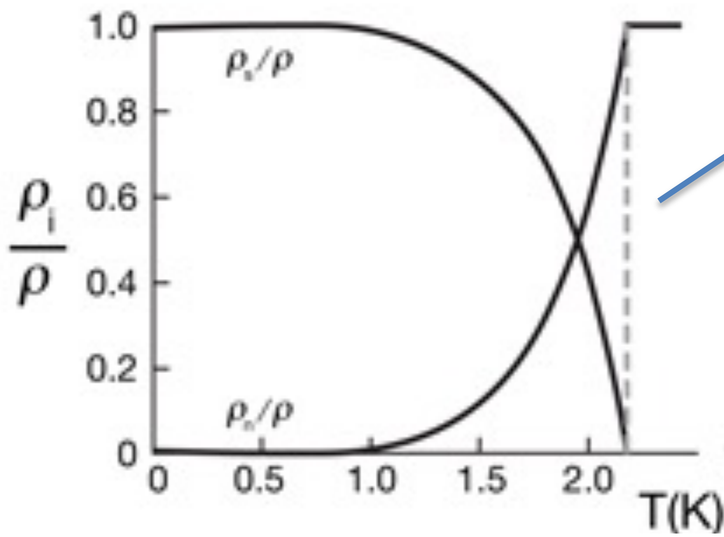
$$f(T) = \left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$$

f = fraction in ground state

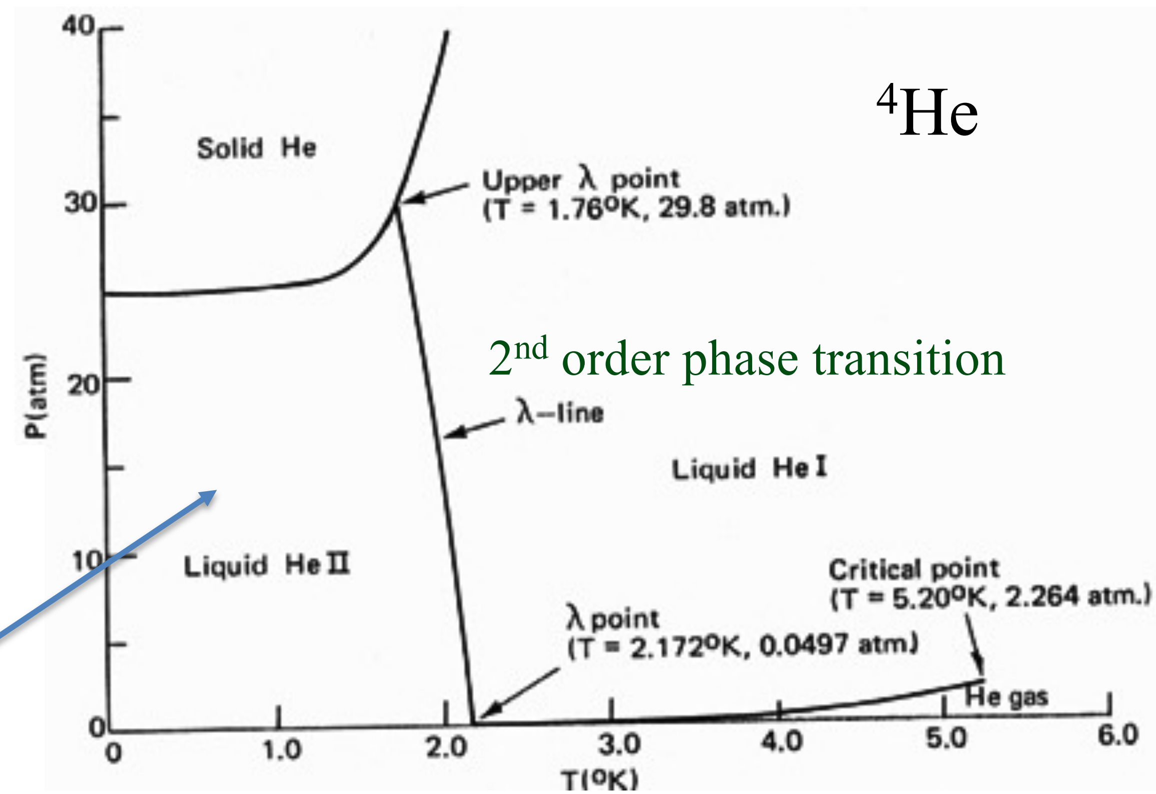
Here μ decreases to keep N constant



Superfluid liquid helium: Example of Bose condensed state in <u>strongly interacting</u> conditions (not "Bose gas")



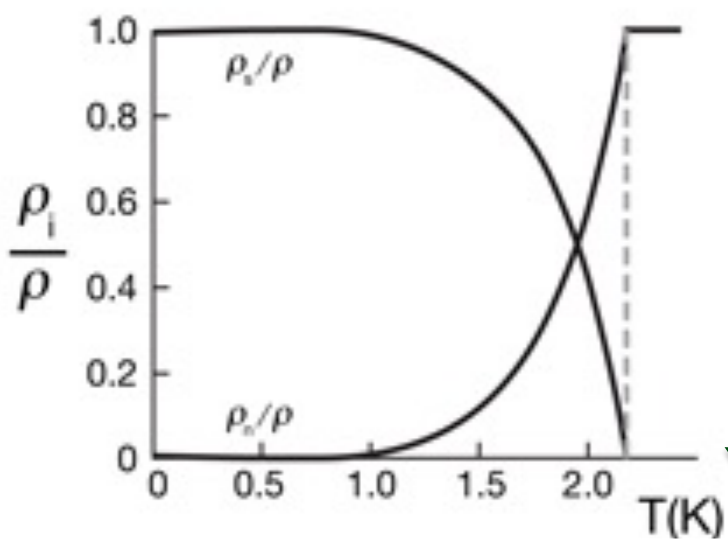
(inertia of oscillator, Andronikashvili & Mamaladze, Rev. Mod. Phys. 1966)



Superfluid: viscosity = 0

Two-fluid picture: <u>superfluid fraction</u> vs. <u>normal fluid fraction</u>

Superfluid liquid helium: Example of Bose condensed state in strongly interacting conditions (not "Bose gas")



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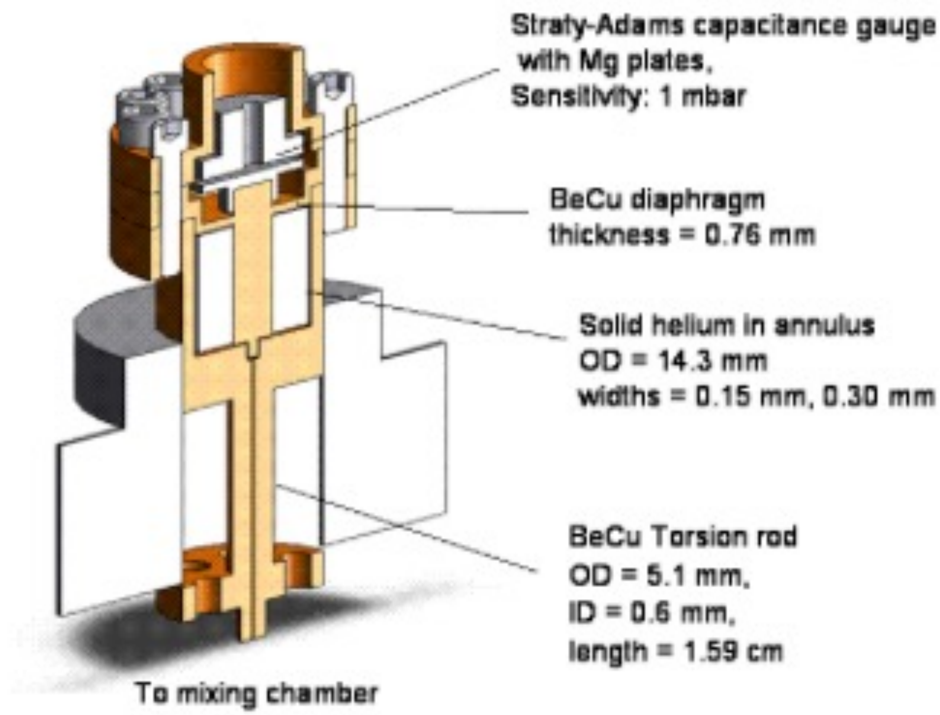
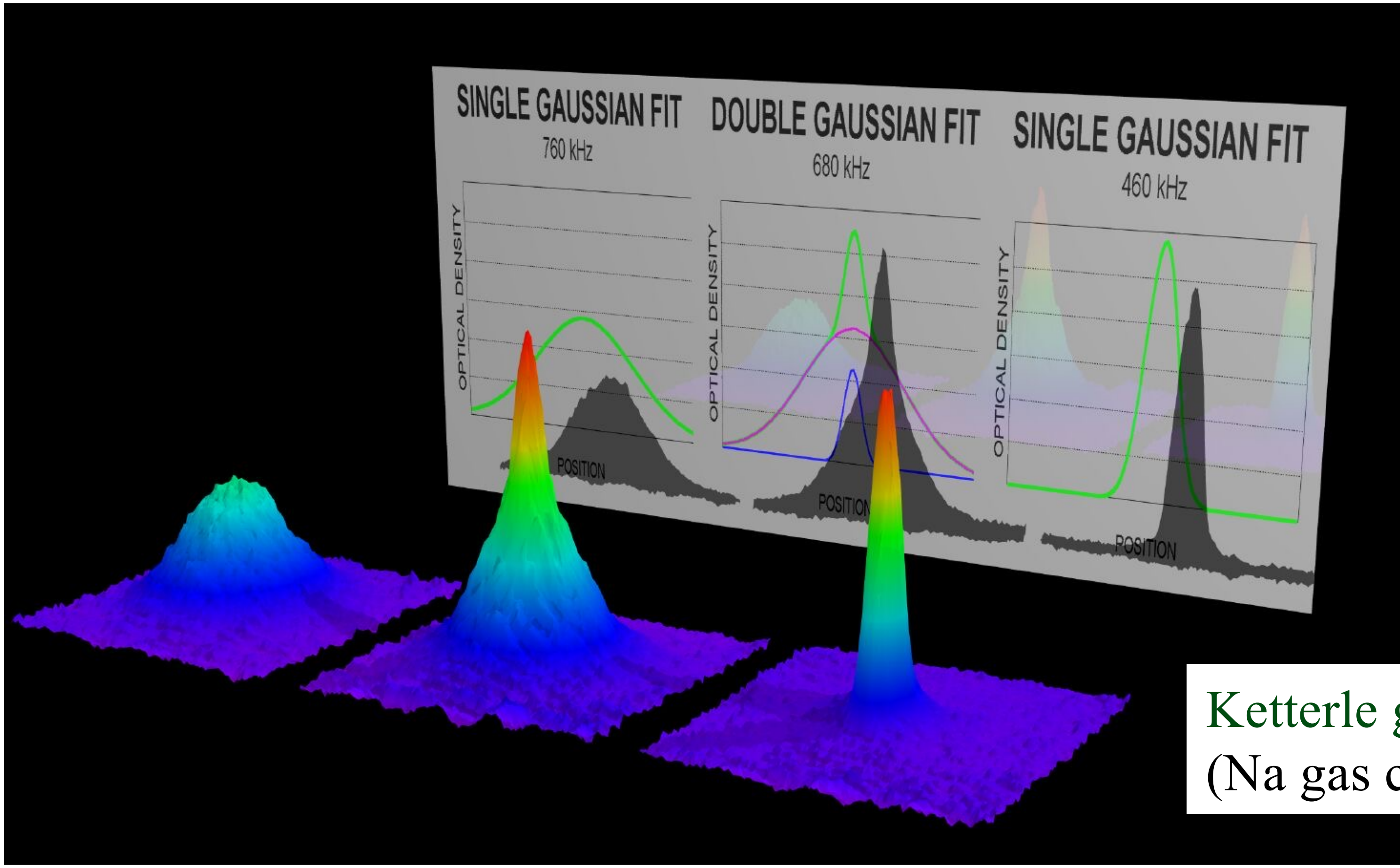


Superfluid: viscosity = 0

Two-fluid picture: superfluid fraction vs. normal fluid fraction

Dilute gases:



Frequency characteristic of harmonic trap induced by applied *B*-field.

***T* ~ 100 nK**

Ketterle group webpage
(Na gas condensation)

Nobel prize 2001 Cornell, Ketterle, Wieman.
Combined optical, magnetic-trap cooling process.
Used harmonic-well trap: small modifications from results using square-well $D(\varepsilon)$.
Measurement of temperature & related properties by imaging particles after turning off trap

Ideal Bose-condensed gas:

$$N = Nf(T) + \int_0^\infty D(\varepsilon)\frac{1}{e^{\beta\varepsilon}-1}d\varepsilon$$

$$D(\varepsilon) = \alpha V\sqrt{\varepsilon} \qquad \alpha = \frac{1}{(4\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}$$

Some thermo (for $T < T_c$)**:**

$$U = \int_0^\infty \frac{D(\varepsilon)\varepsilon}{e^{\beta\varepsilon}-1}d\varepsilon = \alpha V(kT)^{5/2}\int_0^\infty \frac{u^{3/2}du}{e^u-1} \cong 1.78\alpha V(kT)^{5/2}$$

$$\Rightarrow C_V = \frac{5}{2}\frac{U}{T}$$

$$S = \frac{5}{3}\frac{U}{T}$$

Entropy due only to excited states

Condensate $U = 0$; $S = 0$. ("particles in lockstep")


SCIENCE
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Molecule of the Year
the Bose-Einstein Condensate

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Entropy due only to excited states

Condensate $U = 0$; $S = 0$. ("particles in lockstep")

$$\Rightarrow F = U - TS = -\frac{2}{3}U$$

$$P = +\frac{2}{3}\frac{U}{V}$$

& these also give G as expected