Manifold Geometry and Mixing in Observed Atmospheric Flows

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Abstract

In Hamiltonian systems, the hyperbolic points and their associated stable and unstable manifolds determine the nature and location of mixing by the flow. The intersections of the stable and unstable manifolds define lobes, which are the areas in the flow subject to the strong stretching and folding that leads to chaotic Lagrangian motion. Numerical methods have been developed to locate the manifolds and compute the transport due to the lobes. These methods generally require some knowledge of the location of the hyperbolic points in the flow. Many geophysical flows are complex, however, and the location of the hyperbolic points is not apparent. Here a simple, powerful, numerical method, referred to as finite-strain maps, is used to locate the stable and unstable manifolds of complex two-dimensional incompressible flows. The method requires no a priori knowledge of the locations or velocities of hyperbolic points.

The method is tested by applying it to a simple analytical flow field in which the manifold structure is known. The finite-strain maps rapidly collapse to the manifolds, which can be seen to foliate the chaotic region of the flow. The method is also applied to the observed flow in the lower stratosphere, which is two-dimensional and nondivergent to a good approximation. The manifold structure quickly emerges in the finite-strain maps and shows good agreement with known transport properties of the stratosphere. The manifold geometry provides a kinematic explanation for the mixing barriers surrounding the wintertime polar vortices. Finite-strain maps provide a simple, robust method for visualizing manifolds and exploring their statistical properties, as well as a starting place for detailed lobe-dynamics transport calculations. The method should be applicable to a wide range of old and new transport problems in geophysical fluid dynamics.
Introduction

Many problems of current interest involve the transport and mixing of atmospheric properties or constituents. These properties may be dynamically active quantities, such as potential vorticity; or they may be quantities that are effectively passive as far as the dynamics are concerned, such as the concentrations of trace constituents. Lagrangian methods, because they explicitly consider the trajectories of air parcels, have proven to be very helpful in understanding many aspects of transport and mixing.

Many different approaches have been taken to the problem of characterizing Lagrangian motion in a fluid. These include computing measures of the dispersion of particles in a flow (e.g., the effective meridional diffusivity $K_{yy}$) [Bowman, 1993b; Schoeberl et al., 1989], finite-time Lyapunov exponents [Bowman, 1993b; Pierce and Fairlie, 1993; Pierrehumbert, 1991; Pierrehumbert and Yang, 1993], “patchiness” [Malhotra et al., 1999], and various structure functions [Pierrehumbert, 1994]. Progress has been made on some of the underlying mathematical problems of Lagrangian transport [Ottino, 1989; Wiggins, 1992], especially for Hamiltonian systems, and here we describe some numerical methods for applying dynamical systems theory to geophysical mixing problems.

While many geophysical flows are inherently three-dimensional, in some cases the flow can be well approximated as two-dimensional and nondivergent (incompressible). An example of this is the large-scale horizontal flow in the stratosphere. In large parts of the stratosphere, the timescale of diabatic heating is long compared to that for horizontal advection. Therefore, the flow on isentropic surfaces is two-dimensional and incompressible to a good approximation.

In the case of two-dimensional, incompressible flow, the velocity $\mathbf{v}(\mathbf{x},t)$ can be written in terms of a streamfunction $\psi(\mathbf{x},t)$ as
\[ \mathbf{v} = (u, v) = \left( -\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} \right), \]

where \( \mathbf{x} = (x, y) \) is position, \( t \) is time, and \( u \) and \( v \) are the \( x \)- and \( y \)-components of the velocity, respectively. In this case, the equation of motion for a massless particle carried by the flow is

\[ \frac{dx}{dt} = \mathbf{v}(\mathbf{x}, t) \]

or

\[ \frac{dx}{dt} = u = -\frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = v = \frac{\partial \psi}{\partial x}. \]

These equations are in Hamiltonian form, with the streamfunction \( \psi \) playing the role of the Hamiltonian. If \( \psi \) is independent of time, particles remain on streamlines (i.e., trajectories coincide with streamlines) and no mixing occurs. Time-independent systems are said to have one degree of freedom. Such systems are integrable and do not allow for chaotic motion of particles. Time-periodic Hamiltonian systems are said to have one-and-a-half degrees of freedom, while systems with general time-dependence have two degrees of freedom. In general, time-dependent Hamiltonian systems are not integrable, and particle motion can be chaotic, at least in some parts of the domain.

The properties of the solutions to Hamiltonian systems are intimately related to the geometry of the Hamiltonian (the streamfunction in this case). Of particular importance are the stationary solutions or fixed points of the system, that is, points where \( \mathbf{v} = 0 \). Hamiltonian systems have only two kinds of nondegenerate fixed
points: elliptic points or centers, and hyperbolic or saddle points. The characteristic geometry of a prototypical time-independent Hamiltonian system is shown in the top panel of Figure 1. Hyperbolic points are indicated by solid dots; elliptic points, which are surrounded by closed streamlines, are indicated by open dots.

The set of all points in the domain that converge to a hyperbolic fixed point as \( t \to \infty \) comprise the stable manifolds of the fixed point. Similarly, those points that converge to a fixed point as \( t \to -\infty \) comprise the unstable manifolds of the fixed points. In Figure 1 heavy lines indicate the manifolds. In the case of steady flow, shown in the top panel of Figure 1, one of the unstable manifolds of each fixed point coincides exactly with one of the stable manifolds of the other fixed point to form two heteroclinic trajectories. Because it separates the closed trajectories surrounding the elliptic fixed point from the open trajectories outside the cat’s eye, the curve defined by the manifolds is sometimes called a separatrix.

If systems like those in Figure 1 are subjected to a periodic perturbation, the fixed points persist as periodic points, and the associated manifolds take on a complex, intertwined structure like that shown in the bottom panel of Figure 1. The transverse intersection of the manifolds in the perturbed case is one of the characteristics of chaotic systems. With the method of Melnikov it is possible to estimate the rate of transfer of material into and out of the cat’s eye structure for small perturbations [Melnikov, 1963]. The more general theory of lobe dynamics, developed by Wiggins and collaborators [Rom-Kedar and Wiggins, 1990; Wiggins, 1992] based on related ideas by MacKay et al. [1984], provides a method to compute transport for perturbations of arbitrary amplitude. Lobes are the regions bounded by the stable and unstable manifolds between successive intersections of the manifolds. They explicitly delineate the areas of the fluid that are exchanged between the interior and exterior of the cat’s-eye structures due to the time
dependence of the flow. In the periodic case all lobes have equal areas.

Lobe dynamics has been applied to a number of analytical flow fields [Camassa and Wiggins, 1991; Wiggins, 1992], and numerical tools have been developed to carry out lobe dynamics calculations using output from numerical models [Duan and Wiggins, 1997; Miller et al., 1997]. These tools have been used to study cross-jet transport in a nearly-periodic model of a meandering Gulf Stream [Rogerson et al., 1999]. An important aspect of these studies is that the nearly-periodic, wavelike nature of the flow makes it possible to locate the hyperbolic points relatively easily.

The concepts of lobe dynamics have been extended to general aperiodic flows by Haller and Poje [1998], who applied their method to study the pinch-off of a cold-core Gulf Stream ring [Poje and Haller, 1999]. An important result obtained in their lobe-dynamics calculation is that the fluid entrained into a ring is drawn from a long filament lying on the northern flank of the jet; the fluid is not simply pinched off in compact blob. In this case, as in the previous case of the model of the meandering Gulf Stream, the hyperbolic points are relatively easy to locate by direct examination of the flow field.

One of the greatest difficulties in applying lobe dynamics to observed geophysical flows is the inherent complexity of real flows. They can rarely be approximated, for example, as a simple wave plus perturbation. Geophysical flows are quasi-periodic for limited times at best, and they typically contain motions on a wide range of time and space scales simultaneously. In the stratosphere, for example, there are usually no easily identifiable fixed points in the flow. Because there are typically multiple waves with different phase speeds present in the flow, it is generally not possible to find a single rotating reference frame that makes all the hyperbolic structures simultaneously visible. As a result, applying dynamical-
systems methods to observed, rather than idealized, flows has been problematic.

In this paper we present a new numerical method for mapping the stable and unstable manifolds of hyperbolic points in complex, two-dimensional, quasi-nondivergent flows. No a priori knowledge of the character of the flow is required beyond that fact that it be approximately two-dimensional. Once the geometry of the manifolds is known, it is possible to locate the hyperbolic points and to visualize the heteroclinic or homoclinic tangle. Regions within the flow that are foliated by the manifolds should be well mixed, while regions from which the manifolds are excluded should be weakly mixed internally and isolated from well-mixed areas by mixing barriers. This new method makes the essential kinematics of the flow visible, and provides a relatively direct way to locate mixing barriers. Other potential uses for the information obtained from the manifold structure are discussed below.

**Methods and Data**

**Finite strain**

The manifolds are located numerically in the following manner. Consider a grid of particles laid down in the region of a hyperbolic point in the flow, as in Figure 2. Particles are initialized at each of grid intersections. For clarity, only two particles are shown. A heavy line indicates the distance between the pair of particles. If the trajectories of the particles are integrated forward in time, and if the hyperbolic structure is persistent, then pairs of particles that straddle the stable manifold will separate rapidly as the two particles first move in along the stable manifold toward the hyperbolic point and then are drawn apart as they move outward along their respective unstable manifolds. The distance between an adjacent pair of particles divided by their initial separation,
\[ S = \frac{|x_2(t) - x_1(t)|}{|x_2(0) - x_1(0)|}, \quad (4) \]

which we refer to as the ‘finite strain’, grows more rapidly for those pairs of points that straddle the stable manifolds than for those that are simply drawn apart by local shear. On the other hand, pairs of particles that straddle either the unstable manifold or no manifold will undergo little strain as they move forward in time. Thus, forward-in-time calculations should reveal the stable manifolds as regions of large \( S \).

The unstable manifolds can be found by integrating particle positions backward in time. In the figures presented below, the finite strain is plotted for all adjacent pairs of particles on the grid at the initial particle locations using green for the magnitude of the forward strain (stable manifolds) and red for the magnitude of the backward strain (unstable manifolds). We refer to these plots as finite-strain maps.

This approach to mapping the geometry of the flow is similar to a method used to estimate finite-time Lyapunov exponents in which short material line segments are advected by the flow, re-initializing the endpoints of the segments periodically to ensure that the strain estimate is local. For finite-strain maps, however, the endpoints of the material line are not reinitialized. This ensures that, barring too-rapid changes in the flow geometry, the endpoints will continue to straddle the manifolds. Finite-strain maps thus provide a global view of the Lagrangian deformation field, rather than the microscopic view given by Lyapunov exponents.

Straddling

If the locations of the hyperbolic points are known, then a method known as straddling can be used to compute the manifolds [Miller et al., 1997].
material line that straddles, for example, the unstable manifold is initialized near
the hyperbolic point, the hyperbolic character of the flow ensures that the material
line will stretch and approach the manifold as the flow evolves forward in time.
Numerically, new nodes can be inserted into the material line as it stretches [e.g.,
contour advection Norton, 1994; Waugh et al., 1994].

Simple model

We test the finite-strain method with a simple analytical flow for which the
geometry of the stable and unstable manifolds can be anticipated. The flow consists
of a stationary wave on a linear shear, subjected to a sinusoidal perturbation in
time. The streamfunction for the flow is

\[
\psi(x, y, t) = -\frac{\alpha}{2} y^2 - \frac{\nu'}{2\pi} \cos(2\pi x)(1 + \varepsilon \sin\left(\frac{2\pi t}{\tau}\right)),
\]

where \(\alpha\) is the shear of the \(u\)-component of the velocity, \(\nu'\) is the amplitude of the
wave in the \(v\)-component of the velocity; and \(\varepsilon\) and \(\tau\) are the amplitude and period
of the perturbation, respectively. The streamfunction and manifolds of the
perturbed and unperturbed system are similar to those shown in Figure 1. Forward
and backward trajectories for this system are computed numerically using a fifth-
order Runge-Kutta scheme with adaptive step-size adjustment [Bowman, 1995].

Observed atmospheric flows

To apply the method to the atmosphere, we use wind data taken from the
United Kingdom Meteorological Office (UKMO) stratospheric assimilation product.
The UKMO data assimilation system uses operational radiosonde data, TOVS
temperature profiles, and several other types of tropospheric data. The
observations are continuously assimilated into a general circulation model (GCM),
using the Analysis Correction data assimilation scheme [Lorenc et al., 1991]. The
data are output once a day (at 12 Z) on the UARS (Upper Atmosphere Research Satellite) pressure grid (from 1000 hPa to 0.316 hPa) at a horizontal resolution of 2.5° latitude by 3.75° longitude. Further details are given in Swinbank and O'Neill [1994]. Using UKMO wind data, forward and backward Lagrangian trajectories are computed for 10-day periods on isentropic surfaces using a standard fourth-order Runge-Kutta scheme. Details can be found in Bowman [1993b].

**Results**

**Simple model**

Figure 3 shows the results of a finite-strain calculation for the flow in (5). The model parameters are set as follows: \( \alpha = 1, \, v' = 2, \, \epsilon = 0.1, \) and \( \tau = 1. \) Forward and backward trajectories are computed for a mesh of 512x512 particles. The finite strain is computed between all adjacent pairs of particles in the mesh and color coded as described above. The finite strain is plotted at \( t = \tau, \, 2\tau, \, 3\tau, \) and \( 5\tau. \) The \( x^-\)coordinate is chosen to place the hyperbolic point in the middle of the plot. After one perturbation period (\( t = \tau), \) the outline of the cat’s eye is apparent. At \( t = 2\tau, \) the regions of large strain have collapsed to the resolution of the numerical mesh and the first transverse intersections of the manifolds are faintly visible. By \( t = 3\tau, \) several manifold intersections are visible and the shapes of several lobes are well defined. At \( t = 5\tau, \) the heteroclinic tangle around the hyperbolic point is apparent. It is clear that the manifolds foliate a region around the unperturbed manifolds. There are also areas in the interior and exterior of the cat’s eye with no manifolds, as pointed out by Malhotra and Wiggins [1999]. In these areas, KAM-tori-like invariant regions act as barriers to transport. Poincaré sections for this flow reveal the stochastic layer (that is, the region foliated by the manifolds) and the KAM tori inside and outside the cat’s eye. (See del- Castillo-Negrete and Morrison [1993] for
examples.) As expected, increasing the amplitude of the perturbation sufficiently destroys the KAM tori and the mixing barriers. Also as expected, when the model is run with \( \varepsilon = 0 \) (not shown), the flow is steady and the regions of high-strain collapse onto the unperturbed manifolds (the separatrices).

**Observed atmospheric flows**

In order to apply this method to the atmosphere, large-scale horizontal winds on the 500 K potential temperature surface are taken from the UKMO assimilated stratospheric data set described above. Large numbers of particles are initialized on a global grid that has a resolution 2 to 3 greater than the resolution of the gridded wind data. Increasing the grid resolution further has little effect on the results. Particle trajectories are integrated forward and backward in time for 10 days. The great-circle separation of each adjacent pair of particles is computed as a function of time; and the forward and backward finite strain are plotted using green and red, respectively.

The emergence of the manifolds in the finite-strain maps is illustrated in Figure 4. The particle grid is initialized on 11 October 1996. Only the southern hemisphere is shown. At \( t = 0 \) days the forward and backward finite strains are everywhere equal to 1. Equal intensities of red and green produce yellow. After 1 day the broad structures of the stable and unstable manifolds surrounding the polar vortex are already becoming apparent. After 2 days the bands of large strain are much narrower, and by ±4 days the regions of large strain have collapsed nearly to the grid resolution. The manifolds show that on this date the flow is dominated by a zonal wavenumber 2, with hyperbolic points near 45° latitude and 90° and 270° relative longitude. Extending the integration progressively reveals more of the tangle of stable and unstable manifolds on the equatorward side of the cat’s eyes. These manifolds and lobes are responsible for the rapid mixing in the
surf zone. The manifolds also tangle on the poleward side of the cat’s eyes, but the
tangle is confined to a very thin layer near the core of the jet. The finite strain
appears to be capable of capturing the complex manifold structure of the
stratospheric flow.

Transport by lobes

Figure 5 shows the evolution of the manifolds and lobes at 2-day intervals from
1 to 11 October. In this figure, two particular manifolds are also calculated with the
straddling (contour advection) method: a stable manifold of the hyperbolic point on
the left (shown in blue) and an unstable manifold of the hyperbolic point on the
right (shown in magenta). The contour advection results are overlaid on the finite-
strain maps. The close coincidence between the contour advection calculations and
the finite-strain maps confirms that the finite-strain method is indeed locating the
manifolds.

A series of lobes, labeled A through E, can be followed throughout this 10-day
period. Lobes initially “inside” the cat’s eye (A, C, and E) are carried around the
cat’s eye counterclockwise and stretched into the midlatitudes and subtropics as
narrow filaments. These lobes transport air from near the edge of the polar vortex
into middle and low latitudes. Those lobes (B and D) initially “outside” the cat’s eye
move counterclockwise and are entrained, eventually wrapping up into thin layers
near 60° S. This transport of low-latitude air poleward to the edge of the vortex and
the stretching of the lobes is responsible for the layered appearance of the total
ozone field around the ozone hole [Bowman and Mangus, 1993]. Lobe entrainment
and detrainment are responsible for many features seen in stratospheric trace
constituents [Leovy et al., 1985; Randel et al., 1993; Waugh, 1993].

Regions within the flow that are foliated by the manifolds are well mixed,
while regions from which the manifolds are excluded should be weakly mixed
internally and isolated by mixing barriers from the well-mixed areas. Dashed lines in the last panel of Fig. 3 indicate the mixing barriers in the flow. Manifolds from the dominant hyperbolic points do not cross the mixing barrier lines. The mixing barriers are persistent from day to day. Therefore, very little air is exchanged across those lines by the large-scale flow. The faint stable and unstable manifolds south of the mixing barrier are associated with weak, narrow cat's eyes on the poleward side of the jet.

Relationship of manifolds to ozone and potential vorticity

Figure 6 shows the finite-strain superimposed on the total ozone field for 15 October 1996. Total ozone data are from the Total Ozone Mapping Spectrometer (TOMS) instrument on the Advanced Earth Observing Satellite (ADEOS). Data were obtained from the Ozone Processing Team at NASA Goddard Space Flight Center. Low ozone is indicated by darker values; the Antarctic ozone hole is easily apparent. In order to keep the ozone field visible, this map shows only those pairs of particles that experienced the largest strain (the manifolds). This map is typical of the time period prior to the breakdown of the polar vortex and ozone hole. The most important features to note are the tight layers of manifolds wrapped around the vortex and the tangle of stable and unstable manifolds outside the ozone hole in the surf zone. By contrast, there is a noticeable lack of manifolds inside the ozone hole. Because no lobes extend into the ozone hole, there is no large-scale transport of air in or out of the ozone hole. The interior of the vortex is, thus, a region of weak mixing. This is consistent with Bowman [1995; 1996], who showed that the mixing characteristics of the southern hemisphere polar vortex can largely be explained by simple kinematic models of linear waves superimposed on background flows with shear.

The polar vortex weakens in late spring and early summer (usually
between late October and early December), and eventually breaks down completely. This is accompanied by rapid mixing of air throughout the hemisphere. This is reflected in the manifold geometry, as can be seen in Figure 7, which shows the manifolds on selected days superimposed on the potential vorticity (PV) field at 500 K computed from the UKMO data. On 20 October (upper-left) the PV is low inside the vortex, with a strong gradient at the core of the jet (Compare with Figure 6). The strong mixing associated with the manifolds has homogenized the PV field outside the vortex in the surf zone. (Some traces of stable and unstable manifolds can be seen inside the vortex, both here and in Figure 6. Because a wave with a given phase speed should have critical layers on both side of a jet, this is to be expected.) Twenty days later, on 10 November, the vortex is still intact, but the critical layers inside the vortex are becoming more apparent as the vortex decelerates, the shear decreases, and the cat’s eyes widen. By 15 December the vortex is breaking apart. The vortex itself is split into three parts, and some manifolds are beginning to extend from the inside to the outside of the vortex, indicating rapid stretching and folding of lobes throughout the vortex. Only 10 days later (25 December), the vortex has largely vanished. What remains is an irregular spaghetti-like tangle of manifolds associated with short-lived hyperbolic structures.

Conclusions

We present here a simple and powerful method for locating the stable and unstable manifolds in two-dimensional, incompressible fluid flows. The method is tested by applying it to a simple analytical flow field and to observed stratospheric wind data. The method works by computing the distance between initially-nearby pairs of particles in a flow as a function of time. Particles are initialized on a relatively dense mesh. Pairs of particles that straddle the stable and unstable
manifolds experience large separations or 'finite strains' as the particle positions are integrated forward and backward in time, respectively. The principal advantage of this method for locating manifolds, compared to straddling or contour advection methods, is that no a priori knowledge of the location or motion of the hyperbolic points in the flow is required.

Because of the exponential stretching that occurs near the hyperbolic points, the geometry of the manifolds emerges quickly in the flows tested here. In the simple analytical flow model (3), the manifolds and lobes emerge within a few perturbation periods. The heteroclinic tangle is quickly apparent, and the manifolds can be seen to foliate the stochastic layer between the unmixed interior and exterior of the cat's eyes.

In the case of the southern hemisphere stratospheric circulation, the manifolds emerge within a few days and reveal both the mixing barriers around the vortex and the expected tangle of manifolds in the well-mixed surf zone. The existence and location of the hyperbolic points and the manifolds are confirmed by contour advection calculations, and their geometry is consistent with the observed distribution of quasi-conserved tracers such as ozone and potential vorticity. While it has been known for some time that there is a strong mixing barrier around the southern-hemisphere polar vortex, this calculation reveals for the first time the essential underlying geometrical structures in the flow that are responsible for the existence of both the mixing barrier and the surf zone outside the vortex. The results are quite consistent with earlier kinematic studies [Bowman, 1993a; Chen, 1994] and with the dynamical mechanism for the mixing barrier proposed by Bowman [1996].

The ability to visualize the manifold structure in regional or global flow fields holds considerable promise for understanding mixing and transport in
the atmosphere. First, finite-strain maps are a simple tool to locate and identify well-mixed regions, poorly-mixed regions, and the mixing barriers between them. This should be of use both for theoretical transport studies and for applied problems such as pollution transport. Second, finite-strain maps can provide a good starting point for quantitative transport calculations using lobe dynamics. On a more practical note, knowing the locations of the manifolds in an instantaneous flow field should be of considerable use in planning aircraft missions and field experiments of many kinds. Missions designed to revisit air parcels at later times would have much greater ability to select volumes that are unlikely to undergo large deformation and dispersion. This application will require computation of the stable manifolds using forecast flow fields. One study to be undertaken is the comparison of manifold calculations made using forecast winds with those made in hindsight using analyzed winds. The ability to forecast manifold locations could potentially be a very sensitive test of forecast skill.

There are, of course, limitations to the application of these concepts. The current theory applies to Hamiltonian systems. Many atmospheric flows are three-dimensional, however, and these concepts may need modification to be useful in that context. The application of all of these ideas assumes that there are hyperbolic structures in the flow that persist for sufficiently long time to produce well-defined stable and unstable manifolds [Haller and Poje, 1998]. Strongly-forced flows may have hyperbolic structures that appear and disappear too quickly for them to form lobes. Lobe dynamics methods should be applicable in many situations, however, and should provide powerful new tools for studying a wide range of old and new problems.
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Figure 1. Schematic showing the stable and unstable manifolds in prototypical unperturbed and perturbed Hamiltonian systems. For clarity only two of the four perturbed manifolds are shown for each hyperbolic point.

Figure 2. Schematic showing the numerical technique used to locate the stable manifold. The length of a line segment divided by its initial length is referred to as the finite strain. Pairs of points that straddle the stable manifold experience large finite strains, while pairs of points that straddle the unstable manifold or no manifold experience small finite strains. Forward-in-time calculations can thus be used to locate the stable manifolds. Unstable manifolds are located by similar calculations carried out backward in time.

Figure 3. Finite-strain maps showing the emergence of the stable and unstable manifolds for a flow consisting of a perturbed wave on a linear shear. The hyperbolic point is located at the center of the plotting region. Snapshots of the finite strain are plotted at integral intervals of the period of the perturbation $\tau$.

Figure 4. Finite-strain maps showing the emergence of the stable and unstable manifolds for the observed stratospheric flow on the 500 K potential temperature surface. The longitude coordinate is shifted so as to place one of the hyperbolic points at 270°. Note that the maps are not equally spaced in time.

Figure 5. Finite-strain maps of the southern hemisphere stratosphere on the 500 K potential temperature surface. The finite-strain is computed at $t = \pm 10$ days. The magnitudes of the forward and backward finite-strain are indicated by the
intensity of green and red, respectively. Regions where both the forward and backward strains are large appear yellow. Overlaid on the finite-strain field are two material lines computed using contour-advection techniques (blue and magenta lines). These material lines are initialized approximately 20 days before or after the period shown. Dashed lines in the final panel indicate the approximate positions of mixing barriers in the flow.

Figure 6. Stable and unstable manifolds superimposed on the TOMS total ozone field for 19 October 1996.

Figure 7. Stable and unstable manifolds superimposed on the isentropic potential vorticity field on the 500 K potential temperature surface for selected days in 1996 that span the breakdown of the southern hemisphere polar vortex.
Unperturbed Heteroclinic Structure

Perturbed Heteroclinic Structure

Figure 1
Straddling the stable manifold

Straddling the unstable manifold

Figure 2
Figure 3
Mixing barriers

Figure 5
Figure 6

19 October 1996 –7 days
Figure 7

20 October 1996 -7 days

10 November 1996 -7 days

15 December 1996 -7 days

25 December 1996 -7 days