LOOK-BACK TIME, THE AGE OF THE UNIVERSE, AND THE CASE FOR A POSITIVE COSMOLOGICAL CONSTANT

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ABSTRACT

We present explicit expressions for the calculation of cosmological look-back time, for zero cosmological constant and arbitrary density parameter Ω, which, in the limit as redshift becomes infinite, give the age of the universe. The case for non-zero cosmological constant is most easily solved via numerical integration. The most distant objects known at present (approaching redshift $z = 5$) have implied ages of ≈1–2 Gyr after the Big Bang. The range of such age is narrow, in spite of a variety of cosmological models one might choose. We give a graphical representation of a variety of cosmological models and show that a wide range of Hubble constants and values of the age and density of the universe compatible with modern studies are consistent with adoption of a positive cosmological constant.

RÉSUMÉ

Nous présentons dans cet article des expressions explicites permettant le calcul du temps de recul cosmologique ("look-back time"), pour une constante cosmologique égale à zéro et un paramètre de densité Ω arbitraire. Lorsque la limite du décalage vers le rouge tend vers l'infini, on obtient de ces expressions l'âge de l'Univers. Le cas d'une constante cosmologique différente de zéro est plus facile à résoudre via une intégration numérique. Les objets les plus éloignés connus jusqu'à présent (décalage vers le rouge approchant $z = 5$) impliquent des âges approximatifs de ≈1–2 Ga après le Big Bang. Cet intervalle de temps est étroit quelque soit le modèle cosmologique qu'on peut choisir. Nous présentons une représentation graphique d'une variété de modèles cosmologiques et nous montrons qu'un grand intervalle de valeurs compatibles avec des études récentes, pour la constante de Hubble, l'âge et la densité de l'Univers, sont consistantes avec une constante cosmologique positive.

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Assuming the correctness of the standard Big Bang scenario (Peebles *et al.* 1991), the redshift (z) of a distant galaxy or quasar can be related (Longair 1984, equation 15.19) to the cosmic scale factor of the universe (R), as follows:

$$R = \frac{1}{1+z}. \tag{1}$$

$R = 1$ at the present epoch. An object at $z = 1$ emitted its light when the universe was half its present scale ($R = 0.5$). *How long ago* the light was emitted (the look-back time τ) depends on the dynamics of the universe.

The look-back time (following Longair 1984, equation 15.47), assuming for now a zero cosmological constant, is:

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$$\tau = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')^2(\Omega z' + 1)^{1/2}}. \tag{2}$$

Here the density parameter $\Omega = \rho_0/\rho_c$ is the ratio of the density of the universe to the critical density

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-30} H_0^2 \text{ kg m}^{-3}, \tag{3}$$

where H_0 is Hubble's constant in km s^{-1} Mpc^{-1} and G is the gravitational constant.

If the vacuum energy density of the universe is zero (*i.e.* if Einstein's cosmological constant $\Lambda = 0$), $\Omega < 1$ implies a universe with negative curvature which will expand forever; $\Omega > 1$ implies a universe with positive curvature which will eventually recollapse; and $\Omega = 1$ (the "Einstein-de Sitter universe") implies a universe with flat geometry which will expand forever, but which will eventually reach zero expansion rate.

In equations (2) and (3), H_0 is expressed in the usual practical observational units of km s^{-1} Mpc^{-1}, and its value is believed to be somewhere between 50 and 100 km s^{-1} Mpc^{-1}. However, note that H_0 is dimensionally similar to the reciprocal of a time. The reciprocal of Hubble's constant is called the Hubble time T_H:

$$T_H = H_0^{-1}, \tag{4a}$$

where T_H is expressed in s and H_0 in s^{-1}. If H_0 is expressed in km s^{-1} Mpc^{-1} and T_H in gigayears (1 Gyr = 1 billion years = 10^9 years), this becomes

$$T_H = \frac{977.8}{H_0}. \tag{4b}$$

For $H_0 = 50$ km s^{-1} Mpc^{-1}, the Hubble time is 19.6 billion years; for $H_0 = 100$ km s^{-1} Mpc^{-1}, it is 9.8 billion years. It is important to note that the age of the universe can be less than or greater than the Hubble time, depending on the value of Ω and whether or not Einstein's cosmological constant (Λ) is zero. But if $\Lambda = 0$, the age of the universe is necessarily less than the Hubble time, because of the gravitational "braking effect" of the matter in the universe. (In that case the present expansion rate of the universe is necessarily less than it was in the past.)

For an empty universe ($\Omega = 0$) equation 2 has the simple solution:

$$\tau = T_H \left(\frac{z}{1+z}\right). \tag{5}$$

For $\Omega = 1$, equation 2 has another simple solution:

$$\tau = \frac{2}{3} T_H \left[1 - \frac{1}{(1+z)^{3/2}} \right] . \tag{6}$$

For $\Omega > 1$, we can solve equation 2 with the help of Gradshteyn and Ryzhik (1965):

$$\frac{\tau}{T_H} = \frac{(1+\Omega z)^{1/2}}{(\Omega - 1)(1+z)} + \frac{\Omega}{(\Omega-1)^{3/2}} \tan^{-1}\left[\left(\frac{1+\Omega z}{\Omega - 1} \right)^{1/2} \right] - \frac{1}{(\Omega - 1)} - \frac{\Omega}{(\Omega-1)^{3/2}} \tan^{-1}\left[\left(\frac{1}{\Omega - 1} \right)^{1/2} \right] . \tag{7}$$

And for $0 < \Omega < 1$, it can be shown (Gradshteyn and Ryzhik 1965) that the look-back time is:

$$\frac{\tau}{T_H} = \frac{-(1+\Omega z)^{1/2}}{(1-\Omega)(1+z)} - \frac{\Omega}{2(1-\Omega)^{3/2}} \ln\left[\frac{(1+\Omega z)^{1/2} - (1-\Omega)^{1/2}}{(1+\Omega z)^{1/2} + (1-\Omega)^{1/2}} \right] + \frac{1}{(1-\Omega)} + \frac{\Omega}{2(1-\Omega)^{3/2}} \ln\left[\frac{1-(1-\Omega)^{1/2}}{1+(1-\Omega)^{1/2}} \right] . \tag{8}$$

While equations 5 and 6 are often used to describe look-back time, equation 8 is a much more realistic one to use, since many studies imply $\Omega \approx 0.1$ (*e.g.* Ford *et al.* 1981, Press and Davis 1982).

Other solutions mathematically equivalent to equations 7 and 8 have appeared in the literature. For the $\Omega > 1$ case it is easy to show that equation 23 of Sandage's (1961b) parametric method is equal to equation 3.24 *minus* equation 3.22 of Kolb and Turner (1990). For the $0 < \Omega < 1$ case it is not difficult to show that Sandage's equation 24 equals Kolb and Turner's equation 3.25 *minus* equation 3.23, providing one uses the identity:

$$\cosh^{-1}\theta = \sinh^{-1}(\sqrt{\theta^2 - 1}).$$

Then, using trigonometric identities for $\cos(x-y)$, $\cos(x/2)$ and $\sin(x/2)$ it is possible to show that Kolb and Turner's equations 3.24 *minus* 3.22 equals our equation 7 above. (Only the *sum* of the second and fourth terms of our equation 7 equals the *sum* of two of the terms from Kolb and Turner.) For the $0 < \Omega < 1$ case it can be shown that our equation 8 equals Kolb and Turner's equation 3.25 *minus* equation 3.23 providing we use the identity:

$$\cosh^{-1}\theta = \ln[\theta + (\theta^2 - 1)^{1/2}].$$

Conveniently, our equation 8 works out term by term with the solution of Kolb and Turner. Finally, we note that Schmidt and Green (1983) give an expression equivalent to our equation 8 divided by equation 15 (below).

For $\Lambda \neq 0$, following Carroll *et al.* (1992, equation 16) or Krisciunas (1993, equation 27), equation 2 becomes

$$\frac{\tau}{T_H} = \int_0^z \frac{dz'}{(1+z')^2 \left\{ \Omega z' + 1 - \lambda \left[\frac{2z' + (z')^2}{(1+z')^2} \right] \right\}^{1/2}}, \tag{9}$$

where

$$\lambda = \frac{\Lambda c^2}{3H_0^2} \tag{10}$$

is the "reduced" (dimensionless) cosmological constant. Defined this way $1/\sqrt{\Lambda}$ has dimensions of length. This is the "length scale over which the gravitational effects of a nonzero vacuum energy density would have an obvious and highly visible effect on the geometry of space and time" (Abbott 1988). In general equation 9 is most easily solved by means of numerical integration.

In figure 1 we show the look-back time *versus* redshift for four scenarios: $H_0 = 50$ and $\Omega = 0.0$, 0.3, and 1.0 (with $\lambda = 0$); and $H_0 = 80$, $\Omega = 0.12$, $\lambda = 0.88$. The advantages of the last model will become clear shortly.

In figure 2 we show the look-back time, measured in terms of the age of the universe (using equations 14, 15, and 17 below).

According to the inflationary scenario (Narlikar and Padmanabhan 1991), the universe should have flat geometry, a condition satisfied by:

$$\Omega + \lambda = 1. \tag{11}$$

So either: 1) $\Omega = 1$ (and we cannot find the missing mass even from the dynamics of clusters of galaxies); 2) $\Omega \neq 1$ and $\lambda = 1 - \Omega$; or 3) the inflationary paradigm is incorrect.

In the limit as $z \longrightarrow \infty$ for equations 5 through 8, we obtain expressions for the age of the universe (T_0), assuming $\lambda = 0$:

$$T_0 = T_H. \quad (\Omega = 0) \tag{12}$$

$$T_0 = \frac{2}{3} T_H. \quad (\Omega = 1) \tag{13}$$

$$\frac{T_0}{T_H} = \frac{-1}{(\Omega - 1)} + \frac{\Omega}{(\Omega - 1)^{3/2}} \sin^{-1} \left[\left(\frac{\Omega - 1}{\Omega} \right)^{1/2} \right]. \quad (\Omega > 1) \tag{14}$$

$$\frac{T_0}{T_H} = \frac{1}{(1 - \Omega)} - \frac{\Omega}{2(1 - \Omega)^{3/2}} \ln \left[\frac{2 - \Omega}{\Omega} + \frac{2(1 - \Omega)^{1/2}}{\Omega} \right]. \quad (0 < \Omega < 1) \tag{15}$$

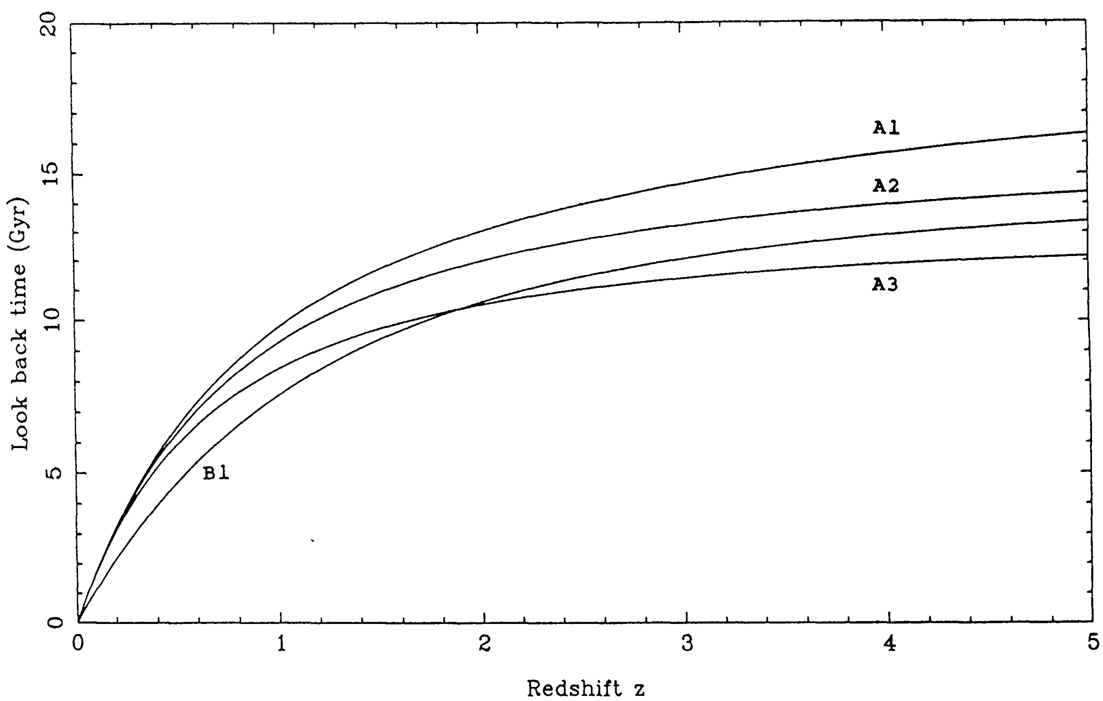

FIG. 1—Look-back time *versus* redshift for four cosmological models. A1, A2, and A3 have $H_0 = 50$, $\lambda = 0$, and $\Omega = 0.0$, 0.3, and 1.0, respectively. Model B1 has $H_0 = 80$, $\lambda = 0.88$, $\Omega = 0.12$.

Our equation 14 is mathematically equivalent to equation 61 of Sandage (1961a), given $q_0 = \Omega/2$ for $\lambda = 0$. Our equation 15 is equal to equation 65 of Sandage (1961a) given the same definition of q_0.

It should be noted that equations 14 and 15 are more easily derived by integrating an expression for the Hubble constant divided by the rate of change of the cosmic scale factor over the age of the universe:

$$\frac{T_0}{T_H} = \int_0^{T_0} \frac{H_0}{(dR/dt)} = \int_0^1 \frac{dR}{\left[\frac{\Omega}{R} + 1 - \Omega + \lambda(R^2 - 1)\right]^{1/2}}, \tag{16}$$

as is done in Krisciunas (1993, Appendix A).

For the special case of $0 < \Omega < 1$ and $\Omega + \lambda = 1$, equation 16 can be solved by making the substitution:

$$R = \left(\frac{\Omega}{1 - \Omega}\right)^{1/3} \sinh^{2/3}\theta$$

and by using the identity

$$\sinh^{-1}\theta = \ln[\theta + (\theta^2 + 1)^{1/2}].$$

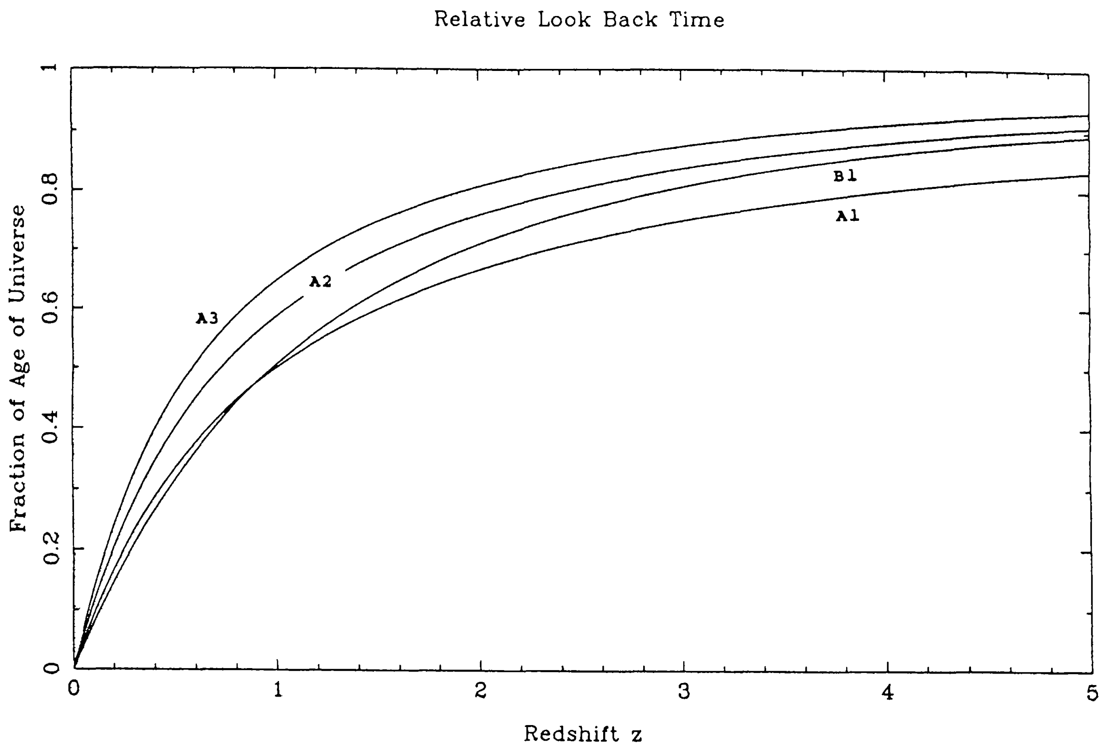

FIG. 2—Look-back time, measured in terms of the age of the universe, *versus* redshift, A1, A2, and A3 have $\lambda = 0$, and $\Omega = 0.0$, 0.3, and 1.0, respectively. Model B1 has $\lambda = 0.88$, $\Omega = 0.12$.

The solution is:

$$\frac{T_0}{T_H} = \frac{2}{3} \frac{1}{(1-\Omega)^{1/2}} \ln \left[\frac{1 + (1-\Omega)^{1/2}}{\Omega^{1/2}} \right]. \quad (0 < \Omega < 1;\ \Omega + \lambda = 1) \quad (17)$$

This is equivalent to equation 3.32 of Kolb and Turner (1990).

Assuming values for Ω and λ allows us to calculate the age of the universe in Hubble time. Assuming H_0 allows us to calculate the Hubble time in billions of years. In figure 3, assuming $\lambda = 0$, we show various loci of points corresponding to a range of Hubble constants, values of the age of the universe, and density parameters.

Now a common idea associated with the construction of the new 8-10 metre class telescopes is that they will "allow us to see 13 billion years into the past, about 1 to 2 billion years after the Big Bang". It turns out that only the second half of that statement is necessarily correct.

The largest observed quasar redshift is presently 4.897 (Schneider *et al.* 1991), and the largest observed galaxy redshift is 3.8 (Chambers *et al.* 1990). Let us consider the look-back time of objects of redshift 5, regarded by many to be the beginning of the era of galaxy formation (Ellis 1987). Using equations 6, 8, 13, 15 and 17, and numerical solutions to 9, we give a number of examples in

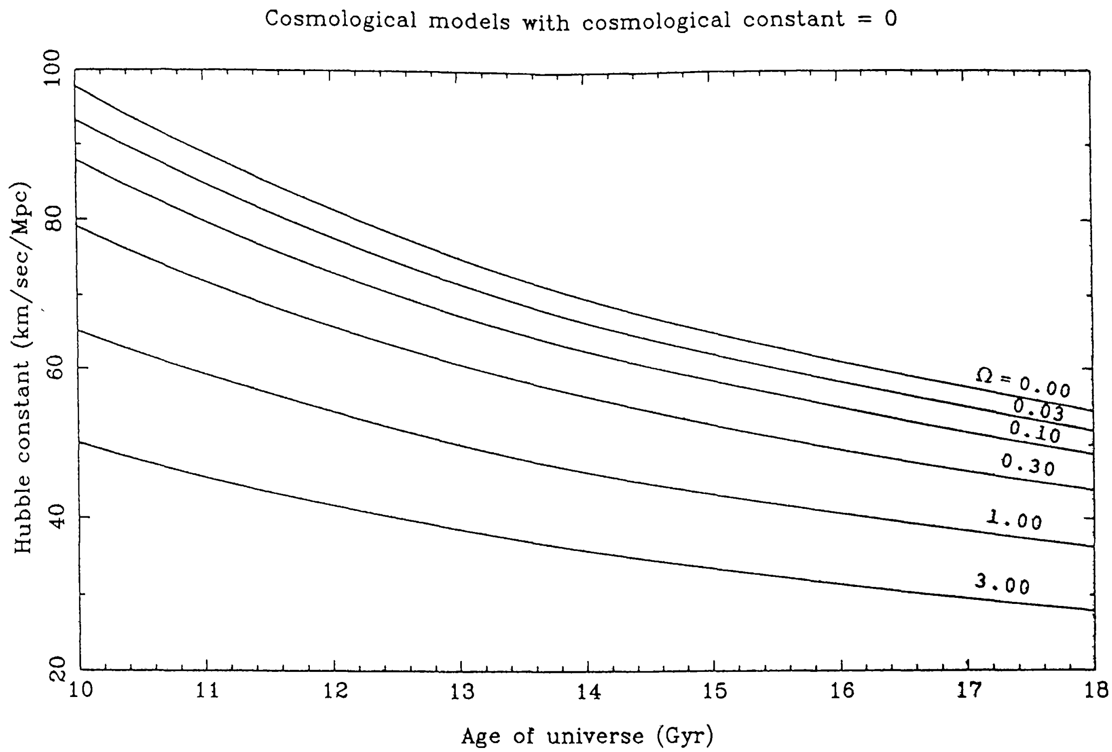

FIG. 3—Loci of constant Ω for a range of Hubble constants and values of the age of the universe (T_0). These models assume a zero cosmological constant.

Table I. For all these cases the age of a $z = 5$ object, reckoned since the Big Bang, is about 1 to 2 billion years, even though the look-back time (depending on the model) ranges from 9.68 to 15.49 Gyr. Because of uncertainties in choosing the right model, we do not really have an accurate handle on the *look-back time* of a $z = 5$ object (because of the range of derived values of the *age* of the universe), but we can say with some certainty that that light originated 1 to 2 Gyr after the Big Bang.

Let us now consider the implications of the loci of points in figure 3. Following Fowler (1987), we agree that our understanding of single star evolution is on a reasonably firm footing, though it should be pointed out that changes in the assumed abundances of certain elements (*e.g.* oxygen) have led to significant revisions in the ages of the oldest known stars in our Galaxy. We could have extended the loci of points in figure 3 further to the right – the universe could be older than 18 Gyr. That would further strengthen the idea that the Hubble constant must have a "small" value or that $\lambda > 0$ (see below).

One often estimates the age of the universe by deriving the age of the oldest stars in our Galaxy and adding a "sensible" incubation time for our Galaxy. Let us suppose for a moment that the age of the oldest stars in our Galaxy is equal to the value derived in the careful study of the globular cluster 47 Tucanae by

TABLE I

LOOK-BACK TIME (τ) OF OBJECT WITH REDSHIFT $z = 5$

H_0 km s^{-1} Mpc^{-1}	Ω	λ	T_0 (Gyr)	$\tau(z = 5)$ (Gyr)	$T_0 - \tau$ (Gyr)
50	1.0	0.0	13.04	12.15	0.89
50	0.1	0.0	17.56	15.49	2.07
80	0.1	0.0	10.98	9.68	1.30
70	0.2290	0.7710	14.50	13.18	1.32
80	0.1186	0.8814	15.00	13.40	1.60
90	0.0613	0.9387	15.50	13.53	1.97

Hesser *et al.* (1987), who find an age of 13.5 ± 0.5 Gyr (internal error; ±2.0 Gyr external error) if our Galaxy formed at the same time as those galaxies that we would observe now at $z = 5$, we can take a representative age-since-the-Big-Bang of 1.5 Gyr from Table I, and add it to the age of 47 Tucanae, 13.5 Gyr, giving an age of the universe, T_0, of 15.0 Gyr.

What is the most valid value to adopt for the Hubble constant? This is a subject too complex to investigate in any detail here, but values of H_0 based on the infrared Tully-Fisher relation cluster around 80 km s^{-1} Mpc^{-1}. Okamura and Fukugita (1992) give a graphical summary. Two recent reviews (Jacoby *et al.* 1992; van den Bergh 1992) give $H_0 = 80 \pm 11$ and $H_0 = 76 \pm 9$, respectively. The Sandage-Tammann value of about 50 km s^{-1} Mpc^{-1} also has its strong advocates (though they seem at present to be the minority).

If $T_0 = 15.0$ Gyr and $H_0 = 80$ km s^{-1} Mpc^{-1} an inspection of figure 3 immediately indicates that the density of the universe must be less than zero! For $\lambda = 0$ and $\Omega \geq 0$, if $T_0 = 15.0$ Gyr, $H_0 \leq 65$. If $H_0 = 80$, then $T_0 \leq 12.2$ Gyr. So either (1) the "most sensible" value of Hubble's constant is wrong; or (2) the oldest stars formed before the epoch of $z = 5$; or (3) we need to revise the models from which we derive the ages of the oldest stars; or (4) $\lambda \neq 0$. Even if we adopt (2) and 47 Tuc formed earlier than galaxies at $z = 5$, the universe must still be older than the age of 47 Tuc, *i.e.* 13.5 Gyr. There is some conflict between $T_0 \geq 13.5$ Gyr and $T_0 \leq 12.2$ Gyr. This conflict is not irreconcilable because of the uncertainty of the 47 Tuc age, although van den Bergh (1992) points out that other studies of globular clusters give ages up to 17 Gyr, which makes the problem worse. But as long as determinations of Hubble's constant give $H_0 \simeq 80$, we must take seriously the possibility that $\lambda > 0$.

A positive cosmological constant is the same as attributing "repulsive" force to the vacuum. As the universe expands, this repulsive force becomes stronger and stronger, as there is more space within which it works. To illustrate the extent of the effect, consider the basic idea of equation 16:

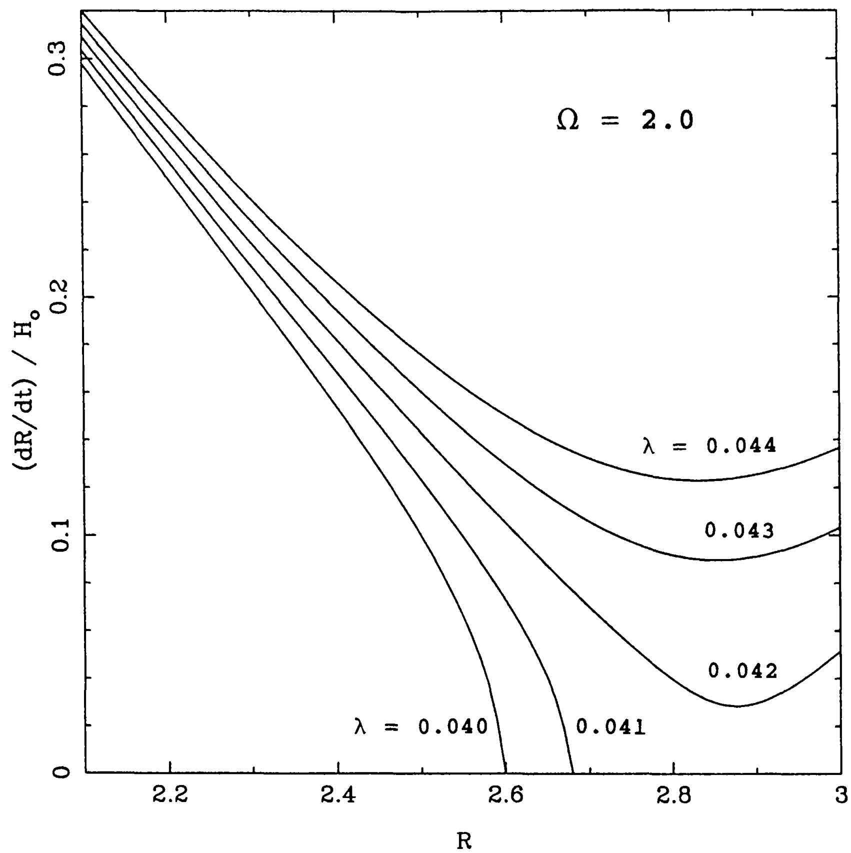

FIG. 4—The rate of change of the cosmic scale factor (divided by the present value of the Hubble constant) *versus* the cosmic scale factor for $\Omega = 2.0$ and various small positive values of the scaled cosmological constant. If the curve intersects the horizontal axis, the model indicates a universe that reaches a maximum scale, then recollapses. Otherwise, the model indicates a universe that expands forever.

$$\frac{(dR/dt)}{H_0} = \left[\frac{\Omega}{R} + 1 - \Omega + \lambda(R^2 - 1)\right]^{1/2}. \quad (18)$$

In figure 4 we graph this function for $\Omega = 2.0$ and various small positive values of λ. One can see that λ only need be as large as $\approx$+0.042 for the rate of change of R to be positive always. In other words, the universe can expand forever, even in cases where $\Omega + \lambda$ is significantly greater than 1.

The effect of a positive cosmological constant is to take the loci of points in figure 3 and pull them up in the diagram (see figure 5). Scanning through the examples in Table I, if $\Omega = 0.1186$, and $\lambda = 1 - \Omega$, we find that $T_0 = 15.0$ Gyr if $H_0 = 80$ km s^{-1} Mpc^{-1}. This satisfies the theoreticians who advocate the verisimilitude of the inflationary scenario, while also allowing our understanding of stellar evolution and the values of H_0 based on the infrared Tully-Fisher

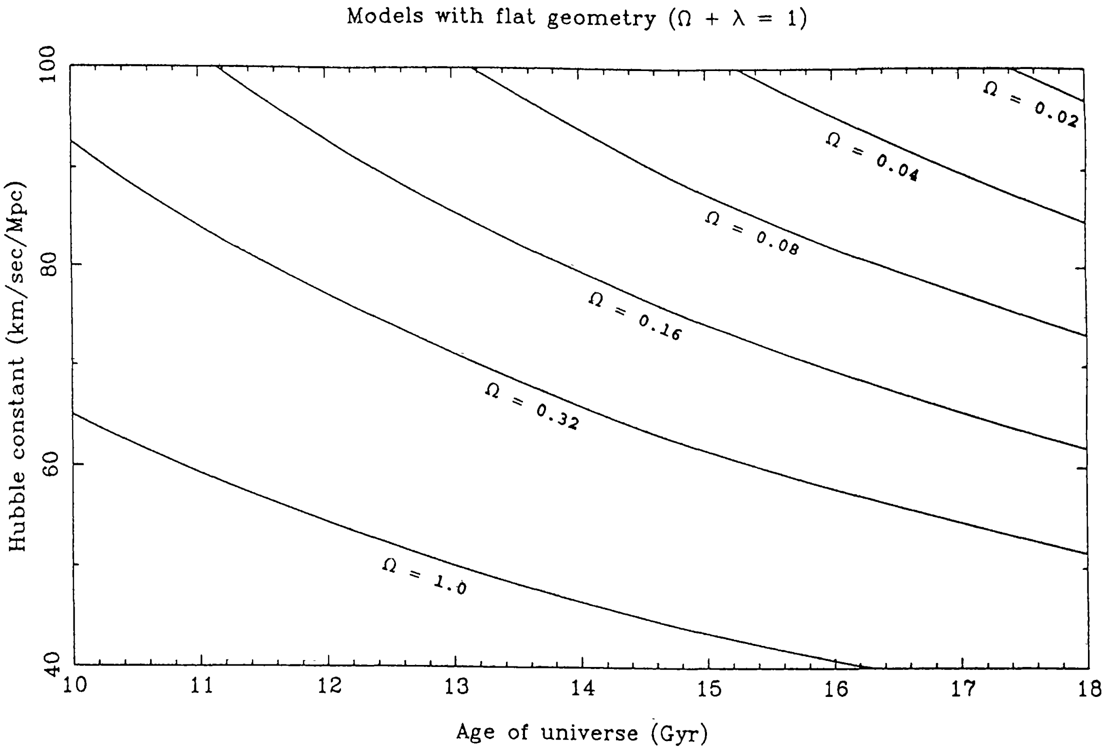

FIG. 5—Loci of constant Ω for a range of Hubble constants and values of the age of the universe (T_0), assuming the correctness of the condition stipulated by the inflationary scenario, $\Omega + \lambda = 1$.

relation to stand as valid. But perhaps this is only a modern example of the ancient Greek method of "saving the phenomenon".[1]

Allowing T_0 to range from 14.5 to 15.5 Gyr, allowing H_0 to range from 70 to 90 km s^{-1} Mpc^{-1}, and assuming that $\Omega + \lambda = 1$, gives $0.06 < \Omega < 0.23$. Such a range of Ω is in accord with studies of large scale structures in the universe (*e.g.* Ford *et al.* 1981, Press and Davis 1982), as well as Big Bang nucleosynthesis studies (Boesgaard and Steigman 1985). Also, $\lambda \approx 0.8$–0.9

[1]For example, the ancient Greeks asserted that a planet's motion must be *uniform* and *circular*. The planet moves uniformly on its circular deferent around the Earth. But, because of the observed effect of retrograde motion, they "saved the phenomenon" of uniform, circular motion by inventing the idea of an epicycle, a smaller circle that turned at a faster rate than the deferent and whose centre rode around on the deferent. Similarly, they invented *eccentric* deferents and *equants* to match the observed planetary positions better with the ephemerides. (See Krisciunas 1988.) As a modern example of "saving the phenomenon", to explain the advance of the perihelion of the orbit of Mercury and retain Newtonian gravity, astronomers postulated the existence of the planet Vulcan. A postulated additional planet certainly worked for the anomalies of the motion of Uranus, which led to the discovery of Neptune. However, for Mercury's motion what was needed was a new theory of gravity, namely General Relativity. We use these two examples to point out that if *ad hoc* hypotheses such as the cosmological constant are "required", perhaps the solution resides in "proper" correction of some of the other data, or a significant revision of the theoretical underpinnings.

does not contradict a recent constraint on the cosmological constant derived from the study of gravitational lens galaxies (Kochanek 1992), that $\lambda < 0.9$.

In choosing a particular cosmological model one must pick *families* of parameters (T_0, H_0, Ω, λ) that satisfy particular mathematical relationships. Not all combinations are allowed. Sensible values of T_0 and H_0 point to a positive value for the cosmological constant. (See Tayler 1986 for a similar discussion.) However, various theoretical discussions (*e.g.* Weinberg 1989) stipulate that the cosmological constant should be identically zero.

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