### Useful Formulas:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(t) = \frac{dx(t)}{dt} )</td>
<td>Velocity</td>
</tr>
<tr>
<td>( a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} )</td>
<td>Acceleration</td>
</tr>
<tr>
<td>( v(t) = \int a(t) , dt )</td>
<td>Position</td>
</tr>
<tr>
<td>( x(t) = \int v(t) , dt )</td>
<td></td>
</tr>
<tr>
<td>(-\frac{dW(r)}{dr})</td>
<td>Force</td>
</tr>
<tr>
<td>( \sum F_x = ma_x )</td>
<td>Net Force</td>
</tr>
<tr>
<td>( \sum F_y = ma_y )</td>
<td></td>
</tr>
<tr>
<td>( W_{a \rightarrow b} = \int_a^b \vec{F}_b \cdot d\vec{x} )</td>
<td>Work</td>
</tr>
<tr>
<td>( W_{a \rightarrow b} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 )</td>
<td>Kinetic Energy</td>
</tr>
<tr>
<td>( \sum E_i = \sum E_f )</td>
<td>Conservation of Energy</td>
</tr>
<tr>
<td>( v_2^2 - v_1^2 = 2a(x_2 - x_1) )</td>
<td></td>
</tr>
<tr>
<td>( \vec{P}(t) = \int \vec{F}(t) , dt = \int m\vec{a}(t) , dt = m\vec{v}(t) )</td>
<td>Conservation of Momentum</td>
</tr>
<tr>
<td>( \sum \vec{P}_i = \sum \vec{P}_f )</td>
<td></td>
</tr>
</tbody>
</table>

### Scores:

<table>
<thead>
<tr>
<th>Question</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (25)</td>
<td></td>
</tr>
<tr>
<td>2 (25)</td>
<td></td>
</tr>
<tr>
<td>3 (25)</td>
<td></td>
</tr>
<tr>
<td>4 (25)</td>
<td></td>
</tr>
<tr>
<td>Bonus (25)</td>
<td></td>
</tr>
<tr>
<td>Total (100) + Bonus</td>
<td></td>
</tr>
</tbody>
</table>
1. A freely moving particle of charge $q_1$ and mass $M_1$ is travelling with an initial velocity $-v_0$ towards a fixed charge $q_2$ a distance $L$ away. The repulsive force on $q_1$ is:

$$F_{12}(x) = \frac{B}{x^2}$$

where $B = kq_1q_2$ is some known constant. Use $B$ (and not $kq_1q_2$) when solving the problem.

![Diagram of particles](image)

a. What is the potential energy function as a function of $x$? Hint: make sure the sign is correct.

$$U(x) = -\int F_2(x) \, dx = -\int \frac{B}{x^2} \, dx = \frac{B}{x} + C$$

b. How far does the charged particle $q_1$ travel before turning around?

**Work energy theorem:**

$$\frac{1}{2} M_1 v_0^2 - \frac{1}{2} M_1 v_0^2 = -(U(x_{\text{min}}) - U(L)) \Rightarrow -\frac{1}{2} M_1 v_0^2 = -B \left( \frac{1}{x_{\text{min}}} - \frac{1}{L} \right) \Rightarrow \frac{B}{x_{\text{min}}} = \frac{1}{2} M_1 v_0^2 + \frac{B}{L}$$

$$x_{\text{min}} = \frac{BL}{2B + \frac{1}{2} M_1 v_0^2 L}$$

c. What is the maximum velocity $v_{\infty}$ that $q_1$ can attain if it is allowed to accelerate forever?

$$\frac{1}{2} M_1 V_{\infty}^2 - \frac{1}{2} M_1 v_0^2 = -(U(\infty) - U(L))$$

$$\frac{1}{2} M_1 V_{\infty}^2 = \frac{1}{2} M_1 v_0^2 + \frac{B}{L}$$

$$V_{\infty}^2 = v_0^2 + \frac{2B}{M_1 L}$$

$$v_0 = \sqrt{v_0^2 + \frac{2B}{M_1 L}}$$
2. (25 points) A pendulum of mass \( M \) has a rigid rod of length \( L \). The wind blows and provides a force:

\[
\bar{F}_w = -M \vec{b} \vec{x}
\]

In this case \( \bar{F}_w \) is not a function of velocity, so \( \vec{b} \) is constant. The gravitational force is of course:

\[
\bar{F}_g = -Mg\hat{y}
\]

a. Draw a free body diagram for the pendulum in its equilibrium position. The equilibrium position is where it is not being accelerated.

b. What is the angle \( \theta_e \) that the equilibrium position makes with the \(-\vec{y}\) axis?

\[
\tan \theta_e = \frac{F_w}{F_g} = \frac{b}{g} \Rightarrow \theta_e = \tan^{-1} \left( \frac{b}{g} \right)
\]

5 pts

c. How much work does it take to move the pendulum from its equilibrium position \( \theta_e \) to an angle \( \theta = 0 \)? You can leave the answer in terms of \( \theta_e \).

\[
W_{\theta_e \to 0} = \int_{x_w}^{x_h} F_w \, dx + \int_{y_h}^{y_h} F_g \, dy = -M b x \bigg|_{x_w}^{x_h} - M g y \bigg|_{y_h}^{y_h} = -M b w + M g h = M \left[ g L (1 - \cos \theta_e) - b L \sin \theta_e \right] = M L \left[ g (1 - \cos \theta_e) - b \sin \theta_e \right]
\]

5 pts

d. If at \( \theta = 0 \) the mass on the pendulum has a velocity \( v_0 \vec{x} \), what is the magnitude of the velocity \( v_f \) at \( \theta_e \)? Leave the answer in terms of \( \theta_e \). Hints: The scalar quantity \( v_f \) happens to be maximum speed.

\[
W_{0 \to \theta_e} = \frac{1}{2} M v_0^2 - \frac{1}{2} M v_f^2 = -W_{\theta_e \to 0}
\]

\[
\frac{1}{2} M v_f^2 = \frac{1}{2} M v_0^2 - W_{\theta_e \to 0}
\]

\[
v_f^2 = v_0^2 - 2L \left[ g (1 - \cos \theta_e) - b \sin \theta_e \right]
\]

\[
v_f = \sqrt{v_0^2 - 2L \left[ g (1 - \cos \theta_e) - b \sin \theta_e \right]}
\]

5 pts
3. (25 points) A person of mass $M$ hangs from a rope in a pendulum system as shown. One end of the rope is attached to a spring with spring constant $k$. The spring is allowed to stretch from its natural equilibrium (zero force) position to a displacement $\Delta y$

a. What is the displacement $\Delta y$ of the spring from its equilibrium?

\[
\begin{align*}
\sum F_y &= 0 \\
T \cos \theta &= T \\
T \sin \theta &= F_s \\
T &= \frac{Mg}{2 \sin \theta} \\
F_s &= -k \Delta y \\
\Delta y &= \frac{Mg}{2k \sin \theta}
\end{align*}
\]

b. How much work does the spring do on the rope as it stretches from its natural equilibrium to a displacement $\Delta y$?

\[
W_{0 \rightarrow \Delta y} = \int_0^{\Delta y} F_s \, dy = -\int_0^{\Delta y} ky \, dy
\]

\[
= -\frac{ky^2}{2} \bigg|_0^{\Delta y} = -k \frac{\Delta y^2}{2}
\]

\[
= -\frac{k}{2} \left( \frac{Mg}{2k \sin \theta} \right)^2
\]
4. (25 points) **The Bowling Alley.** After a gutter ball (or strike or spare), the bowling ball of mass $M$ is lifted to a height $H$ and released from rest onto a frictionless track on the right side of the bowling lane. The bowling ball travels to the Return Station where is slowed down to a return speed $v_R$ at a height $h$ by slowly rotating rubber belts (a friction based assembly).

![Diagram of the bowling alley](image)

(a) How much energy does it take to slow the bowling ball down to the return speed $v_R$?

**Work energy theorem**

\[ W_{\text{net}} = \frac{1}{2} M v_f^2 - \frac{1}{2} M v_i^2 \]

\[ \Rightarrow E_{RS} = -W_{fr} = MgH - Mgh - \frac{1}{2} M v_R^2 \]

(b) If the return station has to slow the ball down in a period of time $\Delta t$, what is the average power $\langle P \rangle$ consumed by the Return Station?

\[ \langle P \rangle = \frac{\Delta W}{\Delta t} = \frac{-W_{fr}}{\Delta t} = \frac{M(gH-gh-\frac{1}{2}v_R^2)}{\Delta t} \]

(c) If the Return Station were not there to slow the ball down, what distance $L$ away from the end of the rail would the ball land? Assume that the height of the incline of angle $\theta$ and the place where the ball lands are both a height $h$.

**Conservation of energy**

\[ \frac{1}{2} M v_h^2 - \frac{1}{2} M v_i^2 = -(Mgh - Mgh) \]

\[ v_h^2 = 2g(H-h) \]

\[ v_h = \sqrt{2g(H-h)} \]

\[ a_y = -g \]

\[ v_y = -gt + v_y = -gt + v_h \sin \theta \]

\[ y(t) = -\frac{1}{2} gt^2 + v_h \sin \theta t + h \]

\[ y(L) = h \rightarrow 0 = -\frac{1}{2} g t_L^2 + v_h \sin \theta L \]

\[ t_L = \frac{2v_h \sin \theta}{g} \]

\[ x(t) = 0 \]

\[ v_x = v_{hx} = v_h \cos \theta \]

\[ x(t) = v_h \cos \theta t_L = L \]

\[ \frac{v_h^2 \cos \theta \sin \theta}{g} = \frac{4(H-h) \cos \theta \sin \theta}{g} = L \]
(25 bonus) This is a one dimensional problem. Two hockey pucks of mass $M_1$ and $M_2$ are on an icy frictionless surface and are being blown by the wind in the $-\hat{X}$ direction. $M_1$ is initially moving to the right with a velocity $v_0$. At the same time, $M_2$ is released from rest at a distance $L$ away from $M_1$. The force of the wind on $M_1$ and $M_2$ are, $F_1 = -M_1b$ and $F_2 = -M_2b$ respectively where $b$ is a known constant.

(a) What is the momentum of each puck as a function of time?

$$p_{1i}(t) = M_1(bt + v_0)$$

$$p_{2i}(t) = -M_2 bt$$

$$p_i(t) = \int F_i(t) dt = -M_i bt + C_i$$

$$=-M_1 bt + M_1 v_0$$

$$=M_1 (-bt + v_0)$$

$$p_{2i}(t) = \int F_2(t) dt = -M_2 bt + C_2$$

$$=-M_2 bt$$

(b) If $v_0 = \sqrt{bL}$, what is the momentum of each puck right before (the exact moment before) they collide? In the final answer, you should replace all $v_0$ with $\sqrt{bL}$ and simplify.

$$p_{1c} = 0$$

$$V_1(t) = \frac{p_{1i}(t)}{M_1} = -bt + v_0$$

$$v_{1c} = \sqrt{b} v_0$$

$$X_{1i}(t) = \int V_1(t) dt = -\frac{1}{2} bt^2 + v_0 t$$

$$X_{1c}(t) = X_{2c}(t)$$

$$p_{2c} = -M_2 \sqrt{bl}$$

$$V_2(t) = \frac{p_{2i}(t)}{M_2} = -bt$$

$$v_{2c} = \sqrt{b} v_0$$

$$X_{2i}(t) = \int V_2(t) dt = -\frac{1}{2} bt^2 + L$$

$$X_{1c}(t_c) = X_{2c}(t_c)$$

$$\frac{1}{2} b v_0^2 + v_0 t_c = -\frac{1}{2} b v_0^2 + L$$

$$t_c = \frac{L}{v_0} = \frac{\sqrt{L}}{b}$$

$$p_{2c} = M_2 (-b \sqrt{L})$$

$$= -M_2 v_0$$
c. Immediately after the collision the wind stops blowing (no more force). What is the momentum of puck 1 after they collide elastically?

\[ P_{1f} = -2 \left( \frac{M_1 M_2}{M_1 + M_2} \right) V_0 \]

**I. Conservation of energy**

\[ \frac{1}{2} M_2 V_0^2 = \frac{1}{2} M_1 V_{1f}^2 + \frac{1}{2} M_2 V_{2f}^2 = \frac{P_{1f}^2}{2M_1} + \frac{P_{2f}^2}{2M_2} \]

**II. Conservation of momentum**

\[ -M_2 V_0 = M_1 V_{1f} + M_2 V_{2f} = P_{1f} + P_{2f} \]

\[ P_{2f} = -(M_2 V_0 + P_{1f}) \quad \text{plug this into energy equation} \]

\[ M_2 V_0^2 = \frac{P_{1f}^2}{M_1} + \frac{(M_2 V_0 + P_{1f})^2}{M_2} \]

\[ = \frac{P_{1f}^2}{M_1} + M_2 V_0^2 + 2 M_2 V_0 P_{1f} + P_{1f}^2 \]

\[ = M_2 V_0^2 + 2 V_0 P_{1f} + P_{1f}^2 \left( \frac{1}{M_1} + \frac{1}{M_2} \right) \]

\[ 0 = 2 V_0 + P_{1f} \left( \frac{M_1 + M_2}{M_1 M_2} \right) \]

\[ P_{1f} = -2 \left( \frac{M_1 M_2}{M_1 + M_2} \right) V_0 \]