Math 166 - Week in Review #11

Section 9.1 - Markov Chains

- Markov Process (or Markov Chain) - a special class of stochastic processes in which the probabilities associated with the outcomes at any stage of the experiment depend *only* on the outcomes of the preceding stage.
- The outcome at any stage of the experiment in a Markov process is called the *state* of the experiment.
- Transition Matrix - A transition matrix associated with a Markov chain with n states is an $n \times n$ matrix T with entries $a_{ij} = P(\text{moving to state } i \mid \text{currently in state } j)$ such that
  1. $a_{ij} \geq 0$ for all i and j.
  2. The sum of the entries in each column of T is 1.

m = # of steps that have been completed

- Any matrix satisfying the two properties above is called a *stochastic* matrix.
- If T is the transition matrix associated with a Markov process, then the probability distribution of the system after m observations (or steps) is given by

$$X_m = T^m X_0$$

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Section 9.2 - Regular Markov Chains

- The goal of this section is to investigate long-term trends of certain Markov chains.
- Regular Markov Chain - A stochastic matrix T is a **regular Markov chain** if the sequence $T, T^2, T^3, \ldots$ approaches a steady state matrix in which all entries are *positive* (i.e., strictly greater than 0).
- It can be shown that a stochastic matrix T is regular if and only if *some* power of T has entries that are all positive.
- **Finding the Steady-State Distribution Vector** - Let T be a regular stochastic matrix. Then the steady-state distribution vector X may be found by solving the vector equation

$$TX = X$$

together with the condition that the sum of the elements of the vector X must equal 1 (i.e., $x_1 + x_2 + \cdots + x_n = 1$).

Section 9.4 - Game Theory and Strictly Determined Games

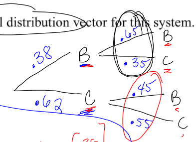
- **Zero-sum Game** - a game in which the payoff to one party results in an equal loss to the other.
- The entries in a payoff matrix represent the earnings of the row player.
- **Maximin Strategy**
  1. For each **row** of the payoff matrix, find the smallest entry in that row and underline it.
  2. Find the largest underlined number and note the row that it is in. This row gives the row player's "best" move.
- **Minimax Strategy**
  1. For each **column** of the payoff matrix, find the largest entry in that column and circle it.
  2. Find the smallest circled number and note the column that it is in. This column gives the column player's "best" move.
- Strictly Determined Game - A strictly determined game has the following properties:
  1. There is an entry in the payoff matrix that is simultaneously the smallest entry in its row and the largest entry in its column (i.e., there is one entry that is both underlined and circles at the same time). This entry is called the **saddle point** for the game.
  2. The optimal strategy for the row player is precisely the maximin strategy and is the row containing the saddle point. The optimal strategy for the column player is the minimax strategy and is the column containing the saddle point.
- The saddle point of a strictly determined game is also referred to as the **value of the game**.

Title: Dec 4-6:19 PM (1 of 9)

1. Acme Taxi services the Bryan/College Station area. At 8am on one particular day, 38% of all of their taxi cabs were in Bryan and the rest were in College Station. According to Acme's records, 35% of all passengers picked up in Bryan ask to be driven to College Station, and 45% of all passengers picked up in College Station ask to be dropped off in Bryan.

(a) Find the transition matrix and initial distribution vector for this system.

State 1: Bryan (B)
State 2: College Station (C)



$$T = \begin{array}{c c} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{bmatrix} .65 & .45 \\ .35 & .55 \end{bmatrix} \end{array}, \quad X_0 = \begin{bmatrix} .38 \\ .62 \end{bmatrix}$$

(b) What is the distribution of the taxi cabs after each cab has transported 1 passenger? After transporting 8 passengers?

$$X_1 = TX_0 = \begin{bmatrix} .526 \\ .474 \end{bmatrix}$$

→ 52.6% of all cabs are in Bryan and 47.4% are in C.S. after 1 passenger has been transported.

$$X_8 = T^8 X_0 = \begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$$

56.25% are in Bryan and 43.75% are in C.S. after 8 passengers have been transported.

(c) Is this a regular Markov chain? If yes, find its steady state vector and explain the meaning of its entries. If no, explain why.

Transition matrix is stochastic (all entries ≥ 0 and columns sum to 1) AND all entries must be strictly greater than 0 in some power of T.

In this example, T is regular.

$$TX = X$$

X = steady-state distrib. vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} .65 & .45 \\ .35 & .55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} .65x_1 + .45x_2 &= 1x_1 \\ -x_1 \qquad\qquad &\quad\; -x_1 \\ .35x_1 + .55x_2 &= 1x_2 \end{aligned} \longrightarrow \begin{aligned} -.35x_1 + .45x_2 &= 0 \\ .35x_1 - .45x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

additional requirement →

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} -.35 & .45 & 0 \\ .35 & -.45 & 0 \\ 1 & 1 & 1 \end{array}\right] \\ 3\times 3 \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 9/16 \\ 0 & 1 & 7/16 \\ 0 & 0 & 0 \end{array}\right] \end{array} \qquad x_1 = \frac{9}{16}, \quad x_2 = \frac{7}{16}$$

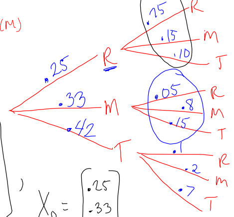
Steady-state vector is $X = \begin{bmatrix} 9/16 \\ 7/16 \end{bmatrix}$.

Eventually, $\frac{9}{16}$ of all taxi cabs will be in Bryan and $\frac{7}{16}$ of all cabs will be in C.S.

2. The town of Gonzales has three restaurants that sell tacos. A study found that 75% of those who dine at Reyna's Taco Hut during a particular week will return to Reyna's in the following week, but 15% will dine at Matamoros Taco Hut in the following week, and the rest will go to Mr. Taco. Of those who dine at Matamoros Taco Hut in a particular week, 5% will then go to Reyna's Taco Hut, 15% will then go to Mr. Taco in the next week, and the rest will return to Matamoros. Of those who dine at Mr. Taco in a particular week, 10% will go to Reyna's, 20% will go to Matamoros Taco hut, and the rest will return to Mr. Taco in the next week. This week, 25% of those craving tacos went to Reyna's Taco Hut, 33% went to Matamoros Taco Hut, and the rest went to Mr. Taco.

(a) Find the transition matrix and initial state vector for this Markov chain.

State 1: Reyna's Taco Hut (R)
State 2 : Matamorus TacoHut (M)
State 3 : Mr. Taco (T)



$$T = \begin{bmatrix} .75 & .05 & .1 \\ .15 & .8 & .2 \\ .1 & .15 & .7 \end{bmatrix}, \quad X_0 = \begin{bmatrix} .25 \\ .33 \\ .42 \end{bmatrix}$$

(b) What percent of customers will visit the three restaurants during the next week? What about 5 weeks from now? 15 weeks from now?

$$X_1 = TX_0 = \begin{bmatrix} .2460 \\ .3855 \\ .3685 \end{bmatrix}$$

24.6% will visit Reyna's, 38.55% will visit Matamoros, & 36.85% — — Mr. Taco in the next week.

$$X_5 = T^5 X_0 = \begin{bmatrix} .2276 \\ .4596 \\ .3128 \end{bmatrix}$$

5 weeks from now, 22.76% will dine at Reyna's, 45.96% will dine at Matamoros, and 31.28% will dine at Mr. Taco.

$$X_{15} = T^{15} X_0 = \begin{bmatrix} .2184 \\ .4725 \\ .3090 \end{bmatrix}$$

15 weeks from now, 21.84% will dine at Reyna's, 47.25% will dine at Matamoros, and 30.9% will dine at Mr. Taco

(c) Is this a regular Markov chain? If yes, find its steady state vector and explain the meaning of its entries. If no, explain why.

T is regular.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$TX = X$ where X = steady-state vector

$$\begin{bmatrix} .75 & .05 & .1 \\ .15 & .8 & .2 \\ .1 & .15 & .7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$.75x_1 + .05x_2 + .1x_3 = x_1 \rightarrow -.25x_1 + .05x_2 + .1x_3 = 0$

$.15x_1 + .8x_2 + .2x_3 = x_2 \rightarrow .15x_1 - .2x_2 + .2x_3 = 0$

$.1x_1 + .15x_2 + .7x_3 = x_3 \rightarrow .1x_1 + .15x_2 - .3x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$$\left[\begin{array}{ccc|c} -.25 & .05 & .1 & 0 \\ .15 & -.2 & .2 & 0 \\ .1 & .15 & -.3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12/55 \\ 0 & 1 & 0 & 26/55 \\ 0 & 0 & 1 & 17/55 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the steady-state vector is $X = \begin{bmatrix} 12/55 \\ 26/55 \\ 17/55 \end{bmatrix}$, meaning:

Eventually, $\frac{12}{55}$ of all customers will dine at Reyna's, $\frac{26}{55}$ of all customers will dine at Matamoros and $\frac{17}{55}$ of all customers will dine at Mr. Taco.

3. Which of the following are regular stochastic matrices? For each stochastic matrix that is regular, find the steady state distribution vector.

all entries ≥ 0 and columns add to 1.

Some power of T has entries all > 0.

(a) $T = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$

Look at T^2: $T^2 = \begin{bmatrix} .75 & .5 \\ .25 & .5 \end{bmatrix} \leftarrow$ all > 0, so T itself is $\boxed{\text{regular.}}$

Steady-state vector X:

$$TX = X$$

$$\begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} .5x_1 + x_2 &= x_1 \\ .5x_1 &= x_2 \end{aligned} \longrightarrow \begin{aligned} -.5x_1 + x_2 &= 0 \\ .5x_1 - x_2 &= 0 \end{aligned} \rightarrow \left[\begin{array}{cc|c} -.5 & 1 & 0 \\ .5 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$$

additional requirement $\longrightarrow x_1 + x_2 = 1$

So the steady-state vector is $X = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

(b) $T = \begin{bmatrix} 0.4 & 0 & 0.1 \\ 0.1 & 1 & 0.3 \\ 0.5 & 0 & 0.6 \end{bmatrix}$

Looking at powers of T, we see that the 2nd column is always $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, T is **not** regular.

(Section 9.4)

4. Each of the following matrices represents the payoff in a two-person zero-sum game. Determine the maximin and minimax strategies for each player. If there is a saddle point, find it and determine the value of the game.

(a) $\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$

1) In ea. row, underline the smallest #.

Maximin: Row player plays Row2

2) In each col., circle the largest #.

Minimax: Col. player plays Col2.

(no saddle pt.)

(b) $\begin{bmatrix} 5 & -8 \\ 3 & 1 \end{bmatrix}$

Maximin: Row player plays Row 2

Minimax: Col. player plays Col 2.

Is a saddle pt at $a_{22} = 1$.

The value of the game is 1.

This game favors the row player

(c) $\begin{bmatrix} -1 & 5 \\ 1 & 4 \\ 0 & -1 \end{bmatrix}$

Maximin: Row player plays R_2

Minimax: Col player plays Col 1.

Saddle pt at $a_{21} = 1$

Value of game is 1

game favor row player

(d) $\begin{bmatrix} 4 & 3 & -1 \\ 2 & 5 & -5 \\ 1 & 0 & 2 \end{bmatrix}$

Maximin: Row 3

Minimax: Col 3

no saddle point

(Section 9.4)

5. Robert and Camille play a game in which each casts a fair four-sided die at the same time. If the sum of the numbers landing up is less than 4, then Camille pays Robert \$10. If the sum of the numbers landing up is greater than 4, then Robert pays Camille \$2. However, if the sum is exactly 4, then no payoff is made to either player. Find the payoff matrix for this two-player zero-sum game.

$$\text{Robert} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \text{Camille} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 10 & 10 & 0 & -2 \\ 10 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{bmatrix} \end{array}$$

(section 9.4)

6. In the game of "paper, rock, scissors," two players simultaneously show a hand signal representing one of paper, rock, and scissors. In this game, "paper" beats "rock" (since paper can smother the rock), "rock" beats "scissors" (since a rock can crush scissors), and "scissors" beats "paper" (since scissors can cut paper). Suppose two people play this game and each time, the loser must pay the winner \$1.

   (a) Write the payoff matrix for this game.

Col

$$\text{Row}\quad \begin{array}{c} \\ \text{paper} \\ \text{rock} \\ \text{scissors} \end{array} \begin{array}{c} \begin{array}{ccc} \text{paper} & \text{rock} & \text{scissors} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

   (b) Determine the maximin and minimax strategies for the row and column players. Is this game strictly determined? ← when you have a saddle pt.

Row player can play R1, R2, or R3.

Col. player can play C1, C2, or C3.

No saddle pt, so not strictly det'd.

(Section 9.4)

7. TV stations R and C each have a quiz show and sitcom to schedule for their 1pm and 2pm time slots. If they both schedule their quiz shows at 1pm, then station R will take \$3,000 in advertising revenue away from station C. If they both schedule their quiz shows at 2pm, then station C will take \$2,000 in advertising revenue from R. If they choose different hours for the quiz show, then R will take \$5,000 in advertising from C by scheduling it at 2pm and \$2,000 by scheduling it at 1pm.

(a) Give the payoff matrix.

$$
\begin{array}{cc}
 & C \\
R & \begin{array}{ccc}
 & Q_{1pm} & Q_{2pm} \\
Q_{1pm} & \left[\begin{array}{c} 3000 \\ 5000 \end{array}\right. & \left.\begin{array}{c} 2000 \\ -2000 \end{array}\right] \\
Q_{2pm} & &
\end{array}
\end{array}
$$

$\leftarrow R_1\, C_2$

(b) Is this game strictly determined? If yes, give the optimal strategies for R and C and state the value of the game.

Yes since there is a saddle pt.

R should schedule the Quiz show at 1pm.

C should schedule their quiz show at 2pm

\$2000 = value of game

(favors the R station)

(section 9.4)

8. A farmer is trying to decide whether or not to expand his production of corn to a higher level. He has determined that if he expands his corn production and the growing season is drier than normal, he will have a profit of \$3,500. If he expands production and the growing season has an average amount of rainfall, then he will make a profit of \$6,000. If he expands production and the growing season is wetter than usual, he will make a profit of \$7,500. If he does not expand production and the growing season is drier than normal, average, or wetter than average, he will make profits of \$2,500, \$3,500, and \$4,000.

(a) Represent this information in the form of a payoff matrix.

weather

$$\text{farmer}\quad \begin{array}{r c} & \begin{array}{ccc} \text{dry} & \text{avg.} & \text{wet} \end{array} \\ \begin{array}{r} \text{expand} \\ \text{not exp.} \end{array} & \begin{bmatrix} \underline{3500} & 6000 & 7500 \\ \underline{2500} & 3500 & 4000 \end{bmatrix} \end{array}$$

(b) Assuming that the weather for the coming growing season is unpredictable, determine whether or not the farmer should expand his corn production.

Since the maximin strategy is Row 1, the farmer should expand production.