

## **Math 166 - Week in Review #1**

### **Section 2.1 - Systems of Linear Equations**

- When a system of linear equations has only two variables, each equation represents a line and “solving the system” means finding all points the lines have in common.
- For any system of linear equations (with finitely many variables), there are only 3 possibilities for the solution: (1) a unique solution, (2) infinitely many solutions, or (3) no solution.
- If a system of equations has infinitely many solutions, you **MUST** give the parametric solution for the system.

Welcome to Math 166  
Week-in-Review!

(Heather Ramsey)

## Section 2.2 - Systems of Linear Equations: Unique Solutions

- Row-Reduced Form of a Matrix

1. Each row consisting entirely of zeros lies below any other row having nonzero entries.
2. The first nonzero entry in each row is 1 (called a **leading 1**).
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column contains a leading 1, then the other entries in that column are zeros.

**NOTE:** We only consider the coefficient side (the left side) of an augmented matrix when determining whether the matrix is in row reduced form.

- Gauss-Jordan Row Operations

1. Interchange any two rows ( $R_i \leftrightarrow R_j$ ).
2. Replace any row by a nonzero constant multiple of itself ( $cR_i$ ).
3. Replace any row by the sum of that row and a constant multiple of any other row ( $R_i + aR_j$ ).

- Gauss-Jordan Elimination Method

1. Write the given linear system in augmented matrix form.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry of the first row (the  $a_{11}$  entry) is nonzero. The  $a_{11}$  entry is the first “pivot element.”
3. Pivot about  $a_{11}$ , i.e., transform the  $a_{11}$  entry to a 1 and then transform the other elements in the first column into zeros using the 3 row operations.
4. Now interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row (the  $a_{22}$  entry) is nonzero. This is the second pivot element.
5. Pivot about  $a_{22}$ .
6. Continue in a similar manner until the the left side of the augmented matrix is in row-reduced form.

- *Pivoting* on an entry in a matrix means transform the pivot element into a 1 and then transform all other entries in the same column into 0's.

## Section 2.3 Systems of Linear Equations: Underdetermined and Overdetermined Systems

- Using RREF to Solve Systems of Equations

STEP 1: Check the final matrix to see if there is no solution. (If the system has no solution, state so and stop here.

Otherwise, go on to Step 2.)

STEP 2: Circle the leading 1's.

a) If each variable has a leading 1 in its column, then there is a unique solution.

b) Otherwise, there are (potentially) multiple solutions and each variable not having a leading one in its

column is a parameter. **NOTE: If the variables in the system of equations**

**represent quantities or**

**units of some items, then you must consider whether it is necessary to put**

**restrictions on the**

**parameters (and do so if it is necessary).**

- Overdetermined systems have more equations than unknowns. These systems can have a unique solution, infinitely many solutions, or no solution.
- Underdetermined systems have fewer equations than unknowns. These systems can only have infinitely many solutions or no solution.
- If a system of equations has infinitely many solutions, you should represent the solutions in parametric form.

1. Solve the system of equations

$$\begin{aligned} 2\left(x - \frac{3}{2}y = -2\right) &\rightarrow 2(2x - 3y = -4) \\ 6\left(\frac{2}{3}x - \frac{5}{6}y = \frac{7}{3}\right) &\rightarrow 4x - 5y = 14 \\ &\rightarrow \begin{array}{r} 4x - 5y = 14 \\ + -4x + 6y = 8 \\ \hline y = 22 \end{array} \end{aligned}$$

Substitute into

$$2x - 3y = -4$$

$$2x - 3(22) = -4$$

$$\begin{aligned} 2x &= 62 \\ x &= 31 \end{aligned}$$

$$\begin{array}{l} \boxed{x=31} \\ \boxed{y=22} \end{array} \text{ or } \boxed{(31, 22)}$$

2. Solve the system of equations

$$\begin{cases} 3x - 6y = 18 \\ -2x + 4y = -12 \end{cases} \leftarrow \text{solve for } y$$

$$-6y = -3x + 18$$

$$y = \frac{1}{2}x - 3$$

$$-2x + 4\left(\frac{1}{2}x - 3\right) = -12$$

$$-2x + 2x - 12 = -12$$

$$-12 = -12$$

$$0 = 0 \text{ True}$$

Infinitely many solutions.

Now find the parametric solution.

Let  $x = t$ , where  $t$  is any real #.

(Now solve for  $y$ .)

$$y = \frac{1}{2}x - 3 = \frac{1}{2}t - 3$$

$$(x, y) = \left(t, \frac{1}{2}t - 3\right) \text{ parametric solution.}$$

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2<sup>nd</sup> option: Let  $y = k$ , where  $k$  is any real #.  
(Now solve for  $x$ .)

$$3x - 6y = 18$$

$$3x = 6y + 18$$

$$x = 2y + 6$$

$$x = 2k + 6$$

$$(x, y) = (2k + 6, k)$$

3. (a) Find the value of  $k$  so that the given system has no solution.

parallel  $\rightarrow$  same slope

$7x - 5y = -3$   $\leftarrow$  a line

$3x + ky = 15$   $\leftarrow$  a line

slope  $\downarrow$   
 $y = mx + b$

$7x - 5y = -3$   
 $-5y = -7x - 3$   $\leftarrow$  y-int for 1st line  
 $y = \frac{7}{5}x + \frac{3}{5}$

set equal  $\rightarrow$   
 $m = \frac{7}{5}$

$ky = -3x + 15$   $\leftarrow$  y-int.  
 $y = -\frac{3}{k}x + \frac{15}{k}$   $\rightarrow$   $m = -\frac{3}{k}$

$\frac{7}{5} = -\frac{3}{k}$

$7k = -15$   
 $k = -\frac{15}{7}$

(b) Is it possible to find a value of  $k$  so that the system has infinitely many solutions? Explain.

Substitute  $k$  to find y-intercept of the 2nd line.

$y = -\frac{3}{k}x + \frac{15}{k}$   $\leftarrow$  y-int

y-int =  $\frac{15}{k} = \frac{15}{-15/7}$   
 $= \frac{18}{1} \cdot -\frac{7}{18}$

y-int for 1st line was  $\frac{3}{5}$ .  $\leftarrow$  not equal  $\rightarrow$  = -7  $\leftarrow$  y-int for 2nd line.

No value of  $k$  makes slopes equal and the same time the y-ints equal.

(c) For what value(s) of  $k$  will the system have a unique solution?

We need slopes of the two lines to be different to have a unique solution. Since  $k = -\frac{15}{7}$  is the only value of  $k$  the slopes equal, then any  $k \neq -\frac{15}{7}$  gives a unique solution.



For the next 3 exercises, set up the system of equations but do not solve.

4. (49, 50, 54 of Finite Mathematics by Lial, et. al.) The U-Drive Rent-A-Truck Co. plans to spend \$6 million on 200 new vehicles. Each van will cost \$20,000, each small truck \$30,000, and each large truck \$50,000. Past experience shows that they need twice as many vans as small trucks. How many of each kind of vehicle can they buy?

Let  $x$  equal the number of vans.  
Let  $y$  equal the number of small trucks.  
Let  $z$  equal the number of large trucks.

$$x + y + z = 200$$
$$20000x + 30000y + 50000z = 6000000$$

"Twice as many vans as small trucks"  
REWORD!

The number of vans is 2 times the # of small trucks.

$$x = 2y$$

$$\begin{aligned} x + y + z &= 200 \\ 20000x + 30000y + 50000z &= 6000000 \\ x - 2y &= 0 \end{aligned}$$

5. A cashier has a total of 96 bills in his register in one-, five-, and ten-dollar denominations. If he has three times as many fives as ones, and if the number of ones and fives combined is half of the number of tens he has, how many bills of each denomination does he have in his register?

Let  $x$  = the number of ones

Let  $y$  = the number of fives.

Let  $z$  = the number of tens.

$$x + y + z = 96$$

"Three times as many fives as ones"  
Reword:

the # of fives is three times the # of ones.

$$y = 3x$$

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$$x + y = \frac{1}{2} z$$

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$$\begin{aligned} x + y + z &= 96 \\ y &= 3x \\ x + y &= \frac{1}{2} z \end{aligned}$$

6. Random, Inc. makes picture collages in three sizes. A small collage requires 30 minutes of cutting time and 36 minutes of pasting time. A medium collage requires 60 minutes of cutting time and 54 minutes for pasting. A large collage requires 90 minutes for cutting and 72 minutes for pasting. There are 380 labor hours available for cutting and 330 labor hours available for pasting each week. If the company wants to run at full capacity and wants to make twice as many small collages as medium collages, how many collages of each size should be made each week?

Let  $x$  = the number of small collages.  
 Let  $y$  = the number of medium collages.  
 Let  $z$  = the number of large collages.

|              | <u>small</u> | <u>medium</u> | <u>large</u> | <u>Amt Avail</u> |
|--------------|--------------|---------------|--------------|------------------|
| cutting time | 30 min       | 60 min        | 90 min       | 380 hrs * 60     |
| pasting time | 36 min       | 54 min        | 72 min       | 330 hrs * 60     |

22800 min  
 19800 min

$$\begin{aligned} 30x + 60y + 90z &= 22800 \\ 36x + 54y + 72z &= 19800 \\ x &= 2y \end{aligned}$$

Twice as many small collages as medium.  
 Reword

The # of small collages is 2 times the # of medium.

$$x = 2y$$

7. For each of the following, determine if the matrix is in row reduced form. If it is, interpret its meaning as a solution to a system of equations. If the matrix is not in row reduced form, perform Gauss-Jordan row operations until it is and then interpret the solution.

(a) 
$$\begin{bmatrix} \textcircled{1} & 0 & | & 5 \\ 0 & \textcircled{1} & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{cases} x = 5 \\ y = 3 \end{cases} \quad \boxed{(5, 3)} \text{ a unique solution}$$

$0x + 0y = 0$   
 $0 = 0$

(b) 
$$\begin{bmatrix} \textcircled{1} & 0 & -4 & | & 2 \\ 0 & 1 & 3 & | & -5 \\ 3 & 0 & -12 & | & 6 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} \textcircled{1} & 0 & -4 & | & 2 \\ 0 & \textcircled{1} & 3 & | & -5 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\downarrow \downarrow$  no leading 1 in z column  
 In row-reduced form.  
 $x - 4z = 2$   
 $x = 4z + 2$   
 $x = 4t + 2$

$R_3 - 3R_1$   
 Let  $z = t$ , where  $t$  is any real #.

$$\begin{aligned} & [3 \ 0 \ -12 \ 6] - 3[1 \ 0 \ -4 \ 2] \\ &= [3 \ 0 \ -12 \ 6] + [-3 \ 0 \ 12 \ -6] \\ &= [0 \ 0 \ 0 \ 0] \end{aligned}$$

$y + 3z = -5$   
 $y = -3z - 5$   
 $y = -3t - 5$

$$\boxed{(4t + 2, -3t - 5, t)}$$

Infinitely many solutions.

(c) 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -3 & -9 & 16 \end{array} \right] \xrightarrow{R_3 + 3R_2} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$R_3 + 3R_2$

$0x + 0y + 0z = 1$

$0 = 1$  FALSE

No solution

$$\begin{aligned} & [0 \ -3 \ -9 \ 16] + 3[0 \ 1 \ 3 \ -5] \\ &= [0 \ -3 \ -9 \ 16] + [0 \ 3 \ 9 \ -15] \\ &= [0 \ 0 \ 0 \ 1] \end{aligned}$$

(d) 
$$\left[ \begin{array}{ccc|c} x & y & z & \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array} \right] \text{ Is in row-reduced form}$$

Let  $y = t$ , where  $t$  is any real #.

(Now solve for  $x$  and  $z$ .)

$x + 3y = 5$

$x = -3y + 5$

$x = -3t + 5$

$z = -7$

$(x, y, z) = (-3t + 5, t, -7)$

Infinitely many solutions.

8. For the next two word problems do the following:  
 I) Define the variables that are used in setting up the system of equations.  
 II) Set up the system of equations that represents this problem. III) Solve for the solution.  
 IV) If the solution is parametric, then tell what restrictions should be placed on the parameter(s). Also give three specific solutions.

(a) The management of a private investment club has a fund of \$300,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that the management is considering have been classified into three categories: high-risk, medium-risk, and low-risk. Management estimates that high-risk stocks will have a rate of return of 16 percent per year, medium-risk stocks, 10 percent per year, and low-risk stocks, 4 percent per year. The investment in medium-risk stocks is to be twice the investment in stocks of the other two categories combined. If the investment goal is to have an average rate of return of 11 percent per year on the total investment, determine how much the club should invest in each type of stock.

Let  $x$  = the amount invested in high-risk stocks.  
 Let  $y$  = the amount invested in medium-risk stocks.  
 Let  $z$  = the amount invested in low-risk stocks.

$$x + y + z = 300000$$

interest earned = 11% of the total investment

$$.16x + .10y + .04z = .11(300000)$$

$$.16x + .10y + .04z = 33000$$

$$y = 2(x + z)$$

The investment in medium risk is to be two times the invest in the other 2 combined

$$y = 2(x + z)$$

$$y = 2x + 2z$$

$$-2x + y - 2z = 0$$

$$x + y + z = 300000$$

$$.16x + .10y + .04z = 33000$$

$$-2x + y - 2z = 0$$

$$\begin{bmatrix} x & y & z & | & \\ 1 & 1 & 1 & | & 300000 \\ .16 & .10 & .04 & | & 33000 \\ -2 & 1 & -2 & | & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} x & y & z & | & \\ 1 & 0 & 0 & | & 75000 \\ 0 & 1 & 0 & | & 200000 \\ 0 & 0 & 1 & | & 25000 \end{bmatrix}$$

3x4

$$x = 75000$$

$$y = 200000$$

$$z = 25000$$

\$75000 in high-risk stocks  
 \$200000 in medium-risk stocks.  
 \$25000 in low-risk stocks

(b) A convenience store sold 23 sodas one summer afternoon in 12-, 16-, and 20-ounce cups (small, medium, and large). The total volume of soda sold was 376 ounces, and the total revenue was \$48. If the prices for small, medium, and large sodas are \$1, \$2, and \$3 respectively, how many of each size did the store sell that day?

(pg. 70-72, Finite Mathematics by Lial, et al.)

Let  $x$  = the number of small sodas sold.  
 Let  $y$  = the number of medium sodas sold.  
 Let  $z$  = the number of large sodas sold.

$$\begin{aligned} x + y + z &= 23 \\ 12x + 16y + 20z &= 376 \\ x + 2y + 3z &= 48 \end{aligned}$$

$$\begin{bmatrix} x & y & z & | & \\ 1 & 1 & 1 & | & 23 \\ 12 & 16 & 20 & | & 376 \\ 1 & 2 & 3 & | & 48 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} x & y & z & | & \\ 1 & 0 & -1 & | & -2 \\ 0 & 1 & 2 & | & 25 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{aligned} x - z &= -2 \\ x &= z - 2 \\ x &= t - 2 \\ y + 2z &= 25 \\ y &= -2z + 25 \\ y &= -2t + 25 \end{aligned}$$

Let  $z = t$ , where  $t$  is an integer that is greater than or equal to 0.

(Now solve for  $x$  and  $y$ ).  $2 \leq t \leq 12$

restriction

$$x = t - 2 \geq 0$$

$+2 \quad +2$

$$t \geq 2$$

$$y = -2t + 25 \geq 0$$

$$-2t + 25 \geq 0$$

$$-2t \geq -25$$

$$t \leq 12.5$$

These two restrictions give us  
 $t = 2, 3, 4, \dots, 12$

Parametric Solution:

$$(x, y, z) = (t - 2, -2t + 25, t)$$

where  $t = 2, 3, 4, \dots, 12$ .

9. Perform the first pivot in the Gauss-Jordan Elimination Method. Indicate all row operations used.

*use this*

*1st pivot element*

$$\frac{1}{3} \left( \begin{array}{ccc|c} 3 & -6 & 0 & 9 \\ 7 & 4 & -8 & 6 \\ 2 & -3 & 1 & 9 \end{array} \right) \xrightarrow{\frac{1}{3} R_1} \begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 7 & 4 & -8 & 6 \\ 2 & -3 & 1 & 9 \end{array}$$

$R_2 - 7R_1$

$$\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 18 & -8 & -15 \\ 2 & -3 & 1 & 9 \end{array} \xrightarrow{R_3 - 2R_1} \begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 18 & -8 & -15 \\ 0 & 1 & 1 & 3 \end{array}$$

$R_2 - 7R_1$

$$[7 \ 4 \ -8 \ 6] - 7[1 \ -2 \ 0 \ 3]$$

$$[7 \ 4 \ -8 \ 6] + [-7 \ 14 \ 0 \ -21]$$

$$= [0 \ 18 \ -8 \ -15]$$