Quantum Network Coding

Salah A. Aly

Department of Computer Science
Texas A& M University

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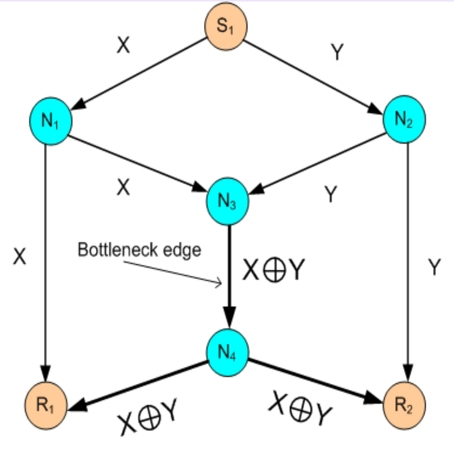
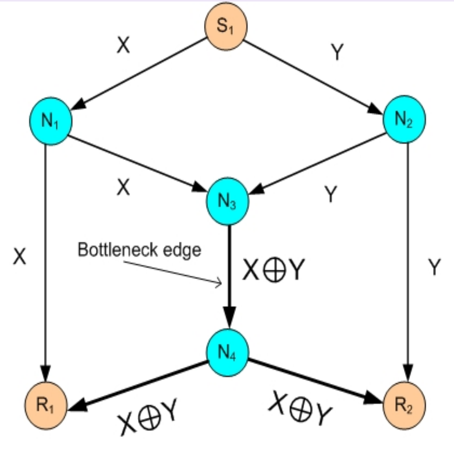
Network coding example

- In this butterfly network, there is a source S_1 and two receivers R_1 and R_2.
- The goal is to send X and Y from S_1 to both R_1 and R_2 simultaneously.
- The idea of network coding is that instead of routing the messages X and Y at node N_3, we rather encode and send them in one message.



Network coding example

- The receiver R_1 is able to decode the message $X \oplus Y$.
- There is a bottleneck edge (N_3, N_4). With traditional routing, we cannot achieve max capacity because of that edge, but with network coding, we can achieve the max capacity.



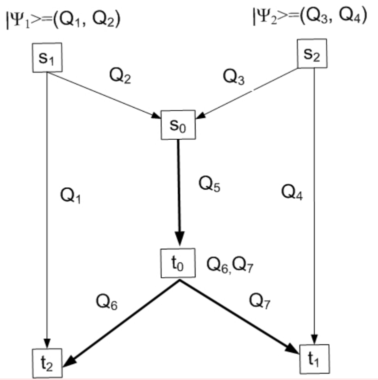
Why network coding

- Network coding rate (NCR) can be much larger than flow rate (FR) in some networks: i.e Butterfly and Wheatstone networks: NCR is 1 and FR is 1/2. Even we can do better in some networks 1/n FR to n NCR.
- With network coding, the max capacity can be achieved as the field size is bounded by certain value.
- Network coding offers robustness and adaptability for the network.

From classical to quantum networks

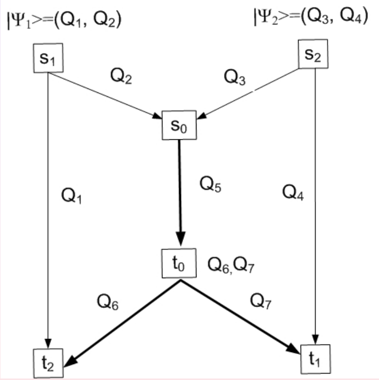
Can we increase the throughput of a quantum network using network coding? Consider the butterfly quantum network.

- The goal is to send $|\psi_1\rangle$ from s_1 to both t_1 and t_2, (similarly, send $|\psi_2\rangle$ from s_2 to both t_1 and t_2).
- Exact transmission is impossible because Qubits are continuous information.



Quantum network coding

- So, we can not do quantum network coding without errors.
- The problem can not be the same as classical network due to nature of quantum information:
  - No cloning principle
  - Quantum information is continuous



Quantum network coding

- The main result in this paper is that quantum network coding is possible if approximation is allowed.
- It is natural to study approximate or probabilistic methods to deal with qubits.
- We will find a way to deal with no cloning principle, so we can copy the information locally.

Theorem (Main Result)

There exists a quantum protocol whose fidelities at nodes t_1 and t_2 are $1/2 + 200/19863$ and $1/2 + 180/19863$, respectively.

Definition (Trace Distance)

Given two quantum states ρ and σ, the trace distance is defined as $D(\rho, \sigma) = \frac{1}{2}\sum_i |\rho_i - \sigma_i|$.

The trace distance can also be computed between density operators. If $\rho = \sum_i r_i|i\rangle\langle i|$ and $\sigma = \sum_i s_i|i\rangle\langle i|$, Then

$$D(\rho, \sigma) = \frac{1}{2}Tr|\sum_i (r_i - s_i)|i\rangle\langle i||.$$

Example:

$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ and $\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$.

$$\begin{aligned} D(\rho, \sigma) &= \frac{1}{2}Tr|(3/4 - 2/3)|0\rangle\langle 0| + (1/4 - 1/3)|1\rangle\langle 1|| \\ &= 1/2(1/12 + 1/12) = 1/12 \end{aligned}$$

Definition (Fidelity)

Given two quantum states ρ and σ, the fidelity is defined as $F(\rho,\sigma) = Tr(\sqrt{\rho^{1/2}\sigma\rho^{1/2}})$.

For mixed states, the trace distance is a function of the fidelity as

$$D(\rho,\sigma) \leq \sqrt{1 - F(|\rho\rangle, |\sigma\rangle)^2}$$

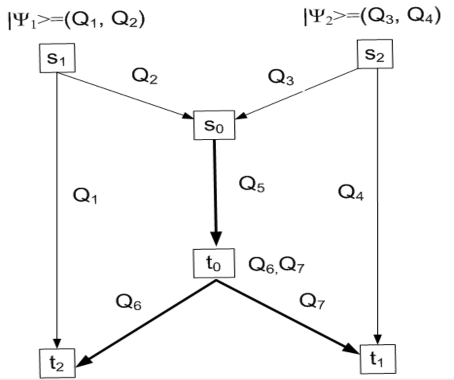
and the inequality holds for pure quantum states.

We measure the quality of data at node t_1 by the fidelity between the original state $|\psi_1\rangle$ at s_1 and the output state ρ at t_1, (similarly between t_2 and s_2).

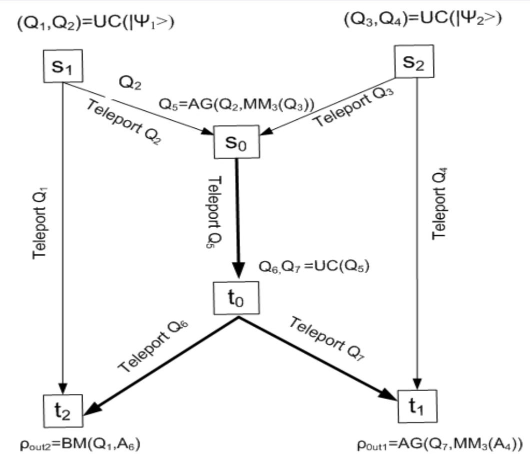
Our goal is to achieve a fidelity strictly $> 1/2$.

- For coherent states, the classical teleportation limit is $F = 0.5$, light polarization states $F = 0.67$.
- If we consider a mixed state, then the fidelity is $1/2$.
- Quantum Teleportation has fidelity $F = 1$ theoretically perfect.

In order to show this Theorem, we can not directly copy qubits at s_1 and s_2, also we can not directly encode at s_0. Rather, we need some building blocks to deal with qubits (i.e XQQ protocol)
Universal Cloning (UC),
3D Measurement (MM_3),
Approximated Group Operation (AG),
3D Bell Measurement (BM)



XQQ protocol



XQQ protocol description

So, the goal is to send quantum information (states) via quantum channel and achieve a fidelity greater than $1/2$.

- Universal cloning at node s_1 to get $(Q_1, Q_2) = UC(|\psi_1\rangle)$, also UC at s_2 to get $(Q_3, Q_4) = UC(|\psi_2\rangle)$. then make teleportation to Q_1, Q_2, Q_3, Q_4.
- At node s_0, apply 3DMM to Q_3 to obtain 3 classical bits $r_1 r_2 r_3$. Then choose one value of the group AG that has 8 elements (i.e. I,X,Y,Z, Inv, Inv X, Inv Y, Inv Z).
- At node t_1, apply the reverse operations of these 8 operation to decode the received q. state.
- At node t_2, recover the 3 bits $r_1 r_2 r_3$ as a key by comparing Q_1 and Q_6.

XQQ Protocol Overview

XQQ Protocol steps.

Input Q.States $|\psi_1\rangle$ at s_1 and $|\psi_2>$ at s_2;
Output Q. States ρ^1_{out} at t_1 and ρ^2_{out} at t_2.

- Step 1. Apply universal cloning at the sources, $(Q_1, Q_2) = UC(|\psi_1\rangle)$ at S_1, and $(Q_3, Q_4) = UC(|\psi_2>)$ at s_2.
- Step 2. $Q_5 = AG(Q_2, MM_3(Q_3))$ at s_0.
- Step 3. $(Q_6, Q_7) = UC(Q_5)$ at t_0.
- Step 4. Finally decoding at nodes t_1 and t_2. $\rho^1_{out} = AG(Q_7, MM_3(Q_4))$, and $\rho^2_{out} = BM(Q_1, Q_6)$.

□

Universal Cloning (UC)

Recall the approximated cloning by Buzek and Hillery known as universal (1,2) cloning $(C_{(2)})$. Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|01>+|10>)$,
Using trace-preserving completely positive map (TP-CP),
$C_2(|0\rangle\langle 0|) = \frac{2}{3}|00\rangle\langle 00| + \frac{1}{3}|\psi\rangle\langle\psi||$,
$C_2(|0\rangle\langle 1|) = \frac{\sqrt{2}}{3}|\psi\rangle\langle 11| + \frac{\sqrt{2}}{3}|00\rangle\langle\psi||$,
$C_2(|1\rangle\langle 0|) = \frac{\sqrt{2}}{3}|11\rangle\langle\psi|| + \frac{\sqrt{2}}{3}|\psi\rangle\langle 00|$,
$C_2(|1\rangle\langle 1|) = \frac{2}{3}|11\rangle\langle 11| + \frac{1}{3}|\psi\rangle\langle\psi||$
Now, let $\rho_1 = Tr_2 C_{(2)}(|\psi\rangle)$ and $\rho_2 = Tr_1 C_{(2)}(|\psi\rangle)$, then
$\rho_1 = \rho_2 = \frac{2}{3}|\psi\rangle\langle\psi|| + \frac{1}{3}.I/2$ that gives us

$$F(|\psi\rangle, \rho_1) = F(|\psi\rangle, \rho_2) = 5/6$$

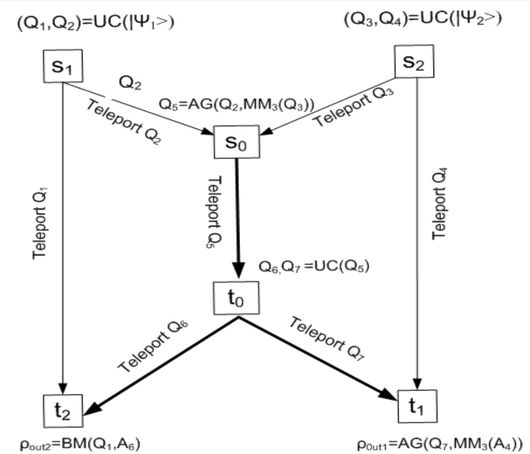
Universal Cloning (UC) can be used instead of Universal (1,2) cloning.

Encoding and Decoding Operations

Approximated Group Operation (AG): Given a mixed state $\rho = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, $Inv\rho = \begin{pmatrix} v_2 & -v_1 \\ -u_2 & u_1 \end{pmatrix}$ Now, we form the abelian group AG, as $\{$I,X,Y,Z,Inv ,Inv X, Inv Y, Inv Z$\}$.

- The approximated group AG is a transformation to encode the quantum state ρ given by $T_{000}(\rho) = \rho$, $T_{011}(\rho) = Z\rho$, $T_{101}(\rho) = X\rho$, $T_{110}(\rho) = Y\rho$, $T_{111}(\rho) = Inv\rho$, $T_{001}(\rho) = InvT_{110}(\rho)$, $T_{010}(\rho) = InvT_{101}(\rho)$, $T_{100}(\rho) = InvT_{011}(\rho)$.
- At node t_1, apply the reverse operations of these 8 operations to decode the received quantum state.
- At node t_2, recover the 3 bits $r_1r_2r_3$ as a key by comparing Q_1 and Q_6.

XQQ protocol



Computing Fidelity $F(|\psi_1\rangle, \rho^1_{out})$

1. 1 Let ρ be a quantum state, a map C is called *p-shrinking*, if C transforms ρ to $p.\rho + (1-p)\frac{I}{2}$ for some probability p.
2. The idea is to discretize one of the two quantum states into one classical bits, then encode the other quantum state.
3. Let $\rho_2 = |\psi_2\rangle\langle\psi_2|$, then compute the maps.
   $C_1 : |\psi_1\rangle \longrightarrow Q_2$, $C_2[\rho_2] : Q_2 \longrightarrow Q_5$
   $C_3 : Q_5 \longrightarrow Q_7$, $C_4[\rho_2] : Q_7 \longrightarrow \rho^1_{out}$.
4. The universal copy is *2/3-shrinking*.
5. If C is a *p-shrinking* and C' is a *p'-shrinking*, then $C \circ C'$ is a *pp'-shrinking*.
6. $C_4[\rho_2] \circ C_2[\rho_2]$ is a $\frac{100}{2187}$*-shrinking*.

Computing Fidelity $F(|\psi_1\rangle, \rho_{out}^1)$ (cont.)

7. If C is a *p-shrinking*, $F(\rho, C(\rho)) \geq 1/2 + p/2$ for any state ρ.
8. The final result is given by $C_{s_1t_1} = C_4[\rho_2] \circ C_3 \circ C_2[\rho_2] \circ C_1$, and its Fidelity shown in this Lemma.
9. The total probability of $C_{s_1t_1}$ is $(2/3)(2/3)(100/2187) = \frac{400}{19683}$

Lemma ($F(|\psi_1\rangle, \rho_{out}^1)$)

There exists a quantum protocol whose fidelity between s_1 and t_1 is $1/2 + 200/19683$.

For any $|\psi_1\rangle$, $F(|\psi_1\rangle, C_{s_1t_1}(|\psi_1\rangle)) \geq 1/2 + 200/19683$.

Computing Fidelity $F(|\psi_2\rangle, \rho_{out}^2)$

Similar to the previous computations.

Lemma ($F(|\psi_2\rangle, \rho_{out}^2)$)

There exists a quantum protocol whose fidelity between s_2 and t_2 is $1/2 + 180/19683$.

Summary of the Results on this paper:

- We can send any quantum state $|\psi_1\rangle$ from s_1 to t_1. and $|\psi_2\rangle$ from s_2 to t_2 with fidelity strictly greater than 1/2.
- If one of $|\psi_1\rangle$ and $|\psi_2\rangle$ is classical then the fidelity can be improved to 2/3.
- Similarly, improvements is also possible if $|\psi_1\rangle$ and $|\psi_2\rangle$ are restricted to only a finite number of states.

Theorem (Main Result)

There exists a quantum protocol whose fidelities at nodes t_1 and t_2 are $1/2 + 200/19863$ and $1/2 + 180/19863$, respectively.

Is that the right approach to do quantum network coding?

- In this protocol, one of the two quantum states has to be measured, i.e. destroying value of the state.
- Which one to measure and which one to send exactly?
- What kind of application to use approximated quantum network coding?

Open questions and discussion:

- Finding upper bounds on the Fidelity rather than 1.
- Comparing classical and quantum network coding. i.e sending classical and quantum information in the same network and measure the performance.
- Applying quantum network coding to different network topology.

These are just natural questions arising from the paper of Hayashi, et. al. Some other questions arise: We can do *Quantum-assisted network coding* by sending classical information over quantum channels. We can study if this gives us a better results than we can do classically.

Questions....