Network Coding Capacity of Random Wireless Networks Under a Signal-to-Interference-and-Noise Model

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Network Coding (NC)

- Traditionally, information flow in networks was treated like fluid through pipes.
- Unicast transmission
  - Maximum transmission rate bounded by minimum cut size.
  - Min-Cut Max-Flow Theorem
- Multicast transmission
  - Maximum flow rate cannot be achieved by routing.
  - Even if each source-destination pair has minimum cut with same size.
  - Some links shared for different source-destination pairs.
- Network coding
  - Nodes encode received messages and forward to next-hop neighbors.
  - Maximum flow rate can be achieved.

Network Coding Capacity (Previous Work)

- NC capacity determined by size of minimum cut.
  - Topology is fixed and known.
- [Ramamoorthy, Shi and Wesel] (Allerton 03, IEEE Info. Tran. 05)
  - Random graphs (RG) and random geometric graphs (RGG).
  - Size of minimum cut between source and any destination is not fixed.
- [Aly, Kapoor, Meng and Klappenecker] (Netcoding 07)
  - Generalized random geometric graph.
  - Tighter upper and lower bounds on NC capacity
- Summary
  - Concentration behavior of minimum cut size in RG or RGG.
  - Independent, bidirectional links — no interference.

Motivation for Our Work

We compute NC capacity by

- Proposing more realistic model.
- Considering noise and interference.
- Assuming various transmission powers.
- Assuming dependency between nodes.
- Considering multi-sources multi-destinations scenario.

SINR Model

- More sophisticated and realistic models for connectivity needed.
  - Noise, interference, and heterogeneity of transmission power
- Signal-to-Interference-and-Noise-Ratio (SINR) model
  - Existence of link from node i to node j depends on ability to decode transmitted signal from i to j.
  - Determined by SINR

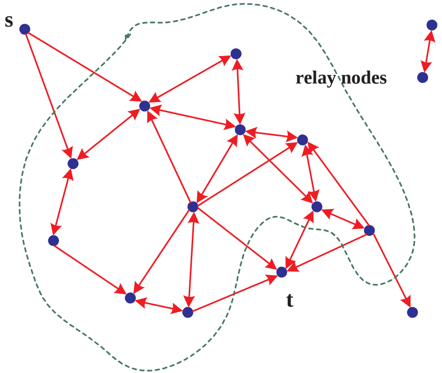
$$\beta_{ij} = \frac{P_i L(d_{ij})}{N_0 + \gamma \sum_{k \neq i,j} P_k L(d_{kj})}$$

  - P_i: transmission power of node i.
  - d_{ij}: Euclidean distance between i and j.
  - N_0: power of background noise.
  - γ: inverse of system processing gain, narrowband ($=$1), wideband($<$1).
  - $L(\cdot)$: path-loss function (signal attenuation function).

Dense SINR Network Model

- Dense network model with SINR — $G(\mathcal{X}, \mathcal{P}, \gamma)$
  - n nodes uniformly distributed at random in unit two-dimensional torus $\mathcal{B}_1 = [1, 1]^2$.
  - Node i transmits with power $P_i \sim f_P(p)$.
  - Link from i to j exists iff $\beta_{ij} \geq \beta$ (threshold).
  - Each link is directional (not necessarily bidirectional).
  - Different links are not independent.
- Will show NC capacity has a sharp concentration when network scale is large enough.

Directed Multi-hop Network



- Single-source single-destination transmission in a directed SINR graphs

Link Capacity

- Capacity of link (i, j)

$$C_{ij} = \begin{cases} R & \beta_{ij} \geq \beta, \\ 0 & \beta_{ij} < \beta. \end{cases}$$

  - $\beta_{ij} \geq \beta$, link (i, j) has capacity R, i.e., i can transmit data at rate R packets/sec to j without any error.
  - An error correcting code at physical layer corrects errors once SINR threshold is met.

- Average Link Capacity

$$\bar{C} = E_X E_P[C_{ij}] = \int_0^{\frac{p_{max}}{N_0}} C_{ij} dF_{\beta_{ij}}(\tau)$$

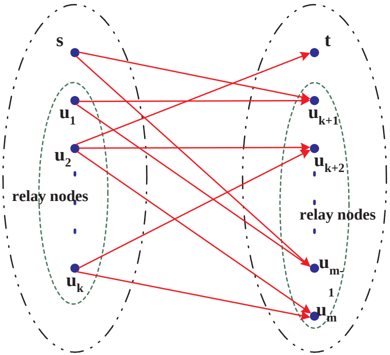
where $F_{\beta_{ij}}(\cdot)$: c.d.f. of β_{ij},
and determined by $f_P(\cdot)$, $\mathcal{X}$ and $L(\cdot)$.

Single-Source Multiple-Destination Transmission

- Source, destinations and relays:
  - Let s be the source node
  - l destination nodes, $\mathcal{R} = \{t_1, \ldots, t_l\}$.
  - m relay nodes, $\mathcal{R} = \{u_1, \ldots, u_m\}$.
- Capacities:
  - C_{si}: link capacity from s to relay u_i
  - C_{ij}: link capacity from relay u_i to another relay u_j.
  - C_{it_j}: link capacity from relay u_i to destination t_j.

Cut and Its Capacity

- An s-t-*cut* of size k for s and $t \in \mathcal{T}$:
  - A partition of relay nodes as two sets V_k and V_k^c.
  - $|V_k| = k, |V_k^c| = m - k$, $V_k \cup V_k^c = \mathcal{R}$ and $V_k \cap V_k^c = \emptyset$.



- Capacity of s-t-cut:

$$C_k = \sum_{u_i \in V_k^c} C_{si} + \sum_{u_j \in V_k} \sum_{u_i \in V_k^c} C_{ji} + \sum_{u_j \in V_k} C_{jt}$$

$$E[C_k] = E_X E_P[C_k] = [m + k(m-k)]\bar{C},$$

Network Coding Capacity

- $C_{s,t}$ — minimum cut capacity among all s-t-cuts:

$$C_{s,t} = \min_{0 \leq k \leq m} C_k$$

- $C_{s,\mathcal{T}}$ — network coding capacity for given source node s and sets of destination nodes $\mathcal{T} = \{t_1, \ldots, t_l\}$ and relay nodes $\mathcal{R} = \{u_1, \ldots, u_m\}$:

$$C_{s,\mathcal{T}} = \min_{t \in \mathcal{T}} C_{s,t}$$

- $\sum_{k\neq j} L(d_{kj}) = \sum_{k\neq j} L(||\mathbf{X}_k - \mathbf{X}_j||)$:
  - Random variable depending on locations of all nodes

- Define

$$J(j) \triangleq \sum_{k\neq j} L(d_{kj}), \quad \text{for all } j$$

$$I(j) \triangleq \sum_{k\neq j} P_k L(d_{kj}), \quad \text{for all } j$$

- For any j

$$\begin{aligned} E[L] &\triangleq E[L(j)] = E_{\mathbf{X}_i}[L(d_{ij})], \quad \text{constant} \\ E[J] &\triangleq E[J(j)] = E_X\left[\sum_{k\neq j} L(d_{kj})\right] \\ E[I] &\triangleq E[I(j)] = E_X E_P\left[\sum_{k\neq j} P_k L(d_{kj})\right] \end{aligned}$$

Constant Transmission Power

- All nodes transmit with a constant power P_0 — $G(\mathcal{X}, P_0, \gamma)$.
- SINR

$$\beta_{ij} = \frac{L(d_{ij})}{N_0/P_0 + \gamma \sum_{k \neq i,j} L(d_{kj})} = \frac{L(d_{ij})}{N_0/P_0 + \gamma J(j) - \gamma L(d_{ij})}$$

- For different j's, $J(j)$'s are not independent.
- Sharp concentration behavior in large scale wireless networks.

Concentration of Interference

Lemma 1: n nodes in network, for all $j = 1, 2, ..., n$

$$\Pr(J(j) \le (1-\epsilon_1)E[J]) = O\left(\frac{1}{n^2}\right),$$

and

$$\Pr(J(j) \ge (1+\epsilon_1')E[J]) = O\left(\frac{1}{n^2}\right)$$

where $\epsilon_1 = \sqrt{\frac{4\ln n}{(n-1)E[L]}}$ and $\epsilon_1' = \sqrt{\frac{6\ln n}{(n-1)E[L]}}$.

- Proved by Chernoff Bound
- Duo to uniform random distribution

Coupling with Another Two Graphs

- Define another two types of SINR models coupled with $G(\mathcal{X}, P_0, \gamma)$
- $G'(\mathcal{X}, P_0, \gamma)$ — same point process and P_0, different SINR:

$$\beta'_{ij} = \frac{L(d_{ij})}{N_0/P_0 + (1+\epsilon'_1)\gamma E[J] - \gamma L(d_{ij})}$$

- $G''(\mathcal{X}, P_0, \gamma)$ — same point process and P_0, different SINR:

$$\beta''_{ij} = \frac{L(d_{ij})}{N_0/P_0 + (1-\epsilon_1)\gamma E[J] - \gamma L(d_{ij})}$$

- Domination relationship
- C'_{ij} and C''_{ij} capacity of link (i,j) in $G'(\mathcal{X}, P_0, \gamma)$ and $G''(\mathcal{X}, P_0, \gamma)$.
- C'_{ij} and C''_{ij} are asymptotically equal to C_{ij}.

Concentration of Cut Capacity

Lemma 2: For any $0 < \epsilon < 1$,

$$\Pr(C_k \leq (1-\epsilon)E[C_k']) \leq \exp\left\{-\frac{E[C_k']\epsilon^2}{2}\right\}\left(1 - O\left(\frac{1}{n}\right)\right),$$

where $E[C_k'] = [m + k(m-k)]\bar{C}'$, and

$$\Pr(C_k \geq (1+\epsilon)E[C_k'']) \leq \exp\left\{-\frac{E[C_k'']\epsilon^2}{3}\right\}\left(1 - O\left(\frac{1}{n}\right)\right),$$

where $E[C_k''] = [m + k(m-k)]\bar{C}''$.

- C_{ij}' and C_{ij}'' are asymptotically equal to C_{ij}.
- $E[C_k']$ and $E[C_k'']$ are asymptotically equal to $E[C_k]$
- C_k concentrates at $E[C_k]$.

Concentration of Network Coding Capacity

Theorem (Lower Bound on NC)

When n is sufficiently large, with high probability,

$$\Pr(C_{s,\mathcal{T}} \geq (1-\epsilon'_\alpha)E[C_0]) = 1 - O\left(\frac{l}{m^\alpha}\right),$$

where $\epsilon'_\alpha = \sqrt{\frac{2\alpha \ln m}{E[C_0]}}$ *for* $\alpha > 0$ *and* $E[C_0] = m\bar{C}$*.*

Theorem (Upper Bound on NC)

When n is sufficiently large, with high probability,

$$\Pr(C_{s,\mathcal{T}} \leq (1+\epsilon''_\alpha)E[C_0]) = 1 - O\left(\frac{1}{m^\alpha}\right),$$

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- Therefore, $C_{s,\mathcal{T}}$ concentrates at $E[C_0] = m\bar{C}$

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- Therefore, $C_{s,\mathcal{T}}$ concentrates at $E[C_0] = m\bar{C}$

Heterogeneous Transmission Power

- Node i transmits with a random power P_i.
- SINR

$$\beta_{ij} = \frac{P_i L(d_{ij})}{N_0 + \gamma \sum_{k \neq i,j} P_k L(d_{kj})} = \frac{P_i L(d_{ij})}{N_0 + \gamma I(j) - \gamma P_i L(d_{ij})}.$$

- For different j's, $J(j)$'s are not independent.
- Capacity of link (i, j) is a constant R, independent of SINR β_{ij}

Lemma 3: n nodes in network, for all $j = 1, 2, ..., n$

$$\Pr(I(j) \leq (1 - \epsilon_2)E[I]) = O\left(\frac{1}{n^2}\right)$$

and

$$\Pr(I(j) \geq (1 + \epsilon_2')E[I]) = O\left(\frac{1}{n^2}\right)$$

where $\epsilon_2 = \sqrt{\frac{4 \ln n}{(n-1)E[P]E[L]}}$ and $\epsilon_2' = \sqrt{\frac{6 \ln n}{(n-1)E[P]E[L]}}$.

- Cannot employ same coupling methods as before.
- In $G'(\mathcal{X}, P_0, \gamma)$ (or $G''(\mathcal{X}, P_0, \gamma)$), C_{ij}''s (respectively, C_{ij}'''s) are independent for all $j \neq i$ for given i.
- This independence does not hold because all C_{ij}'s depend on transmission power P_i.

Azuma's Inequality: Let $Z_0, Z_1, \ldots,$ be a martingale sequence s.t. almost surely

$$|Z_i - Z_{i-1}| \le c_i, \quad i = 1, 2, \ldots$$

where c_i may depend on i. Then for $n > 0$ and $\lambda > 0$,

$$\Pr(Z_n \ge Z_0 + \lambda) \le \exp\left\{ -\frac{\lambda^2}{2\sum_{i=1}^n c_i^2} \right\},$$

and

$$\Pr(Z_n \le Z_0 - \lambda) \le \exp\left\{ -\frac{\lambda^2}{2\sum_{i=1}^n c_i^2} \right\}.$$

- Upper bound $|Z_i - Z_{i-1}|$ for all i by some constant, then can apply AI to obtain some bound on a tail probability.
- Y_i's are dependent, need to understand the properties of the Y_i's to see if we can bound $|Z_i - Z_{i-1}|$.

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- Define

$$\begin{aligned} r'_{min} &= L^{-1}\Big(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1+\epsilon'_2)E[I]}{p_{min}}\Big),\\ r'_{max} &= L^{-1}\Big(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1+\epsilon'_2)E[I]}{p_{max}}\Big),\\ r''_{min} &= L^{-1}\Big(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1-\epsilon_2)E[I]}{p_{min}}\Big),\\ r''_{max} &= L^{-1}\Big(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1-\epsilon_2)E[I]}{p_{max}}\Big). \end{aligned}$$

- Node inside circle centered at $\mathbf{X}_i$ with radius r'_{min} (r''_{min}) is connected to i by bidirectional link.
- node outside circle centered at $\mathbf{X}_i$ with radius r'_{max} (r''_{max}) is not connected to i.
- $N(r'_{min}, r'_{max})$ and $N(r''_{min}, r''_{max})$ — number of nodes in $\mathcal{A}(\mathbf{X}_i, r'_{min}, r'_{max})$ and $\mathcal{A}(\mathbf{X}_i, r''_{min}, r''_{max})$
- Exists constant $\eta > 0$ independent of n

$$N(r'_{min}, r'_{max}) \leq \eta, \quad \text{and} \quad N(r''_{min}, r''_{max}) \leq \eta$$

Lemma 4: For any $0 < \epsilon < 1$,

$$\Pr(C_k \leq (1-\epsilon)E[C_k']) \leq \exp\left\{-\frac{[m+k(m-k)]\bar{C}'^2\epsilon^2}{2(\eta+1)^2R^2}\right\},$$

where $E[C_k'] = [m+k(m-k)]\bar{C}'$, and

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- $C_{s,\mathcal{T}}$ concentrates at $E[C_0] = m\bar{C}$

Multiple Sources Problem

- Multiple sources and multiple destinations transmissions
  - $\mathcal{S} = \{s_1, \ldots, s_h\}$ the set of source nodes
  - An $\mathcal{S}$-t-cut of size k between $\mathcal{S}$ and $t \in \mathcal{T}$ is a partition of relay nodes into V_k and V_k^c, $|V_k| = k, |V_k^c| = m - k$, $V_k \cup V_k^c = \mathcal{R}$ and $V_k \cap V_k^c = \emptyset$.
  - Cut capacity

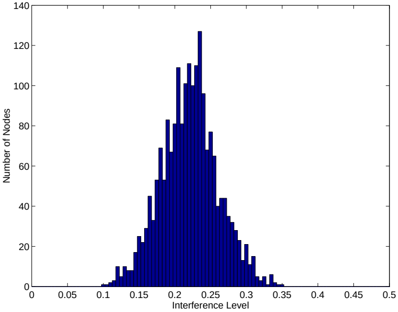
$$C_{\mathcal{S},t} = \min_{0 \le k \le m} C_k$$

  - NC capacity:

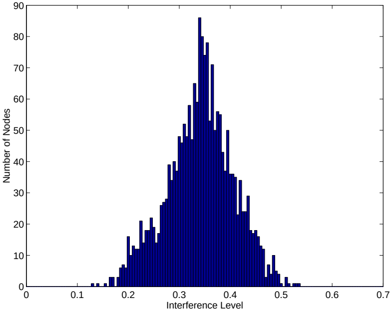
$$C_{\mathcal{S},\mathcal{T}} = \min_{t \in \mathcal{T}} C_{\mathcal{S},t}$$

- Same concentration results.

Simulation



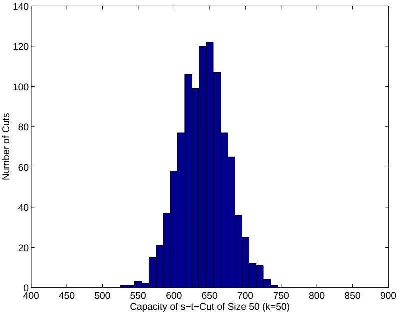
(a) Constant Transmission Power



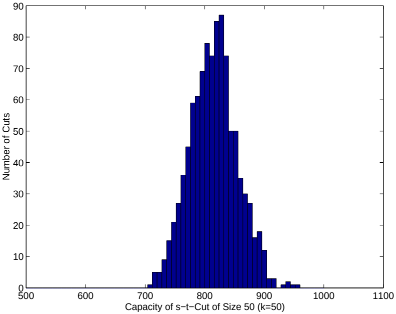
(b) Heterogeneous Transmission Power

- Interferences in networks with constant and heterogeneous transmission power

Simulation



(c) Constant Transmission Power



(d) Heterogeneous Transmission Power

- Capacity of cut with size 50 in networks with with constant and heterogeneous transmission power

Conclusion

- Studied network coding capacity for SINR model
  - Constant transmission power: translate to independent models by coupling methods
  - Heterogeneous transmission power: employ Azuma's inequality to bound tail probability of dependent random variables
- Considered multi-source multi-destinations scenario.
- Showed concentration behavior of both cases

Thank you...