Improving Productivity by Periodic Performance Evaluation: A Bayesian Stochastic Model

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We model the situation where the productivity of members of a group, such as a salesforce, is periodically evaluated; those whose performance is sub-par are dismissed and replaced by new members. Individual productivity is modeled as a random variable, the distribution of which is a function of an unknown parameter. This parameter varies across the members of the group and is specified by a prior distribution. In this manner, the heterogeneity in the group is explicitly accounted for. We model the situation as a parameter adaptive Bayesian stochastic control problem, and use dynamic programming techniques and the appropriate optimality equations to obtain solutions. We prove the existence of an optimal policy in the general case. Further, for the case when the sales process can be characterized by a Beta-Binomial or a Gamma-Poisson distribution, we show that the optimal policy is of the threshold type at each evaluation period, depending only on the accumulated performance up to a given period. We present a computational procedure to solve for the optimal thresholds. Results of computational experiments show that an increase in the heterogeneity of the group can lead to more stringent levels of minimal acceptable performance.

(Stochastic Productivity; Unknown Distribution Parameters; Periodic Evaluations; Bayesian Stochastic Control; Structured Optimal Policy; Dynamic Programming)

1. Introduction

Productivity often varies widely among members of a group such as a salesforce, recruiters for a company or agency, or faculty of an academic department. For example, dollar sales achieved per year, recruit contracts per month, or number of publications in an academic year can vary widely, with obvious “stars” at one level of productivity and poor producers at another. Managers traditionally attempt to increase group productivity by dismissing or reassigning individuals with low productivity, and replacing them with (promising) individuals whose productivity has not yet been fully determined. For example, salespersons who do not meet quota requirements are fired and replaced by new, possibly untested, individuals; Navy recruiters with poor records are rotated into different assignments (Carroll et al. 1986), and replaced by newly trained recruiters; academic departments replace faculty members who have not published “enough” with fresh Ph.Ds. In this paper, we have excluded situations where unproductive individuals are replaced by “established stars” from other institutions, although the conceptual framework can be used to incorporate those cases. Whatever the arena, however, productivity is not a deterministic process. Even after controlling for factors such as market potential and experience, the productivity of a
salesperson depends on individual ability as well as chance. Thus, the modeling of productivity, and the determination of criteria for dismissing unproductive individuals are complex stochastic problems.

In this paper we will use a salesforce example for expository convenience and address the following problem. Select a salesperson at random; suppose the number of units sold by him in period \( t \) is a random variable \( Y_t \), with a probability distribution which is a function of an unknown productivity parameter \( \theta \). For example, \( \{Y_t\} \) might be a Poisson random process with \( \theta \) the unknown rate parameter of the Poisson process. The parameter \( \theta \) determines the inherent ability of the salesperson. We assume that \( \theta \) varies across the members of the potential pool of the salesforce and can be characterized by its own probability distribution which summarizes the heterogeneity in ability among members of the salesforce. For example, \( \theta \) might be distributed across the salesforce according to a Gamma distribution. For an individual salesperson, and without any other information, the expected value is the best estimate of \( \theta \). However, as the performance of the salesperson is observed, i.e., as the observations \( Y_1, Y_2, \ldots \), are obtained, the distribution of \( \theta \) can be updated using Bayes' Theorem, and better estimates of the individual's \( \theta \) value obtained. Within this model of a heterogeneous population, random production and Bayesian updating, we develop and compute the optimal policy for identifying and replacing unproductive salespersons.

The work that we present here is motivated by that in Carroll et al. (1986). They showed that the productivity (enlistment contract production per month) for U.S. Navy recruiters, after being adjusted for learning and demotivation effects, can be modeled as a Poisson process, with the rate parameter having a Gamma distribution across the recruiter force. They considered a policy of the form "If a recruiter produces less than \( c^* \) contracts in \( N \) periods, he is rotated out of recruiting service; otherwise he continues to the end of his normal tour of duty." Carroll et al. determined the values of \( N \) and \( c^* \) that maximized average contracts per period, within this type of rule. Such a policy amounts to considering only one evaluation point.

In spite of the pervasive nature of the problem being studied in the paper, the quantitative literature on the subject appears very scarce. Of course, in spirit, the policies presented are similar to process control charts used in quality control (e.g., see Montgomery 1985), where a process is restored to the in-control state once an observation is found to be outside the control limits. This is analogous to firing and replacing a salesperson based on his non-performance in any single period. Variations of the process control charts aim at detecting the out-of-control state of the process based on cumulative observations; see Duncan (1986), and Gordon and Weindling (1975).

The parameter adaptive nature of the Bayesian stochastic control approach used in our formulation falls within the realm of Bayesian adaptive control as presented in, e.g., Aoki (1989), and Bertsekas (1987) §4.4. Our approach is therefore similar in nature to some inventory control models with uncertain demand distributions as studied by Azoury (1985) and Lovejoy (1990).

The paper is organized as follows. In the next section, we discuss the general formulation of the model. In §3, we prove the existence of an optimal policy in the general case. In §4, we derive optimal policies for the special cases when the sales process is Binomial and Poisson, with parameters distributed as Beta and Gamma, respectively. We show that the optimal policy is of a threshold type and depends on the cumulative sales achieved by a salesperson; salespersons who do not meet a specified threshold at the end of each period are replaced. In §5, we develop a special purpose algorithm to determine the optimal threshold levels. Finally, results of several computational experiments are presented and discussed.

2. Model Formulation

We assume that each salesperson is characterized by a productivity parameter \( \theta \), assumed static for simplicity. The sales level achieved by a salesperson is assumed to be either a discrete or a continuous random variable, with \( f(y|\theta) \) representing the probability mass function or the probability density function, respectively, that the salesperson with productivity level \( \theta \) will achieve sales of \( y \) units in a given time period. We make the natural assumption that the expected sales in any given period are uniformly bounded, i.e., for \( Y \), the (random) variable denoting sales in period \( t \) we assume that \( E[Y_t|\theta] \)
\leq M$, for some constant $M > 0$, for all $\theta \in \Theta$. From a mathematical standpoint, this assumption is made for economy in presentation; unbounded rewards can easily be considered at the expense of some more mathematical preliminaries, see Arapostathis et al. (1993). The parameter $\theta$ is unknown to the firm. However, the firm has a prior distribution with density function $G_0(\theta)$ on the value of $\theta$.\footnote{Note that for some distributions the assumption of uniform boundedness i.e., $\mathbb{E}[Y_i | \theta] \leq M$, may not be satisfied. For example, if $Y_i$ is a Poisson random variable and $G_0(\theta)$ is a Gamma distribution, this condition is not satisfied. We can incorporate such situations by truncating the distribution $G_0(\theta)$ at some arbitrarily large value $\bar{\theta} > 0$, such that the tail of the distribution above $\bar{\theta}$ could be neglected. Since $\bar{\theta}$ is chosen to be finite but arbitrarily large, this is not a restriction of any practical significance.} All additional information available to the firm about $\theta$ is contained in the history of observed sales achieved by the salesperson. Using these observations, the firm updates $G_0(\theta)$ at the beginning of each period to obtain an estimate of the salesperson's productivity. Based on this estimate the firm decides either to retain or replace the salesperson.

When the salesperson is replaced or retires, the firm incurs a cost $C$, which includes hiring and training costs for the new employee and severance pay or retirement benefits for the salesperson who is replaced. When a firm replaces a salesperson with a new salesperson, the productivity parameter of the latter is again assumed to have a prior density function $G_0(\theta)$, i.e., the firm hires again from the same pool of (qualified) candidates. Thus, we are considering a scenario where steady state has been achieved, and therefore the observed sales performance of individuals does not significantly impact our belief of $G_0(\theta)$.

We assume that each salesperson has a maximum finite lifetime $N$ within the organization. For example, a successful salesperson may be promoted to a higher managerial position after a few years. Furthermore, all individuals must retire at some point. In addition, consistent with the vast literature on the theory of the firm in economics and marketing, we assume that the firm has an infinite lifetime. Notice that the maximum number of evaluations for each salesperson is $N - 1$. We also show that treating $N$ as a random variable with a geometric distribution is a simple extension and can be handled within our analysis framework.

We model the decision problem as a (parameter adaptive) \textbf{Bayesian stochastic control problem}. The state variable is defined to be the pair $(\theta_t, T_t)$, where $T_t$ is the time the salesperson has been in service and $\theta_t$ is the value of the productivity parameter at time $t$. The variable $T_t$ takes values in the range $\{0, 1, \ldots, N\}$ while $\theta_t$ takes values in some parameter space $\Theta$, which depends on the choice of distribution $f(\cdot | \cdot)$. For example, if the sales process is Binomial then $\theta$ (the probability of success) is in a parameter space $\Theta = [0, 1]$. By assumption, we have the following state transition equations:

\begin{align*}
\theta_{t+1} &= \theta_t, \quad \forall t \quad \text{and} \quad T_{t+1} = T_t + 1, \quad \text{if } T_t < N \text{ and the salesperson is not fired}, \quad \text{(1)}
\end{align*}

\begin{align*}
T_{t+1} &= 0, \quad \text{otherwise.} \quad \text{(2)}
\end{align*}

Note that the state variable $T_t$ is reset to 0 every time a salesperson is fired or retires, and the maximum time a salesperson can be in service is $N$ periods. Further, the state is only partially observable. This is because the firm can only observe $T_t$ but not $\theta_t$.

The decision alternatives available to the firm at the beginning of a time period are: a) fire the salesperson and replace him with a new salesperson; or b) retain the salesperson. Let $U_t$ be the decision taken by the firm at the beginning of time period $t$. Then

\begin{align*}
U_t = \begin{cases} 
1, & \text{if the salesperson is to be replaced}, \\
0, & \text{otherwise.} 
\end{cases} \quad \text{(3)}
\end{align*}

Let the contribution margin per unit of sales be $m$. Note that since $\theta$ is not observable, the true expected reward cannot be computed. Therefore an estimate is obtained by using all the available data to compute $r(G_t, U_t)$ as\footnote{If $y$ is discrete, the integrals are replaced by corresponding sums.}

\begin{align*}
r(G_t, U_t) &= \begin{cases} 
m \int_0^\infty \int_0^\infty y f(y | \theta) G_t(\theta) \, dy \, d\theta, & \text{if } U_t = 0, \\
-C + m \int_0^\infty \int_0^\infty y f(y | \theta) G_0(\theta) \, dy \, d\theta, & \text{if } U_t = 1. 
\end{cases} \quad \text{(4)}
\end{align*}
where \( G_r(\theta) \) is an updated distribution for the unknown parameter.

Denote the sales history of the salesperson by \( Y^n := (Y_1, Y_2, \ldots, Y_n) \). Given that our state variable includes \( T_t \), we can restrict attention to stationary admissible policies, defined as maps \( \pi: (Y^n, T_t) \rightarrow \{0, 1\} \); which prescribe the actions to be taken by the firm at different times, given the available information. Assume that the firm uses a policy \( \pi \in \Pi \), where \( \Pi \) is the set of admissible policies. Let \( R^* \) and \( L^* \) be the expected sales achieved during a lifetime and the expected lifetime respectively for a salesperson randomly drawn from the population. The firm’s objective is assumed to be that of maximizing average productivity, i.e.,

\[
\max_{\pi \in \Pi} \left\{ \frac{mR^* - C}{L^*} \right\}. \tag{5}
\]

Equations (1) to (5) describe our model.

3. Bayesian Optimal Decision Process

The stochastic decision problem with incomplete (or partial) state information defined above can be converted into an equivalent problem with a completely observed state by replacing \( \theta \) by a sufficient statistic (see Aoki 1989, Arapostathis et al. 1993, Bertsekas 1987). Let \( p(\theta|Y^n) \) denote the posterior probability distribution of \( \theta \) given the observations \( Y^n \). The sufficient statistic for \( \theta \) at time \( t \) is defined as a vector \( \beta_t \) such that \( p(\theta|Y^n) = p(\theta|\beta_t) \), i.e., the posterior probability distribution of \( \theta \) given \( Y^n \) only depends on the vector \( \beta_t \). Further, let \( \beta_0 \) be the parameter of the prior distribution for new salespersons \( G_0(\theta) \). Thus, an equivalent completely observable decision process can be formulated in terms of \((\beta_t, T_t)\), which is taken as the new state variable. Note that when a salesperson is replaced, the state \( X_t := (\beta_t, T_t) \) regenerates to the value \((\beta_0, 0)\). In this context (equivalent) admissible policies are maps \( \pi: X_t \rightarrow \{0, 1\} \), we will continue to denote the set of all such equivalent admissible policies by \( \Pi \). We have the following.

\[ \text{THEOREM 1.} \quad \text{For a given policy } \pi \in \Pi, \text{ the average profits over the lifetime of the firm equal the average productivity of the salesforce, i.e.,} \]

\[
\lim_{k \to \infty} \frac{1}{K+1} \mathbb{E}_{X_0} \left[ \sum_{i=0}^{K} r(X_i, U_i) \right] = \frac{mR^* - C}{L^*} \tag{6}
\]

where \( \mathbb{E}_{X_0} \{ \cdot \} \) is the expectation operator induced by \( \pi \in \Pi \) with initial state \( X_0 \).

\[ \text{PROOF.} \quad \text{The proof follows along lines similar to Theorem 7.5 in Ross (1970, pp. 160-161), by noting that the controlled process, under any } \pi \in \Pi, \text{ is a regenerative process and the regeneration time is finite (} \leq N). \quad \square \]

The optimality criteria and optimal values needed in the sequel are given below.

**Discounted Reward:** For a discount factor \( 0 < \alpha < 1 \), the discounted reward incurred by \( \pi \in \Pi \) is given by

\[
J_\alpha(X_0, \pi) := \lim_{k \to \infty} \mathbb{E}_{X_0} \left[ \sum_{i=0}^{K} \alpha^i r(X_i, U_i) \right], \tag{7}
\]

and the optimal \( \alpha \)-discounted value function is defined as

\[
J^*_\alpha(X_0) := \sup_{\pi \in \Pi} \{ J_\alpha(X_0, \pi) \}. \tag{8}
\]

If for all \( X_0 \), \( J_\alpha(X_0, \pi^*) = J^*_\alpha(X_0) \), then \( \pi^* \) is said to be discounted reward-optimal. Similar terminology will be used for other criteria.

**Average Reward:** The long-run expected average reward under \( \pi \in \Pi \) is given by

\[
J(X_0, \pi) := \lim_{k \to \infty} \frac{1}{K+1} \mathbb{E}_{X_0} \left[ \sum_{i=0}^{K} r(X_i, U_i) \right], \tag{9}
\]

and the optimal average reward is defined as

\[
J^*(X_0) := \sup_{\pi \in \Pi} \{ J(X_0, \pi) \}. \tag{10}
\]

The following result establishes the existence of an optimal policy:

\[ \text{THEOREM 2.} \quad \text{There exists an optimal stationary policy which maximizes the average productivity of the salesperson.} \]

\[ \text{PROOF.} \quad \text{Again, we note that the controlled process is a regenerative system, and because the lifetime of the} \]


\[ ^3 \text{In general, the dimension of } \beta_n \text{ can vary with } n, \text{ as the prior distribution } G_0(\theta) \text{ is updated.} \]
salesperson is finite, the regeneration time is finite. Then by Theorem 2.4 in Ross (1983, p. 97) we see that \( J^*_e(X_e) - J^*_e(X_o) \) is uniformly bounded in \( \alpha \) and \( X_e \). This result ensures the existence of bounded solutions for the average reward optimality equation and thus ensures the existence of an optimal stationary policy which maximizes the average rewards (see Ross 1983, Theorem 2.2 and Arapostathis et al. 1993, Theorem 6.1). □

In order to enable us to characterize the structure of the optimal policy, we assume that \( G_0(\theta) \) is a conjugate distribution, i.e., the distribution on \( \theta \) remains in the same family of distributions after it is updated using Bayes’ Theorem. This assumption is satisfied by numerous distributions. For example, if \( f(\cdot | \theta) \) is a normal distribution, then a normal density \( G_0(\theta) \) satisfies the above assumption. Other distribution pairs \( G_0(\theta) \) and \( f(\cdot | \theta) \) satisfying this condition include the Beta-Binomial and the Gamma-Poisson. Both these distributions have been used in the salesforce context, the Gamma-Poisson by Carroll et al. (1986) and the Beta-Binomial by Basu et al. (1985). To justify these models, we consider the following scenarios. The number of potential accounts in a given territory is known. Let \( \bar{M} \) represent the total number of accounts. If we assume that the salesperson’s productivity level determines the ability of the salesperson to close a sale in any given period, then the process can be represented by a Binomial distribution, with success probability being the individual’s productivity parameter \( \theta \). In fact, account penetration ratio, i.e., percentage of total accounts that the salesperson sells to, is a standard measure of salesperson performance (see Churchill et al. 1993, p. 765). Alternatively, if the sales process is conceptualized as an arrival process with the mean arrivals as a measure representing the productivity of the salesperson then the Gamma-Poisson pair is an adequate representation.\(^4\)

By using the observed sales history \( Y^n \) and Bayes’ Theorem, \( \beta \), can be recursively updated; see Bertsekas 1987, pp. 125–127, Aoki 1989, pp. 292–295. Standard results exist in the literature for the posterior distributions of both the Beta-Binomial and the Gamma-Poisson pairs (see DeGroot 1986). For example, if \( (\beta_0, \beta^*_0) \) represent the initial parameters of the Beta distribution and the sales process is Binomial with parameters \( \bar{M} \) and \( \theta \), then the posterior distribution only depends on the total sales and the value of \( T_e \). Here \( \bar{M} \) gives the maximum possible sales in a period, and \( \theta \) the probability of closing a sale in each case. Let \( Z_n = \sum_{i=1}^{n} Y_i \). The formula for the updating in this case is given as follows.

\[
\beta^*_n = \beta^*_0 + Z_n \tag{11a}
\]

\[
\beta^*_n = \beta^*_0 + n\bar{M} - Z_n \tag{11b}
\]

For the Gamma-Poisson case we have:

\[
\beta^*_n = \beta^*_0 + Z_n \tag{12a}
\]

\[
\beta^{*_n} = \beta^*_0 + n \tag{12b}
\]

Thus, the cumulative sales \( Z_n \), and the periods that the salesperson has been in service \( n \), serve as the sufficient statistic for each of these cases.

4. Optimal Policy Characterization

We now consider the special cases when the sales are either Binomial or Poisson and the corresponding prior distribution on \( \theta \) is Beta or Gamma, respectively. We show that the optimal policy is of a threshold type. First, we recall that the conditional density \( G(\cdot | \phi) \) is said to increase in likelihood ratios as \( \phi \) increases if \( G(\cdot | \phi_1) / G(\cdot | \phi_2) \) increases for \( \phi_1 > \phi_2 \); see Ross 1983, p. 146. We now state the following lemmas. The proofs are presented in Appendix A.

**Lemma 1.** For a given value of \( T_e \), in both the Beta-Binomial and the Gamma-Poisson cases, the posterior density of \( \theta \) increases in likelihood ratio as the cumulative sales \( Z_{T_e} \) increases.

**Lemma 2.** For a given value of \( T_e \), in both the Beta-Binomial and the Gamma-Poisson cases, we have that

\[
\int_0^T \sum_y y f(y | \theta) G(\theta | T_e) d\theta
\]

increases as the cumulative sales \( Z_{T_e} \) increases. In other words, the expected production in a period increases as the cumulative sales achieved by a salesperson in \( T_e \) periods increases.
LEMMMA 3. For a given value of cumulative sales $Z_{T_i}$ in both the Beta-Binomial and the Gamma-Poisson cases, the posterior density of $\theta$ decreases in likelihood ratio as the number of periods a salesperson has been in service increases.

LEMMMA 4. For a given value of cumulative sales $Z_{T_i}$ in both the Beta-Binomial and the Gamma-Poisson cases, we have that

$$\int_{\Theta} \sum_y y f(y|\theta)G(\theta | \beta_{T_i}) \, d\theta$$

decreases in $T_i$. In other words, the expected production in a period decreases as the number of periods taken by a salesperson to achieve a given level of cumulative sales increases.

The following theorem establishes the structure of the optimal policy:

THEOREM 3. For both the Beta-Binomial and the Gamma-Poisson, the optimal policy $\pi^*$ is:

$$\pi^*(Z_{T_i}) = U_i^* = \begin{cases} 0, \text{ (Retain) if } Z_{T_i} \geq c^*_i, \\ 1, \text{ (Replace) otherwise,} \end{cases}$$

(13)

where $c_i^*$, $i = 1, 2 \cdots N - 1$ are constants such that $c_1^* \leq c_2^* \leq \cdots \leq c_{N-1}^*$.

PROOF. At any given time, the firm faces the following decision problem

$$\max \left[ \alpha J^*_a(X_0) - C; \alpha J^*_a(X_{T_{i+1}}) \right. \\
\left. + m \int_{\theta} \sum_y y f(y|\theta)G(\theta | \beta_{T_i}) \, d\theta \right],$$

(14)

where $J^*_a(\cdot)$ represents the optimal $\alpha$-discounted expected return function. The first term in the brackets in (14) corresponds to the action $\text{Replace}$, while the second term corresponds to the action $\text{Retain}$. Note that the first term in the brackets in (14) is a constant.

The likelihood ratio ordering in Lemmas 1 and 3 ensures monotonicity of the $\alpha$-discounted value function $J^*_a(\cdot)$, in both $T_i$ and $Z_{T_i}$; see Ross 1983, p. 148. Therefore for a fixed $Z_{T_i}$, the value function decreases as $T_i$ increases. Similarly, for a fixed $T_i$, the value function increases as $Z_{T_i}$ increases. Thus, using Lemmas 2 and 4, we see that the second term in (14) decreases as $T_i$ increases, and increases as $Z_{T_i}$ increases. Therefore, for a fixed $T_i$ there is a value of $Z_{T_i}$, say $c^*_i$, for which both terms in (14) are equal. Hence, the optimal policy is of threshold type:

$$\pi(Z_{T_i}) = U_i^* = \begin{cases} 0, \text{ (Retain) if } Z_{T_i} \geq c^*_i, \\ 1, \text{ (Replace) otherwise.} \end{cases}$$

(15)

Also, since the second term in the brackets in (14) is decreasing in $T_i$, $c_i^*(\alpha) \leq c_2^*(\alpha) \cdots \leq c_{N-1}^*(\alpha)$. Then by a standard argument, letting $\alpha \uparrow 1$, the result follows; see Arapostathis et al. 1993.

The above theorem can be easily extended to consider the case where $N$ is a geometrically distributed random variable. Let $\gamma$ be the probability that a salesperson retires in any given period. Then we have:

COROLLARY 1. When $N$ is a geometrically distributed random variable then the optimal policy is of threshold type:

$$\pi(Z_{T_i}) = U_i^* = \begin{cases} 0, \text{ (Retain) if } Z_{T_i} \geq c^*_i(\gamma), \\ 1, \text{ (Replace) otherwise,} \end{cases}$$

(16)

where $c_1^*(\gamma) \leq c_2^*(\gamma) \cdots \leq c_{N-1}^*(\gamma)$.

The proof is presented in Appendix B.

5. Computation of Optimal Policies

We use the structure of the optimal policy to formulate a special-purpose dynamic programming algorithm, which restricts the search to threshold policies only. To simplify the notation, let

$$J^* = J(X_0, \pi) = \frac{mR^* - C}{L^*};$$

also, let $J^* = J^*(X_0)$. Consider now $J^*(\lambda) = (mR^* - C) - \lambda L^*$. Given $\lambda$ we want to choose the values of $c^*_i$ so as to maximize $J^*(\lambda)$. To search for the optimal threshold values directly, we follow a bisection method similar to that in Thomas (1982) and Miller (1981); see Appendix C for details. By the structure of the optimal policy given in Theorem 3, $J^*(\lambda)$ can be computed via a special purpose dynamic program, with a horizon of $N$ periods, as follows. Let

$$J^*_a(\lambda | Z_{N-1} = z)$$

$$= \sum_{w \in Z} (m \cdot \text{Prob}(Z_N = w | Z_{N-1} = z)(w - z)) - \lambda,$$
and recursively define

\[ J^*_n(\lambda | Z_{n-1} = z) = \sum_{w = z} (m \cdot \text{Prob}(Z_i = w | Z_{n-1} = z)(w - z)) \]

\[ - \lambda + \max_{c_i \in c_n} \left( \sum_{w = c_i} \text{Prob}(Z_i = w | Z_{n-1} = z) \times J^*_n(\lambda | Z_i = w) \right). \]

The first term represents the expected profits in the present period given that the salesperson has achieved a cumulative sales level \( Z_i = z \). The second term in the recursion represents the expected future profits given a critical value \( c_i \). Based on the above recursive relation, we can find \( J^*(\lambda) = J^*_1(\lambda | Z_0 = 0) - C \).

6. Results of Computational Experiments

Several computational experiments were performed, in which we calculated the optimal thresholds for different parameter values of the Beta-Binomial case. Our objective was to explore the impact of the degree of heterogeneity of the salesforce, via the Beta distribution parameters, the cost \( C \) of firing and replacing, and the maximum sales potential \( \bar{M} \) per period. In all experiments, other parameters were held constant as follows: \( N = 10 \) and \( m = 1 \).

To study the impact of heterogeneity two cases of Beta distribution parameters were considered, with \( E(\theta) \) set to \( \frac{1}{2} \) in each case. The parameters chosen were \((\beta_1^a, \beta_2^a) = (5, 10)\) for low heterogeneity and \((\beta_1^b, \beta_2^b) = (1, 2)\) for high heterogeneity. Three values 5, 10, and 20 of the cost \( C \) have been considered. Finally, three values 6, 8, and 10 have been considered for \( \bar{M} \). Thus, we present results for \( 2 \times 3 \times 3 = 18 \) different computational experiments. Note that we are assuming that the maximum sales potential is known, which is in agreement with the marketing literature as discussed earlier.

Table 1 summarizes the results of our computational experiments. Comparing cases 1–9 with cases 10–18, we find that as heterogeneity increases the thresholds also increase; see Figures 1–2 for illustration. Therefore, the higher the uncertainty about a salesperson’s true potential, the more demanding the manager should be in terms of minimal acceptable performance. This of

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course brings to mind the “prove yourself” rule in popular folklore, and also seems to run counter to the “benefit of the doubt” position. Notice from Figures 1–2 that the optimal thresholds follow a relatively linear trend; this seems to agree with the heuristic rule of “x units per period” commonly used in practice which although not optimal appears to be nearly optimal. Also note from cases 4 and 16 that $c_i^* / i$ is not necessarily monotone. In addition, and perhaps somewhat surprisingly, as the heterogeneity increases, so does the optimal profit achievable by the firm; see Figure 3. This can be explained as follows: since our optimality criterion is average performance over the infinite lifetime of the firm, then the higher the heterogeneity, the more demanding the managers should be, given the chances that a new hire may turn out to be a “star” performer. Similarly, as costs of firing and replacing a salesperson increase, the optimal threshold values decrease, i.e., the firm compromises and accepts a mediocre salesperson, if the cost to replace him is too high; see Figure 4. Finally, as
\( \hat{M} \) declines, the optimal threshold values obviously decline too.\(^5\)

\(^5\) The research of Emmanuel Fernández-Gaucherand was partially supported by The Engineering Foundation under grant RI-A-93-10, in part by the National Science Foundation under grant NSF-INT 9201430, and in part by a grant from the AT&T Foundation. The research of M. R. Rao was partially supported by the National Science Foundation under grant NSF-DDM 9001705.

**Appendix A**

In the following proofs the subscripts in \( \beta \) and \( \beta \) should not be interpreted as the value of \( T_i \). The subscripts are used to denote the two parameters of the Beta and the Gamma distribution. Further \( \beta \) denotes the \( i \)th parameter of the prior distribution.

**Proof of Lemma 1.** Using (11) we can write the posterior density for the Beta distribution as:

\[
G(\theta|\beta_1, \beta_2) = \frac{\Gamma(\beta_1^* + \beta_2^* + n\hat{M})\theta^{\beta_1^*+n\hat{M}-1}(1-\theta)^{\beta_2^*+n\hat{M}-n-1}}{\Gamma(\beta_1^* + \hat{M})\Gamma(\beta_2^* + n\hat{M} - Z_\hat{M})}.
\]

(A1)

Let \( z^1 > z^2 \) be values for the cumulative sales achieved in \( n \) periods, then

\[
G(\theta|z^1) = \frac{\beta_1^* + \beta_2^* + n\hat{M}}{\beta_1^* + \hat{M} + Z_\hat{M}} \frac{\theta^{\beta_1^*+n\hat{M}-1}(1-\theta)^{\beta_2^*+n\hat{M}-n-1}}{\theta^{\beta_1^*+Z_\hat{M}-1}(1-\theta)^{\beta_2^*+n\hat{M}-n-Z_\hat{M}}}
\]

(A2)

Let \( K \) be a term which includes all the terms in (A2) which do not involve \( \theta \). Then (A2) can be rewritten as:

\[
G(\theta|z^1) = K\left(1 - \theta\right)^{z_1 - z^2}.
\]

(A3)

Since \( z^1 > z^2 \), the result immediately follows from (A3).

Using (12), for the Gamma-Poisson case we have:

\[
G(\theta|\beta_1, \beta_2) = \frac{(\beta_1^* + n\hat{M})^{\beta_1^*+n\hat{M}-1} \theta^{\beta_1^*+n\hat{M}-1} \exp\left(-\beta_1^* + n\hat{M}\right)}{\Gamma(\beta_1^* + n\hat{M})}
\]

Let \( z^1 > z^2 \), then the likelihood ratio reduces to:

\[
G(\theta|z^1) = (\beta_1^* + n\hat{M})^{\beta_1^*+n\hat{M}-1} \frac{\theta^{\beta_1^*+n\hat{M}-1} \exp\left(-\beta_1^* + n\hat{M}\right)}{\theta^{\beta_1^*+n\hat{M}-1} \exp\left(-\beta_1^* + n\hat{M}\right)}
\]

Since \( z^1 > z^2 \), the likelihood ratio increases as \( \theta \) increases.

**Proof of Lemma 2.** For the Beta-Binomial case,

\[
\sum_y yf(y|\theta)G_\alpha(\theta) = \frac{\beta_1^* + Z_\hat{M}}{\beta_1^* + \beta_2^* + n\hat{M}}.
\]

(A4)

Similarly, for the Gamma-Poisson case we have:

\[
\sum_y yf(y|\theta)G_\alpha(\theta) = \frac{\beta_1^* + Z_\hat{M}}{\beta_1^* + \beta_2^* + n\hat{M}}
\]

(A5)

From A4 and A5 it is clear that for both the Beta-Binomial and the Gamma-Poisson case

\[
\int_0^1 \sum_y yf(y|\theta)G(\theta|\beta_1)\,d\theta
\]

increases as the cumulative sales \( Z_\hat{M} \) increases. \( \square \)

**Proof of Lemma 3.** For the Beta-Binomial case let \( n_1 > n_2 \), and fix the cumulative sales \( z \). Using (A1) the likelihood ratio can be written as:

\[
\frac{G(\theta|n_1)}{G(\theta|n_2)} = K(1-\theta)^{n_1-n_2}
\]

(A6)

where \( K \) includes all terms without \( \theta \). Since \( n_1 > n_2 \), from (A6) it is clear that the likelihood ratio decreases as \( \theta \) increases.

For the Gamma-Poisson case, the likelihood ratio reduces to:

\[
\frac{G(\theta|n_1)}{G(\theta|n_2)} = \frac{(\beta_1^* + n_1)^{\beta_1^*+n_1-1}\exp(-\theta(n_1-n_2))}{(\beta_1^* + n_2)^{\beta_1^*+n_2-1}\exp(-\theta(n_1-n_2))}
\]

Since \( n_1 > n_2 \), the likelihood ratio decreases as \( \theta \) increases.

**Proof of Lemma 4.** The result immediately follows from A4 and A5. \( \square \)

**Appendix B**

**Proof of Corollary 1.** The expected reward if the salesperson does not retire in this period and the firm decides to replace the salesperson is given by

\[
J^*_* = \frac{J^*_*}{C}.
\]

(B1)

If the firm retains the salesperson at time \( t \) then the expected value of such an action is

\[
m\int_0^1 \sum_y yf(y|\theta)G(\theta|\beta_1)\,d\theta
\]

+ \( \alpha(1 - \gamma)J^*_* + \alpha\gamma[j^*_*(X_0, \gamma) - C].
\]

(B2)

The manager replaces the salesperson if the value of the term in (B1) exceeds the value in (B2). After straightforward manipulations, the problem is:

\[
\max \left[ J^*_*(X_0, \gamma) - C ; \alpha_0J^*_*(X_{ret}, \gamma) \right]
\]

(B3)

where \( n = m/1 - \alpha\gamma \) and \( \alpha_0 = \alpha(1 - \gamma)/(1 - \alpha \gamma) \). Equation (B3) is similar to (14) and therefore by similar arguments the optimal policy is of a threshold type:

\[
\pi(Z_\hat{M}) = \begin{cases} 0, & \text{(Retain) if } Z_\hat{M} > c^*_1(\alpha, \gamma), \\ 1, & \text{(Replace) otherwise,} \end{cases}
\]

(B4)

where \( c^*_1(\alpha, \gamma) < c^*_1(\alpha, \gamma) < \cdots < c^*_N(\alpha, \gamma) \). Then letting \( \alpha \to 1 \), the result follows; see Arapostathis et al. 1993. \( \square \)
Appendix C

Bisection Method
Given \( \lambda \) we want to choose the values of \( c \), so as to maximize \( J^*(\lambda) \). Consider now a value \( \bar{\lambda} \) in the interval \([a, b]\) containing the unknown \( J^* \). If \( J^*(\bar{\lambda}) > 0 \), then \( \bar{\lambda} < J^* \); suppose \( \bar{\lambda} \geq J^* \), and let \( \pi^* \) be a strategy such that \( J^*(\bar{\lambda}) = J^*(\lambda^*) \); then \( J^*(\lambda) = (mR^+ - C) - \bar{\lambda}L^* > 0 \), and hence \( J^* > \bar{\lambda} \geq J^* \), violating optimality of \( J^* \). Therefore, \( \bar{\lambda} < J^* \) and \( \lambda \) is to be increased from \( \bar{\lambda} \). Similarly, if \( J^*(\bar{\lambda}) < 0 \), then \( \bar{\lambda} > J^* \) and \( \lambda \) is to be decreased from \( \bar{\lambda} \). Finally if \( J^*(\bar{\lambda}) = 0 \) then \( \bar{\lambda} = J^* \).

Hence, by the bisection method, we can find \( \lambda^* \) so that \( J^*(\lambda^*) = 0 \). It remains for us to find \( J^*(\lambda) \).

References


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