EXAMPLES OF SUPPORT VARIETIES FOR HOPF ALGEBRAS WITH NONCOMMUTATIVE TENSOR PRODUCTS

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Abstract. The representations of some Hopf algebras have curious behavior: Nonprojective modules may have projective tensor powers, and the variety of a tensor product of modules may not be contained in the intersection of their varieties. We explain a family of examples of such Hopf algebras and their modules, and classify left, right, and two-sided ideals in their stable module categories.

1. Introduction

Over the past few decades, the theory of support varieties has become one of the cornerstones of the representation theory of finite groups. Its success has led to the introduction of similar methods in the representation theory of restricted Lie algebras, finite group schemes, and complete intersections, as well as in various other branches of representation theory.

One of the main points of the theory is that projectivity (or, in some situations, finite projective dimension) of a module can be detected through the support variety. Complexity, a measure of the rate of growth of a minimal resolution, may also be detected. In categories with tensor products, often the support of a tensor product is the intersection of the supports, allowing one to tell whether a tensor product of modules is projective, and more generally to compute its complexity.

The purpose of this paper is to highlight a family of examples of finite dimensional Hopf algebras A which are neither commutative nor cocommutative, and for which the tensor product of modules exhibits some unusual behaviour. Here are the main features of this family of examples.

- If M and N are A-modules then $M \otimes N$ and $N \otimes M$ are generally not isomorphic.
- The support of $M \otimes N$ need not be contained in the intersection of the supports of M and N.
- One of the two modules $M \otimes N$ and $N \otimes M$ can be projective while the other is not.
- There are examples of modules M which are not projective, but $M \otimes M$ is projective.
- More generally, given $n > 0$ there exists such an A and M with $M^{\otimes(n-1)}$ not projective but $M^{\otimes n}$ projective. In these examples, the complexity of $M^{\otimes i}$ is equal to $n - i$ for $1 \leq i \leq n$; but other decreasing sequences of complexities can be arranged at will.
- Nonetheless there is a formula for the support of $M \otimes N$ in terms of the support of M and of N.

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Using the support, we classify the localising subcategories, the left ideals, the right ideals and the two-sided ideals of the stable module category $\mathsf{StMod}(A)$.

In more detail, let G and L be finite groups, together with an action $L \to \mathrm{Aut}(G)$ of L on G by group automorphisms. Let k be an algebraically closed field of characteristic p dividing the order of G. As an algebra, $A = kG \otimes k[L]$, the tensor product of the group algebra of G and the coordinate ring of L. The coalgebra structure is dual to the algebra structure of the smash product $k[G]\#kL$ using the action of L on G.

Since $k[L]$ is semisimple, its indecomposable representations are all one-dimensional, and correspond to the elements of L. We write k_ℓ for the one dimensional representation corresponding to $\ell \in L$. So every left A-module M has a canonical decomposition

$$M = \bigoplus_{\ell \in L} M_\ell \otimes k_\ell,$$

where $\otimes = \otimes_k$ and each M_ℓ is a kG-module.

In terms of this decomposition, the tensor product of modules is given by the following formula, as we show in Section 2:

$$(M \otimes N)_\ell = \bigoplus_{\ell_1 \ell_2 = \ell} M_{\ell_1} \otimes {}^{\ell_1}N_{\ell_2},$$

where ${}^{\ell_1}N_{\ell_2}$ denotes the conjugate of the kG-module N_{ℓ_2} by the action of the element ℓ_1 on G. Similarly, the dual of an A-module M is shown in Section 2 to be given by

$$(M^*)_\ell = {}^{\ell^{-1}}(M_{\ell^{-1}})^*.$$

These formulas lead to the examples in Section 3 of A-modules whose tensor products have such curious behavior in comparison to better-known settings. In Section 4 we classify ideal subcategories of the stable module category of all A-modules.

Throughout, our modules will be left modules unless otherwise specified. We will only sometimes require them to be finitely generated, and will so indicate.

2. A noncommutative tensor product

We first give explicitly the Hopf algebra structure of A, which goes back to Molnar [7], and then some immediate consequences for its representations. The action of L on G induces an action on $k[G] := \mathrm{Hom}_k(kG, k)$. Let $k[G]\#kL$ (or $k[G] \rtimes L$) denote the resulting smash (or semidirect) product: As a vector space, it is $k[G] \otimes kL$, and multiplication is given by $(p_g \otimes x)(p_h \otimes y) = \delta_{g,{}^x h}\, p_g \otimes xy$ for all $g, h \in G$ and $x, y \in L$, where $\{p_g \mid g \in G\}$ is the basis of $k[G]$ dual to G, δ denotes the Kronecker delta, and ${}^x h$ is the image of h under the action of x. The algebra $k[G]\#kL$ is a Hopf algebra with the tensor product coalgebra structure. Let

$$A := \mathrm{Hom}_k(k[G]\#kL, k),$$

the Hopf algebra dual to $k[G]\#kL$. Then A has the tensor product algebra structure, $A = kG \otimes k[L]$. The comultiplication, dual to multiplication on $k[G]\#kL$, is given by

$$\Delta(g \otimes p_\ell) = \sum_{x \in L} (g \otimes p_x) \otimes ({}^{x^{-1}}g \otimes p_{x^{-1}\ell})$$

for all $g \in G$, $\ell \in L$. The counit and coinverse are determined by

$$\varepsilon(g \otimes p_\ell) = \delta_{1,\ell} \quad \text{and} \quad S(g \otimes p_\ell) = {}^{\ell^{-1}}(g^{-1}) \otimes p_{\ell^{-1}}.$$

Note that S^2 is the identity map on A.

If M, N are A-modules, then $M \otimes N$ is an A-module via the coproduct Δ. The k-linear dual $M^* := \mathrm{Hom}_k(M, k)$ is an A-module via the coinverse S: $(a \cdot f)(m) = f(S(a)m)$ for all $a \in A$, $f \in M^*$, $m \in M$.

Theorem 2.1. *Let M, N be A-modules, and $\ell \in L$. Then*

$$(M \otimes N)_\ell = \bigoplus_{\ell_1 \ell_2 = \ell} M_{\ell_1} \otimes {}^{\ell_1}N_{\ell_2}$$

and $(M^*)_\ell = {}^{\ell^{-1}}(M_{\ell^{-1}})^*$.

Proof. It suffices to prove the statement componentwise. Let $y, z \in L$. Define a k-linear map $\phi : (M_y \otimes k_y) \otimes (N_z \otimes k_z) \to (M_y \otimes {}^yN_z) \otimes k_{yz}$ by

$$\phi((m \otimes p_y) \otimes (n \otimes p_z)) = (m \otimes n) \otimes p_{yz}$$

for all $m \in M_y$, $n \in N_z$. Clearly ϕ is a bijection. It is also an A-module homomorphism: Let $g \in G$, $\ell \in L$. Applying the coproduct Δ to $g \otimes p_\ell$, we find

$$\begin{aligned} \phi((g \otimes p_\ell)((m \otimes p_y) \otimes (n \otimes p_z)) &= \delta_{\ell,yz}\ \phi((gm \otimes p_y) \otimes (({}^{y^{-1}}g)n \otimes p_z) \\ &= \delta_{\ell,yz}\ (gm \otimes ({}^{y^{-1}}g)n) \otimes p_{yz}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (g \otimes p_\ell)\phi((m \otimes p_y) \otimes (n \otimes p_z)) &= (g \otimes p_\ell)((m \otimes n) \otimes p_{yz}) \\ &= \delta_{\ell,yz}\ (gm \otimes ({}^{y^{-1}}g)n) \otimes p_{yz}. \end{aligned}$$

Let $\psi : {}^{y^{-1}}(M_{y^{-1}})^* \otimes k_{y^{-1}} \to (M_y \otimes k_y)^*$ be the k-linear map defined by

$$\psi(f \otimes p_{y^{-1}}) = \widetilde{f}$$

where $\widetilde{f}(m \otimes p_y) = f(m)$ for all $m \in M_y$ and $y \in L$. Clearly ψ is a bijection. It is also an A-module homomorphism: Let $g \in G$ and $\ell \in L$. Then

$$\psi((g \otimes p_\ell)(f \otimes p_{y^{-1}})) \ = \ \delta_{\ell,y^{-1}}\ \psi({}^{{}^yg}f \otimes p_{y^{-1}}) \ = \ \delta_{\ell,y^{-1}}\ \widetilde{{}^{{}^yg}f}$$

where $\widetilde{{}^{{}^yg}f}(m \otimes p_y) = {}^{{}^yg}f(m) = f(({}^yg)^{-1}m)$. On the other hand,

$$(g \otimes p_\ell)(\psi(f \otimes p_{y^{-1}})) \ = \ (g \otimes p_\ell)(\widetilde{f})$$

where

$$\begin{aligned} ((g \otimes p_\ell)(\widetilde{f}))(m \otimes p_y) &= \sum_{x \in L}(g \otimes p_x)(\widetilde{f}(S({}^{x^{-1}}g \otimes p_{x^{-1}\ell})(m \otimes p_y))) \\ &= (g \otimes p_{y\ell})\widetilde{f}({}^{\ell^{-1}}(g^{-1})m \otimes p_y) \ = \ \delta_{\ell,y^{-1}} f(({}^yg)^{-1}m). \quad \square \end{aligned}$$

We remark that a more complicated formula holds for the tensor product of modules of the dual, $k[G]\#kL$, of A. For example, see [10, Theorem 4.8] for a formula in the more general setting of abelian extensions of Hopf algebras.

3. Support varieties of finite dimensional modules

We recall a definition of support varieties for finite dimensional modules, adapted from Snashall and Solberg [9]. (A more general definition, suitable for infinite dimensional modules, will be recalled in the next section.) We illustrate with examples some ways in which their behavior is different from the cocommutative case. For this purpose, *assume from now on that G is a p-group.* (The more general case is slightly more complicated.) Recall that the group cohomology ring, $H^*(G,k) := \mathrm{Ext}^*_{kG}(k,k)$, may be regarded as a quotient of the Hochschild cohomology ring, $\mathrm{HH}^*(kG) := \mathrm{Ext}^*_{kG\otimes(kG)^{op}}(kG,kG)$, by a nilpotent ideal, and the quotient map is split by the canonical inclusion $H^*(G,k) \to \mathrm{HH}^*(kG)$ (see, e.g., [8, Theorem 10.1]). It follows that the tensor product $H^*(G,k)\otimes k[L]$ may also be regarded as a quotient of $\mathrm{HH}^*(A) \cong \mathrm{HH}^*(kG)\otimes k[L]$ by a nilpotent ideal, and the quotient map is split by the canonical inclusion $H^*(G,k)\otimes k[L] \to \mathrm{HH}^*(A)$.

For finite dimensional A-modules M and N, consider the action of the Hochschild cohomology ring $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,N)$ given by $-\otimes_A M$ followed by Yoneda composition. By restriction we obtain an action of $\mathrm{H}^*(G,k)\otimes k[L]$, and it is this action that we choose to define support varieties. Let $I_A(M)$ denote the annihilator in $\mathrm{H}^*(G,k)\otimes k[L]$ of this action on $\mathrm{Ext}^*_A(M,M)$. The *support variety* of M is

$$V_A(M) := \mathrm{Proj}(\mathrm{H}^*(G,k)\otimes k[L]/I_A(M)),$$

where Proj denotes the space of homogeneous prime ideals other than the maximal ideal of positive degree elements. These varieties have standard properties as proved in [4, 9]. For example, a finite dimensional A-module M is projective if, and only if, its support variety is empty, and more generally the dimension of $V_A(M)$ is the complexity of M (the rate of growth of a minimal projective resolution).

Remark 3.1. Our definition of support variety is equivalent to that of Snashall and Solberg [9] since we have assumed G is a p-group. It differs from that of Feldvoss and the second author [5], since the cohomology $\mathrm{H}^*(A,k) := \mathrm{Ext}^*_A(k,k)$ is isomorphic to $\mathrm{H}^*(G,k)$, and not to $\mathrm{H}^*(G,k)\otimes k[L]$. It has an advantage over the latter in that it remembers more information about an A-module.

The support variety $V_A := V_A(k)$ has the form

$$\mathrm{Proj}(H^*(G,k)\otimes k[L]) = V_G \times L.$$

We write $V_{G,\ell}(M)$ for $V_G(M_\ell)$, the support variety of the kG-module M_ℓ. Then

$$V_A(M) = \coprod_{\ell\in L} V_{G,\ell}(M)\times \ell \subseteq V_G\times L.$$

For each ℓ, the variety $V_G(M_\ell)$ is the collection of primes containing the kernel of the map from $H^*(G,k)$ to $\mathrm{Ext}^*_{kG}(M_\ell,M_\ell)$ given by $-\otimes M_\ell$.

Our formula for the support of a tensor product follows directly from the tensor product formula of Theorem 2.1 for A-modules and properties of support varieties for kG-modules:

$$V_{G,\ell}(M\otimes N) = \bigcup_{\ell_1\ell_2=\ell} V_{G,\ell_1}(M)\cap {}^{\ell_1}V_{G,\ell_2}(N).$$

Example 3.2. Let G be the Klein 4-group with nonidentity elements a,b,c, and let L be the cyclic group of order 3 with generator ℓ, permuting a,b,c cyclically. We work over a field k of characteristic 2. Let U be the kG-module given by the quotient of the left regular module kG by the ideal generated by $a-1$. Let $A = kG \otimes k[L]$, and consider the following A-modules: $M = U \otimes k_1$ and $N = k \otimes k_\ell$. The support varieties of $M \otimes N$ and $N \otimes M$ are

$$\begin{aligned} V_A(M \otimes N) &= V_G(U) \times \ell, \quad \text{and} \\ V_A(N \otimes M) &= V_A({}^{\ell}U \otimes k_\ell) \;=\; {}^{\ell}V_G(U) \times \ell \;\neq\; V_A(M \otimes N), \end{aligned}$$

since $V_G(U) \neq {}^{\ell}V_G(U)$. For comparison, note that $V_A(M) \cap V_A(N) = \varnothing$. We thus see that the variety of the tensor product depends on the order and is *not* contained in the intersection of the varieties. The same is true even if we use the version of support varieties in [5], contradicting Proposition 2.4(5) in that paper; see [6] for a correction.

By modifying these modules, we obtain an example where $M \otimes N$ is projective while $N \otimes M$ is not: Let $M = U \otimes k_\ell$ and $N = U \otimes k_1$. Then $M \otimes N \cong (U \otimes {}^{\ell}U) \otimes k_\ell$, which has empty variety, and so is projective. On the other hand, $N \otimes M \cong (U \otimes U) \otimes k_\ell$, which has (nonempty) variety corresponding to that of U.

Finally, if $M = U \otimes k_\ell$ as before, then

$$V_A(M \otimes M) = V_G(U \otimes {}^{\ell}U) \times \ell^2 = (V_G(U) \cap V_G({}^{\ell}U)) \times \ell^2 = \varnothing.$$

Therefore $M \otimes M$ is projective while M is not. Moreover, $V_A(M^*) = V_G({}^{\ell}U^*) \times \ell^{-1}$, which differs from $V_A(M)$ (and their support varieties as defined in [5] also differ from each other).

The projectivity of the second tensor power of a nonprojective module is generalized in the next example.

Example 3.3. Let k be a field of positive characteristic p and let $n \geq 2$ be a positive integer. Let $G = (\mathbb{Z}/p\mathbb{Z})^n$, generated by $g_1, \dots, g_n$, and $L = \mathbb{Z}/n\mathbb{Z}$, with generator ℓ, acting on G by cyclically permuting these generators. Let U be the kG-module that is the quotient of kG by the ideal generated by $g_1 - 1, \dots, g_{n-1} - 1$, equivalently U is the trivial module induced from the subgroup $\langle g_1, \dots, g_{n-1} \rangle$. Then

$$V_G(U \otimes {}^{\ell}U \otimes \cdots \otimes {}^{\ell^{n-2}}U) = V_G(U) \cap V_G({}^{\ell}U) \cap \cdots \cap V_G({}^{\ell^{n-2}}U) \neq \varnothing,$$

while

$$V_G(U \otimes {}^{\ell}U \otimes \cdots \otimes {}^{\ell^{n-1}}U) = \varnothing.$$

Therefore, letting $A = kG \otimes k[L]$ and $M = U \otimes k_\ell$, we have

$$V_A(M^{\otimes(n-1)}) \neq \varnothing \quad \text{and} \quad V_A(M^{\otimes n}) = \varnothing.$$

As a consequence, $M^{\otimes n}$ is projective while $M^{\otimes(n-1)}$ is not projective. (Compare with [3, Theorem 6.1], in which a similar phenomenon occurs in the unbounded derived category of a non-Noetherian commutative ring.)

4. Localising subcategories

Next, we turn to applications of support theory to the structure of the stable module category $\mathsf{StMod}(A)$, a triangulated category whose objects are all A-modules (including infinite dimensional ones), and morphisms are all A-module homomorphisms modulo those factoring through projective modules. For our purposes in this section, we require a version of support varieties that works well for infinitely generated modules and agrees with the definition given in the last section for finitely generated modules.

Set $R = \mathrm{H}^*(G,k)\otimes k[L]$. Recall that there is a canonical inclusion of R into the Hochschild cohomology ring $\mathrm{HH}^*(A)$. Composing this with the natural map from $\mathrm{HH}^*(A)$ to the graded center $Z^*(\mathsf{StMod}(A))$, we obtain an action of R on $\mathsf{StMod}(A)$ in the sense of [1] by the first author, Iyengar, and Krause. As in Section 5 of [1], we obtain local cohomology functors $\Gamma_{\mathfrak{p}}\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ for each $\mathfrak{p}$ in the homogeneous prime ideal spectrum of R. We have $\Gamma_{\mathfrak{p}} \neq 0$ if and only if $\mathfrak{p}$ is not a maximal ideal. We define $\mathcal{V}_A$ to be the projective spectrum of nonmaximal homogeneous prime ideals of R, and we define the *support* of an A-module M to be

$$\mathcal{V}_A(M) = \{\mathfrak{p} \mid \Gamma_{\mathfrak{p}}(M) \neq 0\}.$$

In case M is finite dimensional, $\mathcal{V}_A(M)$ coincides with the set of nonmaximal homogeneous primes containing $I_A(M)$, and therefore it determines and is determined by $V_A(M)$.

A *left ideal localising subcategory* of $\mathsf{StMod}(A)$ is a full triangulated subcategory that is closed under taking direct summands, arbitrary direct sums, and tensoring on the left by objects of $\mathsf{StMod}(A)$. (In fact, the latter two properties imply the first; see [2].) Similarly define *right* and *two-sided ideal localising subcategories.*

The main theorem of [2] implies that if we regard $\mathsf{StMod}(kG)$ as a tensor triangulated category, it is stratified by the action of $H^*(G,k)$. Loosely speaking, this means that the tensor ideal localising subcategories are classified in terms of subsets of the variety $\mathcal{V}_G = \operatorname{Proj} H^*(G,k)$, the projective spectrum of homogeneous prime ideals of the cohomology ring $H^*(G,k)$, but not including the maximal ideal of positive degree elements. If we forget the tensor structure, this is still true provided that G is a finite p-group, but not more generally.

Next we describe the classification theorem for tensor ideals in $\mathsf{StMod}(A)$. Since the tensor product formula in Theorem 2.1 gives

$$(k\otimes k_{\ell_1})\otimes(M_\ell\otimes k_\ell)\otimes(k\otimes k_{\ell_2^{-1}}) = {}^{\ell_1}M_\ell\otimes k_{\ell_1\ell\ell_2^{-1}},$$

we introduce an action of $L\times L$ on $\mathcal{V}_G\times L$ as follows.

$$(\ell_1,\ell_2)\colon (x,\ell)\mapsto ({}^{\ell_1}x, \ell_1\ell\ell_2^{-1}).$$

The next theorem is a straightforward consequence of the tensor product formula in Theorem 2.1 and the result [2, Theorem 10.3] of the first author, Iyengar and Krause in the finite group setting.

Theorem 4.1. *Let A be the Hopf algebra described above. Then the theory of support gives the following bijections:*

(i) *The left ideal localising subcategories of* $\mathsf{StMod}(A)$ *and the subsets of the set of orbits of $L\times 1$ on $\mathcal{V}_G\times L$.*

(ii) *The right ideal localising subcategories of* $\mathsf{StMod}(A)$ *and the subsets of the set of orbits of $1\times L$ on $\mathcal{V}_G\times L$.*

(iii) *The two-sided ideal localising subcategories of* $\mathsf{StMod}(A)$ *and the subsets of the set of orbits of $L\times L$ on $\mathcal{V}_G\times L$.*

In each case the correspondence takes a set of orbits to the full subcategory of modules whose support lies in one of the orbits in the set, and it takes a subcategory to the union of the supports of its objects.

Remark 4.2. Abstractly, the orbits in (i) and (ii) both give copies of $\mathcal{V}_G$, but the left ideals are not the same as the right ideals. The orbits in (iii) give the quotient $\mathcal{V}_G/L$, namely the projective spectrum of the fixed points $H^*(G,k)^L$.

There are analogous statements for the category $\mathsf{Mod}(A)$ of all A-modules, an application of [2, Theorem 10.4] for finite groups: The same sets of orbits classify non-zero tensor ideal localising subcategories of $\mathsf{Mod}(A)$. (A full subcategory $\mathcal{C}$ of $\mathsf{Mod}(A)$ is localising if direct sums and direct summands of modules in $\mathcal{C}$ are also in $\mathcal{C}$ and if any two modules in an exact sequence of A-modules are in $\mathcal{C}$, then so is the third. It is a tensor ideal if it is closed under tensor products with modules in $\mathsf{Mod}(A)$.) Similary, one obtains analogous classification results for the categories of finite dimensional modules, $\mathsf{mod}(A)$ and $\mathsf{stmod}(A)$.

References

[1] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.

[2] ———, *Stratifying modular representations of finite groups*, Ann. of Math. **174** (2011), 1643–1684.

[3] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. **1**0 (2008), no. 1, 413–436.

[4] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, and R. Taillefer, *Support varieties for selfinjective algebras*, K-Theory **3**3 (2004), no. 1, 67–87.

[5] J. Feldvoss and S. Witherspoon, *Support varieties and representation type of small quantum groups*, Int. Math. Res. Not. (2010), no. 7, 1346–1362.

[6] ———, *Erratum to: Support varieties and representation type of small quantum groups*, arxiv:1308.4928.

[7] R. Molnar, *Semi-direct products of Hopf algebras*, J. Algebra **4**7 (1977), 29–51.

[8] S. F. Siegel and S. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. **79** (1999), 131–157.

[9] N. Snashall and Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705–732.

[10] S. Witherspoon, *Products in Hochschild cohomology and Grothendieck rings of group crossed products*, Adv. Math. **185** (2004), no. 1, 136–158.

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