

BRAID GROUP REPRESENTATIONS FROM TWISTED QUANTUM DOUBLES OF FINITE GROUPS

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ABSTRACT. We investigate the braid group representations arising from categories of representations of twisted quantum doubles of finite groups. For these categories, we show that the resulting braid group representations always factor through finite groups, in contrast to the categories associated with quantum groups at roots of unity. We also show that in the case of p -groups, the corresponding pure braid group representations factor through a finite p -group, which answers a question asked of the first author by V. Drinfeld.

1. INTRODUCTION

Any braided tensor category \mathcal{C} gives rise to finite dimensional representations of the braid group \mathcal{B}_n . A natural problem is to determine the image of these representations. This has been carried out to some extent for the braided tensor categories coming from quantum groups and polynomial link invariants at roots of unity [7, 8, 9, 10, 11, 12, 13]. A basic question in this direction is: *Is the image of the representation of \mathcal{B}_n a finite group?* In the aforementioned papers the answer is typically “no”: Finite groups appear only in a few cases when the degree of the root of unity is small.

In this paper we consider the braid group representations associated to the (braided tensor) categories $\text{Mod-}D^\omega(G)$, where $D^\omega(G)$ is the twisted quantum double of the finite group G . We show (Theorem 4.2) that the braid group images are *always* finite. We also answer in the affirmative (Theorem 4.5) a question of Drinfeld: *If G is a p -group, is the image of the pure braid group \mathcal{P}_n also a p -group?*

The contents of the paper are as follows. In Section 2 we record some definitions and basic results on braided categories, and Section 3 is dedicated to the needed facts about $D^\omega(G)$. Then we prove our main results in Section 4. The last section describes some open problems suggested by our work.

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2. BRAIDED CATEGORIES AND BRAID GROUPS

In this section we recall some facts about braided categories and derive some basic consequences. For more complete definitions the reader is referred to either [2] or [14].

The **braid group** \mathcal{B}_n is defined by generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations:

- (B1) $\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}$ for $1 \leq i \leq n-2$,
- (B2) $\beta_i \beta_j = \beta_j \beta_i$ if $|i-j| \geq 2$.

The kernel of the surjective homomorphism from \mathcal{B}_n to the symmetric group S_n given by $\beta_i \mapsto (i, i+1)$ is the **pure braid group** \mathcal{P}_n , and is generated by all conjugates of β_1^2 .

Let \mathcal{C} be a k -linear braided category over an algebraically closed field k of arbitrary characteristic. The braiding structure affords us representations of \mathcal{B}_n as follows. For any object X in \mathcal{C} we have braiding isomorphisms $c_{X,X} \in \text{End}(X^{\otimes 2})$ so that defining

$$\check{R}_i := \text{Id}_X^{\otimes(i-1)} \otimes c_{X,X} \otimes \text{Id}_X^{\otimes(n-i-1)} \in \text{End}(X^{\otimes n})$$

we obtain a representation ϕ_X^n of \mathcal{B}_n by automorphisms of $X^{\otimes n}$ by

$$\phi_X^n(\beta_i) = \check{R}_i.$$

Similarly, for any collection of objects $\{X_i\}_{i=1}^n$, one has representations of \mathcal{P}_n on $X_1 \otimes \dots \otimes X_n$. Throughout the paper, when we refer to representations of \mathcal{P}_n and \mathcal{B}_n arising from tensor products of objects in a braided category, these are the representations we mean.

We say that Y is a **subobject** of Z if there exists a monomorphism $q \in \text{Hom}_{\mathcal{C}}(Y, Z)$, and W is a **quotient object** of Z if there exists an epimorphism $p \in \text{Hom}_{\mathcal{C}}(Z, W)$. Because of the functoriality of the braiding, we have the following obvious lemma, which will be used in Section 4.

Lemma 2.1. *(i) If Y is a quotient object or a subobject of Z , then $\phi_Y^n(\mathcal{B}_n)$ is a quotient group of $\phi_Z^n(\mathcal{B}_n)$ and similarly for the restrictions of these representations to \mathcal{P}_n .*

(ii) Let S be a finite set of objects of a braided tensor category \mathcal{C} for which the image of the representation of \mathcal{P}_n in $\text{End}(X_1 \otimes \dots \otimes X_n)$ is finite for all $X_1, \dots, X_n \in S$. Let X be the direct sum of finitely many objects taken from S . Then the image of the representation of \mathcal{B}_n in $\text{End}(X^{\otimes n})$ is finite.

3. THE TWISTED QUANTUM DOUBLE OF A FINITE GROUP

In this section we define the twisted quantum double of a finite group, and give some basic results that we need. For more details, see for example [3, 5, 16].

Let k be an algebraically closed field of arbitrary characteristic ℓ . Let G be a finite group with identity element e , kG the corresponding group algebra, and

$(kG)^*$ the dual algebra of linear functions from kG to k , under pointwise multiplication. There is a basis of $(kG)^*$ consisting of the dual functions δ_g ($g \in G$), defined by $\delta_g(h) = \delta_{g,h}$ ($g, h \in G$). Let $\omega : G \times G \times G \rightarrow k^\times$ be a 3-cocycle, that is

$$\omega(a, b, c)\omega(a, bc, d)\omega(b, c, d) = \omega(ab, c, d)\omega(a, b, cd)$$

for all $a, b, c, d \in G$. The **twisted quantum** (or **Drinfeld**) **double** $D^\omega(G)$ is a quasi-Hopf algebra whose underlying vector space is $(kG)^* \otimes kG$. We abbreviate the basis element $\delta_x \otimes g$ of $D^\omega(G)$ by $\delta_x \bar{g}$ ($x, g \in G$). Multiplication on $D^\omega(G)$ is defined by

$$(\delta_x \bar{g})(\delta_y \bar{h}) = \theta_x(g, h)\delta_{x,gyg^{-1}}\delta_x \bar{gh},$$

where

$$\theta_x(g, h) = \frac{\omega(x, g, h)\omega(h, h, h^{-1}g^{-1}xgh)}{\omega(g, g^{-1}xg, h)}.$$

As an algebra, $D^\omega(G)$ is semisimple if and only if the characteristic ℓ of k does not divide the order of G [16].

The quasi-coassociative coproduct $\Delta : D^\omega(G) \rightarrow D^\omega(G) \otimes D^\omega(G)$ is defined by

$$\Delta(\delta_x \bar{g}) = \sum_{\substack{y, z \in G \\ yz=x}} \gamma_g(y, z)\delta_y \bar{g} \otimes \delta_x \bar{g},$$

where

$$\gamma_g(y, z) = \frac{\omega(y, z, g)\omega(g, g^{-1}yg, g^{-1}zg)}{\omega(y, g, g^{-1}zg)}.$$

The quasi-Hopf algebra $D^\omega(G)$ is quasitriangular with

$$R = \sum_{g \in G} \delta_g \otimes \bar{g} \quad \text{and} \quad R^{-1} = \sum_{g, h \in G} \theta_{ghg^{-1}}(g, g^{-1})^{-1} \delta_g \bar{e} \otimes \delta_h \bar{g}^{-1}.$$

In particular $R\Delta(a)R^{-1} = \sigma(\Delta(a))$ for all $a \in D^\omega(G)$, where σ is the transposition map. If X and Y are $D^\omega(G)$ -modules, then $\check{R} = \sigma \circ R$ provides a $D^\omega(G)$ -module isomorphism from $X \otimes Y$ to $Y \otimes X$. Let $c_{X,Y}$ be this action by \check{R} . Then the category $\text{Mod-}D^\omega(G)$ of finite dimensional $D^\omega(G)$ -modules is a braided category with braiding c .

4. THE IMAGES OF \mathcal{B}_n AND \mathcal{P}_n

In this section we fix a finite group G and a 3-cocycle ω , and prove that the image of \mathcal{B}_n in $\text{End}_{D^\omega(G)}(V^{\otimes n})$ is finite for any positive integer n and any finite dimensional $D^\omega(G)$ -module V . In case G is a p -group, we prove that the image of \mathcal{P}_n in $\text{End}_{D^\omega(G)}(V^{\otimes n})$ is also a p -group.

Remark 4.1. It follows from a theorem of C. Vafa (see [2, Theorem 3.1.19]) and the so-called *balancing axioms* that for braided fusion categories over \mathbb{C} , the images of the braid group generators β_i in the above representations of \mathcal{B}_n always have finite order. This is far from enough to conclude that the image of \mathcal{B}_n is

finite; Coxeter [4] has shown that the quotient of \mathcal{B}_n by the normal closure of the subgroup generated by $\{\beta_i^k : 1 \leq i \leq n-1\}$ is finite if and only if $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$.

The case of general finite groups. Let r and m be positive integers. The **full monomial group** $G(r, 1, m)$ is the multiplicative group consisting of the $m \times m$ matrices having exactly one nonzero entry in each row and column, all of whose nonzero entries are r th roots of unity. It is one of the irreducible complex reflection groups.

Let $r = |G|_{\ell'}$ be the part of $|G|$ not divisible by the characteristic ℓ of k (i.e. $|G| = r\ell^s$ and $(r, \ell) = 1$).

Theorem 4.2. *Let V be a finite dimensional $D^\omega(G)$ -module. Then the image of \mathcal{B}_n in $\text{End}(V^{\otimes n})$ is finite. More specifically, this image is a quotient of a subgroup of $G(r, 1, m)$ for $m = |G|^{2n}$.*

Proof. We will need the following well known lemma, which follows from [15, Theorem 6.5.8]. Let $\mu_r \subset k^\times$ be the set of r -th roots of unity.

Lemma 4.3. *The natural map $H^i(G, \mu_r) \rightarrow H^i(G, k^\times)$ is surjective. In particular, any element in $H^i(G, k^\times)$ may be represented by a cocycle taking values in μ_r .*

Now we turn to the proof of the theorem. As any finite dimensional $D^\omega(G)$ -module is finitely generated, and therefore is a quotient of a finite rank free module, by Lemma 2.1 (i), it suffices to prove the statement when V is a finite rank free module. By Lemma 2.1 (ii), we need only consider the case $V = D^\omega(G)$, the left regular module.

Assume first that $n = 2$. Let $x, y, a, b \in G$. The action of \check{R} on the basis element $\delta_x \bar{a} \otimes \delta_y \bar{b}$ of $D^\omega(G) \otimes D^\omega(G)$ is

$$\begin{aligned} \check{R}(\delta_x \bar{a} \otimes \delta_y \bar{b}) &= \sigma\left(\sum_{g \in G} \delta_g \otimes \bar{g}\right)(\delta_x \bar{a} \otimes \delta_y \bar{b}) \\ &= \sigma(\theta_{xyx^{-1}}(x, b) \delta_x \bar{a} \otimes \delta_{xyx^{-1}} \overline{xb}) \\ &= \theta_{xyx^{-1}}(x, b) \delta_{xyx^{-1}} \overline{xb} \otimes \delta_x \bar{a}. \end{aligned}$$

If $n > 2$, similar calculations show that each \check{R}_i permutes the chosen basis of $D^\omega(G)$ up to scalar multiples of the form $\theta_{xyx^{-1}}(x, b)$. By Lemma 4.3, we may assume that ω and hence θ takes values in the r -th roots of unity. This implies that the image of \mathcal{B}_n in $\text{End}(D^\omega(G)^{\otimes n})$ is contained in $G(r, 1, m)$. \square

Corollary 4.4. *Let \mathcal{C} be a braided fusion category that is group-theoretical in the sense of [6]. Let V be any object of \mathcal{C} . Then the image of \mathcal{B}_n in $\text{End}(V^{\otimes n})$ is finite.*

Proof. Let $Z(\mathcal{C})$ be the Drinfeld center of \mathcal{C} . Since \mathcal{C} is braided, we have a canonical braided tensor functor $F : \mathcal{C} \rightarrow Z(\mathcal{C})$. Thus it suffices to show the result holds for the category $Z(\mathcal{C})$. Since \mathcal{C} is group-theoretical, $Z(\mathcal{C})$ is equivalent to $\text{Mod-}D^\omega(G)$ for some G, ω . Thus the desired result follows from Theorem 4.2. \square

The case of p -groups.

Theorem 4.5. *Suppose that G is a finite p -group and V is a finite dimensional $D^\omega(G)$ -module. Then the image of \mathcal{P}_n in $\text{End}(V^{\otimes n})$ is also a p -group.*

The rest of the subsection is occupied by the proof of Theorem 4.5. We will need a technical lemma:

Lemma 4.6. *Let H be a group with normal subgroups $H = H_0 \supset H_1 \supset \dots \supset H_N = 1$, such that H_i/H_{i+1} is abelian, and $[H_i, H_j] \subset H_{i+j}$, and let I be a subgroup of $\text{Aut}(H)$ that preserves this filtration and acts trivially on the associated graded group. Then I is nilpotent of class at most $N - 1$.*

Proof. Let $L_1(I) = I$, $L_2(I) = [I, I]$, $L_3(I) = [[I, I], I], \dots$, be the lower central series of I . We must show $L_N(I) = 1$.

We prove by induction on n that for any $f \in L_n(I)$ and $h \in H_m$, $f(h) = ha(h)$, where $a(h) \in H_{n+m}$.

The case $n = 1$ is clear, since $f \in I$ acts trivially on H_m/H_{m+1} . Suppose the statement is true for n . Take $g \in I$, $f \in L_n(I)$ and $h \in H_m$ so that: $f(h) = ha(h)$, $g(h) = hb(h)$, where $a(h) \in H_{n+m}$ and $b(h) \in H_{m+1}$. Then $fg(h) = f(h)f(b(h)) = ha(h)b(h)a(b(h))$, while $gf(h) = hb(h)a(h)b(a(h))$.

Since g acts trivially on the associated graded group, $b(a(h)) \in H_{n+m+1}$. Also $a(b(h)) \in H_{n+m+1}$ since $b(h) \in H_{m+1}$, by the induction assumption. Moreover, $a(h)b(h) = b(h)a(h)$ modulo H_{n+m+1} since $[H_i, H_j] \subset H_{i+j}$. Thus, $fg(h) = gf(h)$ in H/H_{n+m+1} , and thus $[f, g](h) = h$ in H/H_{n+m+1} , which is what we needed to show.

Taking $m = 0$ and $n = N - 1$, any $[f, g] \in L_N(I)$ is the identity on $H = H/H_N$, and the lemma is proved. \square

Now we are ready to prove the theorem. Any finite dimensional $D^\omega(G)$ -module is a quotient of a multiple of the left regular $D^\omega(G)$ -module $H = D^\omega(G)$. By Lemma 2.1, it suffices to show that the image of \mathcal{P}_n in $\text{End}(H^{\otimes n})$ is a p -group. By Theorem 4.2, the image K of \mathcal{P}_n is a subgroup of the full monomial group $G(r, 1, m)$, where $r = p^t$ for some t , and $m = |G|^{2n}$. The normal subgroup of diagonal matrices in K is thus a p -group, so it is enough to show that K modulo the diagonal matrices is a p -group. Thus it suffices to assume that $\omega = 1$ and $H = D(G)$.

Computing, we have:

$$\tilde{R}(\bar{a}\delta_x \otimes \bar{b}\delta_y) = \sigma\left(\sum_{g \in G} \delta_g \otimes \bar{g}\right)(\bar{a}\delta_x \otimes \bar{b}\delta_y) = \overline{axa^{-1}b}\delta_y \otimes \bar{a}\delta_x$$

for all $a, b, x, y \in G$. Denote by (g, x) the element $\bar{g}\delta_x$ so that a basis of $H^{\otimes n}$ is:

$$(g_1, x_1) \otimes \dots \otimes (g_n, x_n)$$

with $g_i, x_i \in G$. The braid generator β_i fixes all factors other than the i th and $(i + 1)$ st, and on these it acts by:

$$\begin{aligned} (g_i, x_i) \otimes (g_{i+1}, x_{i+1}) &\mapsto (g_i x_i g_i^{-1} x_{i+1}, x_{i+1}) \otimes (g_i, x_i) \\ &= ([g_i, x_i] x_i g_{i+1}, x_{i+1}) \otimes (g_i, x_i), \end{aligned}$$

where $[a, b]$ denotes the group commutator. This action induces a homomorphism $\psi : \mathcal{B}_n \rightarrow \text{Aut}(\text{Fr}_{2n})$ where Fr_{2n} is the free group on $2n$ generators. Explicitly, $\psi(\beta_i)$ is the automorphism defined on generators $\{g_i, x_i\}_{i=1}^n$ of Fr_{2n} by:

$$\begin{aligned} x_j &\mapsto x_j, & g_j &\mapsto g_j & \text{for } j &\notin \{i, i+1\} \\ x_i &\mapsto x_{i+1}, & x_{i+1} &\mapsto x_i \\ g_i &\mapsto [g_i, x_i] x_i g_{i+1}, & g_{i+1} &\mapsto g_i. \end{aligned}$$

Since G is a p -group, it is nilpotent of class, say, $N - 1$. Note that ψ descends to a homomorphism $\psi_N : \mathcal{B}_n \rightarrow \text{Aut}(\text{Fr}_{2n}/L_N(\text{Fr}_{2n}))$ where $L_N(\text{Fr}_{2n})$ denotes the N th term of the lower central series of Fr_{2n} . Since G is nilpotent of class $N - 1$, the action of \mathcal{B}_n on the set G^{2n} defined above factors through ψ_N . Thus, setting $I = \psi_N(\mathcal{P}_n)$, one sees that the action of \mathcal{P}_n on $H^{\otimes n}$ factors through I , that is, K is a quotient of I .

Let us now show that I is nilpotent. Define a descending filtration on $M = \text{Fr}_{2n}/L_N(\text{Fr}_{2n})$ by positive integers as follows. Let $M_1 = M$. Define degrees on the generators by $\deg(g_i) = 1$ and $\deg(x_i) = 2$ for all i , and define M_j for $j \geq 2$ to be the normal closure of the group generated by $[M_k, M_{j-k}]$ for all $0 \leq k \leq j$ together with the generators of degree at least j . Since M is nilpotent, this filtration is finite. Further, I preserves this filtration and acts trivially on the quotients M_i/M_{i+1} . By Lemma 4.6, I is nilpotent.

It follows that the finite group K is nilpotent. However, K is generated by conjugates of β_1^2 , and we claim that β_1^2 is an element of order a power of p . Indeed, this follows from the fact that if the ground field is \mathbb{C} (which may be assumed without loss of generality, since the double of G is defined over the integers), then the eigenvalues of $c_{X,Y} c_{Y,X}$ for any objects X, Y are ratios of twists, which are computed from characters of G (in [2]), and hence are roots of unity of degree a power of p . Therefore, K is a finite p -group. The theorem is proved.

5. QUESTIONS

We mention some directions for further investigation suggested by these (and other) results. We refer the reader to [6] and [14] for the relevant definitions.

- (1) Suppose G is a p -group. Theorem 4.5 shows that the image of the associated representation of \mathcal{P}_n is also a p -group. What is its nilpotency class relative to that of G ? Some upper bounds can be obtained from the proof of Theorem 4.5, but it is not clear how tight they are.

- (2) The finite groups that appear as images of representations of \mathcal{B}_n associated to quantum groups and link invariants at roots of unity (see [7, 9, 10, 11, 12, 13]) basically fall into two classes: symplectic groups and extensions of p -groups by the symmetric group S_n . Does this hold for the representations of \mathcal{B}_n associated with $\text{Mod-}D^\omega(G)$? In general, is there a relationship between the image of \mathcal{P}_n and G ?
- (3) As a modular category, $\text{Mod-}D^\omega(G)$ gives rise to (projective) representations of mapping class groups of compact surfaces with boundary. Are the images always finite? It is known to be true for the mapping class groups of the torus and the n -punctured sphere (Theorem 4.2). For more general modular categories, the answer is definitely “no,” see [1, Conjecture 2.4].
- (4) Let us say that a braided category \mathcal{C} has property \mathcal{F} if all braid group representations associated to \mathcal{C} have finite images. What class of braided categories have property \mathcal{F} ? Among braided fusion categories, Corollary 4.4 shows that all braided group-theoretical categories (in the sense of [6]) have property \mathcal{F} . Do all braided fusion categories with integer Frobenius-Perron dimension have property \mathcal{F} ?

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