GERSTENHABER BRACKETS FOR SKEW GROUP ALGEBRAS IN POSITIVE CHARACTERISTIC

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ABSTRACT. The deformation theory of an algebra is controlled by the Gerstenhaber bracket, a Lie bracket on Hochschild cohomology. We develop techniques for evaluating Gerstenhaber brackets of semidirect product algebras recording actions of finite groups over fields of positive characteristic. The Hochschild cohomology and Gerstenhaber bracket of these skew group algebras can be complicated when the characteristic of the underlying field divides the group order. We show how to investigate Gerstenhaber brackets using twisted product resolutions, which are often smaller and more convenient than the cumbersome bar resolution typically used. These resolutions provide a concrete description of the Gerstenhaber bracket suitable for exploring questions in deformation theory. We demonstrate with the prototypical example of a Drinfeld Hecke algebra (graded Hecke algebra) in positive characteristic.

1. INTRODUCTION

The Hochschild cohomology space of an associative algebra is a Gerstenhaber algebra under two binary operations, the cup product and the Gerstenhaber bracket. The Gerstenhaber bracket is a Lie bracket controlling the deformation theory of the algebra. Historically, the bracket has been more difficult to compute than the cup product: It is defined in terms of the cumbersome bar resolution and notoriously resists transfer to more convenient resolutions. In general, we lack user-friendly formulas giving the Gerstenhaber bracket explicitly.

We consider the Hochschild cohomology of a skew group algebra (semidirect product algebra) arising from the action of a finite group G on an algebra S. We work in the modular setting, i.e., over a field k of positive characteristic that may divide the group order $|G|$. In this setting, the Hochschild cohomology of $S \rtimes G$ is complicated by the potentially onerous cohomology of kG, in contrast to the characteristic zero case where it is always trivial.

Computations of the Gerstenhaber bracket on $S \rtimes G$ directly using the bar resolution often yield little useful information—the bar resolution itself is too large and unwieldy. It can be a struggle even to describe adequately the Hochschild cohomology using the bar resolution. Thus one seeks a description of the Gerstenhaber bracket in terms of smaller

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resolutions used to compute Hochschild cohomology, a description that is concrete and straightforward to apply in specific examples.

In this note, we consider the flexible *twisted product resolution* of a skew group algebra: one chooses a convenient resolution for S and another for G and then combines them to create a resolution of $S \rtimes G$. We show how to apply new techniques from [4, 11] on Gerstenhaber brackets to twisted product resolutions for skew group algebras from [8, 9]. This approach provides advantages over employing the often unmanageable but traditional bar resolution. Our results apply to various kinds of resolutions, such as those that are differential graded algebras. We produce an explicit description of the Gerstenhaber bracket that should prove user-friendly and we illustrate with an example from deformation theory. This quintessential example using a small transvection group captures the difference between the modular and nonmodular settings, both in the theory of reflection groups and in the theory of graded Hecke algebras (and rational Cherednik algebras, see [3]).

Outline. In Section 2, we recall the twisted product resolution from [8, 9] obtained by twisting a resolution of S with one for G. We summarize notions for resolutions with diagonal maps in Section 3. We present methods of [4, 11] analyzing Gerstenhaber brackets in Section 4 and show how they apply to twisted product resolutions for skew group algebras in Section 5. We focus on counital resolutions but also show how twisting together two differential graded coalgebras produces a third. We further extend results to resolutions that are merely pre-counital (a weaker condition) using Volkov [11]. We illustrate these techniques by showing how to compute some Gerstenhaber brackets concretely for a small transvection group example from [8] in Section 6.

Notation. Throughout, k is a field of arbitrary characteristic and we take tensor products over k unless otherwise indicated. In addition, we follow the standard Koszul sign convention: For graded vector spaces P, P', Q, Q' and graded maps $f : P \to Q$ and $f' : P' \to Q'$, the map $f \otimes f' : P \otimes P' \to Q \otimes Q'$ has signs attached so that for homogeneous p in P and p' in P',

$$(f \otimes f')(p \otimes p') = (-1)^{(\deg p)(\deg f')}\big(f(p) \otimes f'(p')\big) \qquad \textit{(Koszul sign convention).} \tag{1.1}$$

2. Twisted product resolutions

We recall the twisted product resolution from [8, 9]. Consider a finite group G acting on a k-algebra S by automorphisms. Let $A = S \rtimes G$ be the corresponding skew group algebra: As a vector space, $S \rtimes G = S \otimes kG$, and we abbreviate the element $s \otimes g$ by sg ($s \in S$, $g \in G$) when no confusion can arise. Multiplication is defined by

$$(sg) \cdot (s'g') = s(^g s')\, gg' \quad \text{ for all } s, s' \in S \text{ and } g, g' \in G\,.$$

The action of g on s' here is denoted by $^g s'$. We use the enveloping algebra $S^e = S \otimes S^{op}$ of any algebra S to express bimodule actions as left actions.

Compatible resolutions. We consider projective resolutions

(i) $C:\ \ldots \to C_2 \to C_1 \to C_0 \to 0$ of kG as a kG-bimodule, and

(ii) $D:\ \ldots \to D_2 \to D_1 \to D_0 \to 0$ of S as an S-bimodule.

We assume the resolution C carries a G-grading compatible with the G-action and that D carries a G-action compatible with the S-action. This means that each C_i is G-graded with differentials preserving the grading and

$$g_1\big((C_i)_{g_2}\big)g_3 = (C_i)_{g_1g_2g_3} \quad \text{for all } g_1, g_2, g_3 \in G \text{ and all degrees } i, \tag{2.1}$$

and that D is also a resolution of kG-modules with

$$g \cdot (s \cdot d) = {}^g s \cdot (g \cdot d) \quad \text{and} \quad g \cdot (d \cdot s) = (g \cdot d) \cdot s \quad \text{for all } g \in G,\ s \in S,\ d \in D\,. \tag{2.2}$$

This ensures C and D are *compatible* with the twisting map $g \otimes s \mapsto {}^g s \otimes g$ given by the group action (see [9, Definition 2.17] and the twisted product resolution structure below). This is the case, for example, when $C_\bullet$ is the bar or reduced bar resolution of kG and when $D_\bullet$ is the Koszul resolution of a Koszul algebra S (see [9, Prop 2.20(ii)]).

The twisted product resolution. The *twisted product resolution* $X = C \otimes^G D$ of the algebra $S \rtimes G$ is the total complex of the double complex $C \otimes D$,

$$X = C \otimes^G D \quad \text{where} \quad X_n = \bigoplus_{i+j=n} C_i \otimes D_j\,,$$

with each X_n suffused with the additional structure of a $(S \rtimes G)$-bimodule given by

$$s'g' \cdot (c \otimes d) \cdot sg = g'cg \otimes ({}^{(g'hg)^{-1}} s')({}^{g^{-1}}(ds)) \quad \text{for} \quad c \in C_h, d \in D, g, g', h \in G, s, s' \in S\,.$$

The total differential ∂_X on X is then $\partial_X = \partial_C \otimes 1_D + 1_C \otimes \partial_D$ for differentials ∂_C, ∂_D on C, D, respectively (with the Koszul sign convention (1.1)). With this action, X is a resolution of $A = S \rtimes G$, i.e., X provides an exact sequence of A-bimodules (see [9] or [7, §4]):

$$\ldots \to X_2 \to X_1 \to X_0 \to A \to 0\,.$$

When the A-bimodules X_n are all projective as A^e-modules, X is also a projective resolution of A. This occurs, for example, when D is a Koszul resolution of a Koszul algebra and C is the bar resolution of kG. (See [9, Proposition 2.20(ii)].)

3. Resolutions, diagonal maps, and homotopies

We now consider resolutions with their diagonal maps and augmentations.

Counital resolutions. Consider a k-algebra A and a projective resolution P of A as an A-bimodule,

$$\ldots \to P_2 \to P_1 \to P_0 \to 0\,,$$

with differential ∂_P and augmentation map $\mu_P : P_0 \to A$ extended as usual to

$$\mu_P : P \to A \qquad \text{with } \mu|_{P_i} \equiv 0 \text{ for } i > 0 \qquad (\textit{augmentation map}).$$

Throughout, we use the map

$$\mu_P \otimes 1_P : P \otimes_A P \to P \quad \text{defined by} \quad p \otimes p' \mapsto \mu_P(p) \cdot p'$$

for $p, p' \in P$ (and similarly for $1_P \otimes \mu_P$).

Since $P \otimes_A P$ is also a projective resolution of A with the usual total differential $\partial_{P\otimes P} = \partial_P \otimes 1_P + 1_P \otimes \partial_P$ (using the Koszul sign convention (1.1)), there exists a (degree zero) A-bimodule chain map lifting the canonical A-bimodule isomorphism $A \xrightarrow{\sim} A \otimes_A A$, called a diagonal map:

$$\Delta_P : P \to P \otimes_A P \qquad (\textit{diagonal map}).$$

The resolution (P, Δ_P, μ_P) is

coassociative when $(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$ as maps $P \to P \otimes_A P \otimes_A P$, and
counital when $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$ as maps $P \to P$.

The resolution (P, Δ_P, μ_P) is a *differential graded coalgebra* when it is both coassociative and counital. This means that $P = \oplus_i P_i$ has a coalgebra structure compatible with its differential ∂_P. For our main results, it will suffice to assume only that the counital condition holds, and, in fact, that an even weaker condition from [11] holds (see Theorem 5.12).

Homotopy from right to left. The chain maps $\mu_P \otimes 1_P$ and $1_P \otimes \mu_P$ from $P \otimes_A P$ to P both lift the canonical isomorphism $A \otimes_A A \xrightarrow{\sim} A$, and therefore are chain homotopic. Let ϕ_P be a homotopy from $\mu_P \otimes 1_P$ to $1_P \otimes \mu_P$, i.e., a map $\phi_P : P \otimes_A P \to P$ with $\phi_P(P_m \otimes_A P_n) \subset P_{m+n+1}$ satisfying

$$\partial_P \, \phi_P + \phi_P \, \partial_{P\otimes_A P} = \mu_P \otimes 1_P - 1_P \otimes \mu_P \, . \tag{3.1}$$

Example 3.2. The bar resolution B of the algebra A is a differential graded coalgebra. Indeed, for $B_n = A \otimes A^{\otimes n} \otimes A$, a diagonal map $\Delta_B : B \to B \otimes_A B$ is defined by

$$\Delta_B(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{j=0}^{n} (a_0 \otimes \cdots \otimes a_j \otimes 1) \otimes_A (1 \otimes a_{j+1} \otimes \cdots \otimes a_{n+1}) \tag{3.3}$$

for $a_0, \ldots, a_{n+1}$ in A. This map is coassociative and counital. One choice of homotopy $\phi_B : B \otimes_A B \to B$ from $\mu_B \otimes 1_B$ to $1_B \otimes \mu_B$ is defined by

$$\begin{aligned} &\phi_B\big((a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p) \otimes_A (a'_p \otimes a_{p+1} \otimes \cdots \otimes a_{n+1})\big) \\ &\qquad = (-1)^{p-1} \, a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p a'_p \otimes a_{p+1} \otimes \cdots \otimes a_{n+1} \qquad \text{for all } a_i, a'_p \in A \, . \end{aligned} \tag{3.4}$$

Koszul resolutions of Koszul algebras are also differential graded coalgebras [1]. The Koszul resolution P of a Koszul algebra embeds into the bar resolution, however the above map ϕ_B does not preserve the image. Instead, a homotopy ϕ_P may be found directly in this case; see [4, §4], [5, §3.2], or [2, §4] for some examples.

4. Gerstenhaber brackets

The Gerstenhaber bracket for an algebra A is defined using the bar resolution of A, but we seek descriptions in terms of more convenient resolutions used to compute Hochschild cohomology. In this section, we summarize some results of [4, 11] and develop additional techniques for computing Gerstenhaber brackets in the modular setting. Contrast with [5, 6], where the characteristic of the underlying field was 0.

Definition of the Gerstenhaber bracket. The Gerstenhaber bracket for A is defined on cochains on the bar resolution B of A. Identify each space of cochains $\text{Hom}_{A^e}(B_n, A)$ with $\text{Hom}_k(A^{\otimes n}, A)$ via the canonical isomorphism. Then the Gerstenhaber bracket

$$[\ ,\ ] : \text{Hom}_k(A^{\otimes n}, A) \times \text{Hom}_k(A^{\otimes m}, A) \to \text{Hom}_k(A^{\otimes(n+m-1)}, A)$$

on cochains is defined by

$$[f, f'] \ = \ f \circ f' - (-1)^{(n-1)(m-1)} f' \circ f$$

where, for a_i in A, the circle product $(f \circ f')(a_1 \otimes \cdots \otimes a_{n+m-1})$ is

$$\sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f\big(a_1 \otimes \cdots \otimes a_{i-1} \otimes f'(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}\big) .$$

Gerstenhaber bracket on other resolutions. Suppose (P, Δ_P, μ_P) is a resolution of A. The Gerstenhaber bracket may be defined directly at the chain level on P using [4, Theorem 3.6] or [11, Corollary 4.5]; we recall how a homotopy ϕ_P (see (3.1)) may give the bracket explicitly.

We first define $\Delta_P^{(2)} : P \to P \otimes_A P \otimes_A P$ by

$$\Delta_P^{(2)} = (1_P \otimes \Delta_P)\Delta_P.$$

For any cochain $f \in \text{Hom}_{A^e}(P_n, A)$, extend f to all of P by defining $f \equiv 0$ on P_m with $m \neq n$. Note that as f is graded of degree $-n$, the Koszul sign convention (1.1) implies that the map $1_P \otimes f \otimes 1_P$ on $P \otimes_A P \otimes_A P$ satisfies

$$(1_P \otimes f \otimes 1_P)(x \otimes y \otimes z) = (-1)^{ni}\, x \otimes f(y) \otimes z \qquad \text{for } x \in P_i,\ y, z \in P.$$

The next theorem is [4, Theorem 3.6] extended to counital resolutions. The proof given in [4] holds under the hypothesis that the resolution is a differential graded coalgebra; Volkov [11, Corollary 4.5] extended this result to more general resolutions by introducing a correction term for the circle product.

Theorem 4.1. *Let (P, Δ_P, μ_P) be a counital resolution of an algebra A. The Gerstenhaber bracket $[\ ,\ ]$ of any elements in Hochschild cohomology $\text{HH}^\bullet(A)$ is given at the cochain level on P by the map $[\ ,\ ]_P$ on cocycles defined by*

$$[f, f']_P = f \circ_P f' - (-1)^{(n-1)(m-1)} f' \circ_P f$$

where $f \circ_P f'$ (similarly $f' \circ_P f$) is the composition

$$f \circ_P f' : P \xrightarrow{\Delta_P^{(2)}} P \otimes_A P \otimes_A P \xrightarrow{1_P \otimes f' \otimes 1_P} P \otimes_A P \xrightarrow{\phi_P} P \xrightarrow{f} A.$$

for $f \in \text{Hom}_{A^e}(P_n, A)$ and $f' \in \text{Hom}_{A^e}(P_m, A)$.

Proof. Since P is counital, Δ_P is a diagonal 2-approximation and $\Delta_P^{(2)}$ is a diagonal 3-approximation in the terminology of Volkov [11] (see his Remark 4.6). Volkov [11] adds a correction term to the circle operation of [4, Theorem 3.6] in order to extend that theorem (see his Section 4 just before Corollary 4.5). This correction term involves a homotopy for

$$(\mu_P \otimes 1_P \otimes 1_P - 1_P \otimes 1_P \otimes \mu_P)\Delta_P^{(2)} ,$$

but the counital property immediately implies that

$$(\mu_P \otimes 1_P \otimes 1_P - 1_P \otimes 1_P \otimes \mu_P)\Delta_P^{(2)} = 0$$

since

$$(\mu_P \otimes 1_P \otimes 1_P)(\Delta_P \otimes 1_P)\Delta_P = ((\mu_P \otimes 1_P)\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes 1_P)\Delta_P = \Delta_P$$

whereas

$$(1_P \otimes 1_P \otimes \mu_P)(\Delta_P \otimes 1_P)\Delta_P = (\Delta_P \otimes \mu_P)\Delta_P = \Delta_P(1_P \otimes \mu_P)\Delta_P = \Delta_P\,.$$

Thus Volkov's correction term vanishes and the conclusion of [4, Theorem 3.6] holds by [11, Corollary 4.5]. □

5. Gerstenhaber bracket for twisted product resolutions

We show in this section that a twisted product resolution of $S \rtimes G$ constructed from two counital resolutions of S and kG is again a counital resolution. We also show that if the two resolutions are in fact differential graded coalgebras, then so is the twisted product resolution. We then give the Gerstenhaber bracket for the twisted product resolution in terms of the maps describing the Gerstenhaber brackets on the separate resolutions individually.

Throughout this section and next, we fix

- a bimodule resolution (C, Δ_C, μ_C) of G and
- a bimodule resolution (D, Δ_D, μ_D) of S, producing
- a twisted product resolution $X = C \otimes^G D$ of $A = S \rtimes G$.

We assume that C and D are compatible with the G and S actions as in Section 2 (see (2.1) and (2.2)) with Δ_C, μ_C preserving the grading and also with with Δ_D, μ_D both kG-module homomorphisms. This is the case, for example, if C is the bar (or reduced bar) resolution of kG and D is the Koszul resolution of a Koszul algebra (see [9, Proposition 2.20(ii)]).

Twisted comultiplication. In the next lemmas, we use diagonal maps Δ_C and Δ_D for C and D to produce a diagonal map $\Delta_X : X \to X \otimes_A X$.

Lemma 5.1. *Define a twisting map* $\tau : C \otimes D \to D \otimes C$ *by*

$$\tau_{i,j}(c \otimes d) = (-1)^{ij}\ ({}^g d \otimes c) \quad \textit{for all } c \in (C_i)_g \textit{ and } d \in D_j\,. \tag{5.2}$$

Then τ *extends to a well-defined chain map*

$$1_C \otimes \tau \otimes 1_D : (C \otimes_{kG} C) \otimes (D \otimes_S D) \longrightarrow (C \otimes^G D) \otimes_{S \rtimes G} (C \otimes^G D)\,.$$

Proof. Consider the map

$$C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D \longrightarrow (C \otimes^G D) \otimes_{S \rtimes G} (C \otimes^G D),$$

where the latter map is the canonical surjection. Calculations show that the composition of these two maps is kG-middle linear in the first two arguments and S-middle linear in the last two arguments, and so it induces a well-defined map as claimed. A calculation confirms that it is a chain map. □

Proposition 5.3. *Let $X = C \otimes^G D$ be the twisted product resolution of $S \rtimes G$ for resolutions (C, Δ_C, μ_C) and (D, Δ_D, μ_D) of kG and S, respectively, as above. Then X has diagonal map $\Delta_X : X \to X \otimes_A X$ and augmentation map $\mu_X : X \to A$ given by*

$$\Delta_X = (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D) \quad \text{and} \quad \mu_X = \mu_C \otimes \mu_D\,.$$

If C and D are both counital, then so is X. If C and D are both coassociative, then so is X. Thus if C and D are differential graded coalgebras, then so is X.

Proof. Observe that Δ_X is a chain map, i.e., $\Delta_X\,\partial_X = (\partial_X \otimes 1_X + 1_X \otimes \partial_X)\Delta_X$, for ∂_X the differential on X. This follows from the fact that τ, Δ_C, Δ_D are all chain maps.

We verify that Δ_X is counital when Δ_C and Δ_D are each counital. We use the extra assumption that μ_C preserves the G-grading and μ_D is a kG-module homomorphism as well as the definition of the $S \rtimes G$-bimodule structure on $C \otimes^G D$:

$$\begin{aligned}(\mu_X \otimes 1_X)\Delta_X &= (\mu_C \otimes \mu_D \otimes 1_C \otimes 1_D)(1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D)\\ &= (\mu_C \otimes 1_C \otimes \mu_D \otimes 1_D)(\Delta_C \otimes \Delta_D) = ((\mu_C \otimes 1_C)\Delta_C) \otimes ((\mu_D \otimes 1_D)\Delta_D)\\ &= 1_C \otimes 1_D \quad = \quad 1_X,\end{aligned}$$

and, similarly, $(1_X \otimes \mu_X)\Delta_X = 1_X$.

Lastly, we check that Δ_X is coassociative when Δ_C and Δ_D are each coassociative. Our hypotheses on the G-grading of C and G-action on D imply that the diagonal maps Δ_C and Δ_D commute suitably with the twisting τ, for example, on $C \otimes C \otimes D \otimes D$,

$$\begin{aligned}&(\Delta_C \otimes \Delta_D \otimes 1_C \otimes 1_D)(1_C \otimes \tau \otimes 1_D)\\ &\quad = (1_C \otimes 1_C \otimes 1_D \otimes \tau \otimes 1_D)(1_C \otimes 1_C \otimes \tau \otimes 1_D \otimes 1_D)(\Delta_C \otimes 1_C \otimes \Delta_D \otimes 1_D).\end{aligned}$$

For brevity, we omit subscripts on identity maps for C and D:

$$\begin{aligned}&(\Delta_X \otimes 1_X)\Delta_X\\ &= \big((1 \otimes \tau \otimes 1)(\Delta_C \otimes \Delta_D) \otimes 1 \otimes 1\big)(1 \otimes \tau \otimes 1)(\Delta_C \otimes \Delta_D)\\ &= (1 \otimes \tau \otimes 1 \otimes 1 \otimes 1)(\Delta_C \otimes \Delta_D \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1)(\Delta_C \otimes \Delta_D)\\ &= (1 \otimes \tau \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes 1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \tau \otimes 1 \otimes 1)(\Delta_C \otimes 1 \otimes \Delta_D \otimes 1)(\Delta_C \otimes \Delta_D)\\ &= (1 \otimes \tau \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes 1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \tau \otimes 1 \otimes 1)(1 \otimes \Delta_C \otimes 1 \otimes \Delta_D)(\Delta_C \otimes \Delta_D)\\ &= (1 \otimes 1 \otimes 1 \otimes \tau \otimes 1)(1 \otimes \tau \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes \tau \otimes 1 \otimes 1)(1 \otimes \Delta_C \otimes 1 \otimes \Delta_D)(\Delta_C \otimes \Delta_D)\\ &= (1 \otimes 1 \otimes 1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta_C \otimes \Delta_D)(1 \otimes \tau \otimes 1)(\Delta_C \otimes \Delta_D)\\ &= (1_X \otimes \Delta_X)\Delta_X.\end{aligned}$$

□

Remark 5.4. One may check that the map $1 \otimes \tau \otimes 1$ of (5.2) interpolates between the maps of the form $\mu \otimes 1 - 1 \otimes \mu$ for the various complexes, that is,

$$\mu_X \otimes 1_X - 1_X \otimes \mu_X = (\mu_C \otimes 1_C \otimes \mu_D \otimes 1_D - 1_C \otimes \mu_C \otimes 1_D \otimes \mu_D)(1_C \otimes \tau^{-1} \otimes 1_D). \tag{5.5}$$

Twisted homotopy. We now give a theorem describing a homotopy from $\mu_X \otimes 1_X$ to $1_X \otimes \mu_X$ concretely in terms of homotopies from $\mu_C \otimes 1_C$ to $1_C \otimes \mu_C$ and from $\mu_D \otimes 1_D$ to $1_D \otimes \mu_D$ by adapting [2, Lemmas 3.3, 3.4, and 3.5] to our setting.

Theorem 5.6. *Let $X = C \otimes^G D$ be as above with homotopies ϕ_C from $\mu_C \otimes 1_C$ to $1_C \otimes \mu_C$ and ϕ_D from $\mu_D \otimes 1_D$ to $1_D \otimes \mu_D$. Define $\phi_X : X \otimes_A X \to X$ by*

$$\phi_X = \big(\phi_C \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes \phi_D\big)(1_C \otimes \tau^{-1} \otimes 1_D).$$

Then ϕ_X is a homotopy from $\mu_X \otimes 1_X$ to $1_X \otimes \mu_X$.

Proof. For convenience, let $\phi'_X : (C \otimes_{kG} C) \otimes (D \otimes_S D) \to C \otimes D$ be the map

$$\phi'_X = \phi_X(1_C \otimes \tau \otimes 1_D) = \phi_C \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes \phi_D\,.$$

Then on $(C \otimes_{kg} C) \otimes (D \otimes_S D)$,

$$\begin{aligned} \partial_X\, \phi_X\ (1_C \otimes \tau \otimes 1_D) =&\ \partial_X\, \phi'_X \\ =&\ (\partial_C \otimes 1_D + 1_C \otimes \partial_D)\big(\phi_C \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes \phi_D\big) \\ =&\ \partial_C\phi_C \otimes \mu_D \otimes 1_D - \phi_C \otimes \partial_D(\mu_D \otimes 1_D) \\ &\ + \partial_C(1_C \otimes \mu_C) \otimes \phi_D + 1_C \otimes \mu_C \otimes \partial_D\phi_D\,, \end{aligned} \tag{5.7}$$

whereas, since $1_C \otimes \tau \otimes 1_D$ is a chain map from $(C \otimes_{kG} C) \otimes (D \otimes_S D)$ to $X \otimes_A X$,

$$\begin{aligned} \phi_X\, \partial_{X\otimes X}\, (1_C \otimes \tau \otimes 1_D) &= \phi_X\, (1_C \otimes \tau \otimes 1_D)\, \partial_{(C\otimes C)\otimes(D\otimes D)} = \phi'_X\ \partial_{(C\otimes C)\otimes(D\otimes D)} \\ &= \phi'_X\, \big(\partial_{C\otimes C} \otimes 1_{D\otimes D} + 1_C \otimes 1_C \otimes \partial_{D\otimes D}\big) \\ &= \phi_C\partial_{C\otimes C} \otimes \mu_D \otimes 1_D + \phi_C \otimes (\mu_D \otimes 1_D)\partial_{D\otimes D} \\ &\quad - (1_C \otimes \mu_C)\partial_{C\otimes C} \otimes \phi_D + 1_C \otimes \mu_C \otimes \phi_D\partial_{D\otimes D}. \end{aligned} \tag{5.8}$$

Note that the Koszul sign convention with $\deg \phi_C, \deg \phi_D = -1$ and $\deg \mu_C, \det \mu_D = 0$ gives the signs in the above calculation, for example,

$$\begin{aligned} (1_C \otimes \partial_D)(\phi_C \otimes \mu_D \otimes 1_D) &= -\phi_C \otimes \partial_D(\mu_D \otimes 1_D) \quad \text{and} \\ (1_C \otimes \mu_C \otimes \phi_D)(\partial_{C\otimes C} \otimes 1_{D\otimes D}) &= -(1_C \otimes \mu_C)\partial_{C\otimes C} \otimes \phi_D \quad \text{whereas} \\ (1_C \otimes \partial_D)(1_C \otimes \mu_C \otimes \phi_D) &= +(1_C \otimes \mu_C \otimes \partial_D\phi_D)\,. \end{aligned}$$

This is because

$$(-1)^{\deg c + \deg c'}(1_C \otimes \mu_C)(c \otimes c') = (-1)^{\deg c}(1_C \otimes \mu_C)(c \otimes c') = (-1)^{\deg c}\mu_C(c') \cdot c$$

for all homogeneous c, c' in C, as $\mu_C(c') = 0$ unless $\deg c' = 0$, and likewise for D.

The second term of (5.7) cancels with the second term of (5.8) as $\mu_D \otimes 1_D$ is a chain map; likewise, the third terms cancel as $\mu_C \otimes 1_C$ is a chain map. Hence

$$\begin{aligned} &(\partial_X\, \phi_X\ + \phi_X\, \partial_{X\otimes X})(1_C \otimes \tau \otimes 1_D) \\ &\qquad = (\partial\phi_C + \phi_C\partial) \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes (\partial\phi_D + \phi_D\partial) \\ &\qquad = (\mu_C \otimes 1_C - 1_C \otimes \mu_C) \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes (\mu_D \otimes 1_D - 1_D \otimes \mu_D) \\ &\qquad = \mu_C \otimes 1_C \otimes \mu_D \otimes 1_D - 1_C \otimes \mu_C \otimes 1_D \otimes \mu_D, \end{aligned}$$

and, by equation (5.5),

$$\begin{aligned} \partial\phi_X + \phi_X\partial &= (\mu_C \otimes 1_C \otimes \mu_D \otimes 1_D - 1_C \otimes \mu_C \otimes 1_D \otimes \mu_D)(1_C \otimes \tau^{-1} \otimes 1_D) \\ &= \mu_X \otimes 1_X - 1_X \otimes \mu_X. \end{aligned}$$

□

Twisted Gerstenhaber bracket. The next theorem gives the Gerstenhaber bracket on a twisted product resolution $X = C \otimes^G D$ for C and D compatible resolutions as in Section 2. It includes the case when C and D are differential graded coalgebras resolving kG and S. Note that the twisting map τ in the theorem is from Lemma 5.1.

Theorem 5.9. *Let $X = C \otimes^G D$ be a twisted product resolution of $A = S \rtimes G$ for counital resolutions (C, Δ_C, μ_C) and (D, Δ_D, μ_D) of kG and S, respectively, as above. Then X is a counital resolution of A, and the Gerstenhaber bracket of elements of Hochschild cohomology $\mathrm{HH}^{\bullet}(A)$ represented by cocycles $f \in \mathrm{Hom}_{A^e}(X_n, A)$ and $f' \in \mathrm{Hom}_{A^e}(X_m, A)$ is represented by the cocycle*

$$[f, f'] = f \circ_X f' - (-1)^{(n-1)(m-1)} f' \circ_X f \,,$$

where $f \circ_X f'$ (similarly $f' \circ_X f$) is the composition

$$X \xrightarrow{(1_X \otimes \Delta_X)\Delta_X} X \otimes_A X \otimes_A X \xrightarrow{1_X \otimes f' \otimes 1_X} X \otimes_A X \xrightarrow{\phi_X} X \xrightarrow{f} A \tag{5.10}$$

with

$$\Delta_X = (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D) \quad \text{and}$$
$$\phi_X = \big(\phi_C \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes \phi_D\big)(1_C \otimes \tau^{-1} \otimes 1_D) \,.$$

Proof. We combine Lemmas 5.1, Proposition 5.3, and Theorem 5.6 with Theorem 4.1. □

Example 5.11. In case $S = S(V) \cong k[x_1, \ldots, x_n]$, the symmetric algebra on a finite dimensional vector space V, we take D to be the Koszul resolution for which a choice of ϕ_D has been made in [4, §4] (see also [5, §3.2]). We may take C to be the bar or reduced bar resolution of kG for some applications, with homotopy ϕ_C as defined by equation (3.4).

Generalization to pre-counital resolutions. The above arguments hold more generally using results of Volkov [11] if we replace the counital hypothesis with a weaker condition. We say a resolution (P, Δ_P, μ_P) is *pre-counital* when

$$(\mu_P \otimes 1_P \otimes 1_P)(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes 1_P \otimes \mu_P)(\Delta \otimes 1_P)\Delta_P \,.$$

Note that all counital resolutions are pre-counital (see the proof of Theorem 4.1). Theorem 5.9 holds in the more general setting of pre-counital resolutions:

Theorem 5.12. *Suppose (C, Δ_C, μ_C) and (D, Δ_D, μ_D) are resolutions of kG and S, respectively, as above. If C and D are both pre-counital, then the twisted product resolution $X = C \otimes^G D$ is also pre-counital and the conclusion of Theorem 5.9 holds.*

Proof. We first verify that X is pre-counital with respect to the diagonal map $\Delta_X : X \otimes_A X \to X$ and augmentation $\mu_X : X \to A$ from Proposition 5.3. To check that $(\mu_X \otimes 1_X \otimes 1_X)(\Delta_X \otimes 1_X)\Delta_X$ coincides with $(1_X \otimes 1_X \otimes \mu_X)(\Delta_X \otimes 1_X)\Delta_X$, we recall that (see the proof of Proposition 5.3)

$$\begin{aligned}
&(\Delta_X \otimes 1_X)\Delta_X \\
&\quad = (1_C \otimes \tau \otimes \tau \otimes 1_D)(1_C \otimes 1_C \otimes \tau \otimes 1_D \otimes 1_D)\big((\Delta_C \otimes 1_C)\Delta_C \otimes (\Delta_D \otimes 1_D)\Delta_D\big) \,.
\end{aligned}$$

We apply $\mu_X \otimes 1_X \otimes 1_X = (\mu_C \otimes \mu_D \otimes 1_C \otimes 1_D \otimes 1_C \otimes 1_D)$ on the left and note that

$$\begin{aligned}&(\mu_C \otimes \mu_D \otimes 1_C \otimes 1_D \otimes 1_C \otimes 1_D)(1_C \otimes \tau \otimes \tau \otimes 1_D)(1_C \otimes 1_C \otimes \tau \otimes 1_D \otimes 1_D)\\ &\qquad = (1_C \otimes \tau \otimes 1_D)(\mu_C \otimes 1_C \otimes 1_C \otimes \mu_D \otimes 1_D \otimes 1_D)\,,\end{aligned}$$

whereas

$$\begin{aligned}&(1_C \otimes 1_D \otimes 1_C \otimes 1_D \otimes \mu_C \otimes \mu_D)(1_C \otimes \tau \otimes \tau \otimes 1_D)(1_C \otimes 1_C \otimes \tau \otimes 1_D \otimes 1_D)\\ &\qquad = (1_C \otimes \tau \otimes 1_D)(1_C \otimes 1_C \otimes \mu_C \otimes 1_D \otimes 1_D \otimes \mu_D)\,.\end{aligned}$$

To finish the check, we use the fact that C is pre-counital and substitute the expression $(1_C \otimes 1_C \otimes \mu_D)(\Delta_C \otimes 1_C)\Delta_C$ for $(\mu_C \otimes 1_C \otimes 1_D)(\Delta_C \otimes 1_C)\Delta_C$, and likewise for D.

The bracket formula of Theorem 4.1 then holds with the assumption that X is pre-counital instead of counital using [11, Corollary 4.5] (as pre-counital implies that the correction term in the proof of Theorem 4.1 vanishes). We then use Lemma 5.1 and Theorem 5.6 to describe the bracket as in Theorem 4.1. Note that $\Delta_X^{(2)} = (\Delta_X \otimes 1_X)\Delta_X$ is a diagonal 3-approximation (in the terminology of [11]) since Δ_X is a chain map. □

6. A small transvection group example

We end by demonstrating how to use a twisted product resolution to compute Gerstenhaber brackets explicitly via Theorem 5.9. We also see how computation of explicit brackets can shed light on questions in deformation theory (see [8]). We illustrate with the prototype example of a graded Hecke algebra (or rational Cherednik algebra) in positive characteristic (see [3] and [8]). In the nonmodular setting, these algebras have parameters supported only on the identity group element and on bireflections; in the modular setting, parameters can also be supported on reflections. All reflections in a finite linear group G acting in the modular setting are either diagonalizable or act as in this example. We include some explicit details to illustrate how to evaluate the maps in Theorem 5.9 concretely. We find both a nonzero and a zero Gerstenhaber bracket.

Group action and twisted product resolution. Say $\mathrm{char}(k) = p > 0$ and consider the cyclic group $G \simeq \mathbb{Z}/p\mathbb{Z}$ acting on $V = k^2$ with basis v, w generated by

$$g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \text{so that} \quad {}^g v = v \;\text{ and }\; {}^g w = v + w.$$

We work in the twisted product resolution $X = C \otimes^G D$ of $S(V) \rtimes G$ obtained from twisting the reduced bar resolution C of kG with the Koszul resolution D of $S(V)$:

$$X_n = \bigoplus_{i+j=n} X_{i,j} \quad \text{for} \quad X_{i,j} = kG \otimes (\overline{kG})^{\otimes i} \otimes kG \otimes S(V) \otimes \textstyle\bigwedge^j V \otimes S(V).$$

Here, $C_n = kG \otimes (\overline{kG})^{\otimes n} \otimes kG$ with $\overline{kG} = kG/k1_G$ and $D_n = S(V) \otimes \bigwedge^n V \otimes S(V)$. Then C and D satisfy the conditions specified in Section 3, and Theorem 5.9 applies.

Cochains. Consider cochains on the resolution X:

$$\begin{aligned}&\kappa \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{0,2}, S(V) \rtimes G),\\ &\lambda \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{1,1}, S(V) \rtimes G), \text{ and}\\ &\delta \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{0,1}, S(V) \rtimes G)\end{aligned}$$

defined by (with subscripts on the tensor signs suppressed for brevity)

$$\begin{aligned} \lambda\big((1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) &= 0, \\ \lambda\big((1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) &= ig^{i-1}, \\ \kappa\big((1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)\big) &= g, \\ \delta\big((1_G \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) &= v, \\ \delta\big((1_G \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) &= 0 \end{aligned}$$

for $0 \le i \le p-1$, with all other values determined by these. One can check directly that λ and κ are 2-cocycles and that δ is a 1-cocycle for X. We will show that

$$[\delta, \kappa] \neq 0 \quad \text{and} \quad [\lambda, \lambda] = [\lambda, \kappa] = 0\,.$$

The diagonal maps. We give some values of the diagonal maps at play in finding the Gerstenhaber brackets. The diagonal map Δ_C on the reduced bar resolution of kG is deduced from (3.3). For example, after identifying g^i with its image in $\overline{kG}$,

$$\begin{aligned} \Delta_C(1_G \otimes g^i \otimes 1_G) &= (1_G \otimes 1_G) \otimes_{kG} (1_G \otimes g^i \otimes 1_G) + (1_G \otimes g^i \otimes 1_G) \otimes_{kG} (1_G \otimes 1_G), \quad \text{and} \\ \Delta_C(1_G \otimes 1_G) &= (1_G \otimes 1_G) \otimes_{kG} (1_G \otimes 1_G)\,. \end{aligned}$$

The diagonal map Δ_D is found from embedding the Koszul into the bar resolution and then using (3.3). For example, we identify $v \wedge w$ with $v \otimes w - w \otimes v$ and observe that

$$\begin{aligned} \Delta_D(1_S \otimes v \wedge w \otimes 1_S) &= (1_S \otimes 1_S) \otimes_S (1_S \otimes v \wedge w \otimes 1_S) \\ &\quad + (1_S \otimes v \otimes 1_S) \otimes_S (1 \otimes w \otimes 1) - (1_S \otimes w \otimes 1_S) \otimes_S (1_S \otimes v \otimes 1_S) \\ &\quad + (1_S \otimes v \wedge w \otimes 1_S) \otimes_S (1_S \otimes 1_S)\,, \quad \text{and} \\ \Delta_D(1_S \otimes v \otimes 1_S) &= (1_S \otimes 1_S) \otimes_S (1_S \otimes v \otimes 1_S) + (1_S \otimes v \otimes 1_S) \otimes_S (1_S \otimes 1_S)\,. \end{aligned}$$

Homotopies. Let $\phi_C : C \otimes_{kG} C \to C$ be the homotopy from $\mu_C \otimes 1$ to $1 \otimes \mu_C$ from (3.4). We choose the homotopy $\phi_D : D \otimes_S D \to D$ from $\mu_D \otimes 1$ to $1 \otimes \mu_D$ given in [4, Definition 4.3] and record a few values here for later use:

$$\begin{aligned} &\phi_D((1 \otimes w \otimes 1) \otimes_S (v \otimes 1)) = 1 \otimes v \wedge w \otimes 1, && \phi_D((1 \otimes v) \otimes_S (1 \otimes w \otimes 1)) = 0, \\ &\phi_D((1 \otimes 1) \otimes_S (1 \otimes v \otimes 1)) = 0, && \phi_D((1 \otimes v \otimes 1) \otimes_S (1 \otimes 1)) = 0\,. \end{aligned}$$

Nonzero bracket. We use Theorem 5.9 to show explicitly that $[\delta, \kappa] = \kappa$. First note that $[\delta, \kappa]$ is zero on all components of X except possibly $X_{0,2}$. We consider the composition (5.10) with $f' = \delta$ and $f = \kappa$ to find $\kappa \circ_X \delta$. As a first step, we apply the map $(\Delta_X \otimes 1_X)\Delta_X$ to the element $(1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)$ of $X_{0,2}$, where, recall

$$\Delta_X = (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D)\,.$$

Direct calculation confirms that

$$\begin{aligned}(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1) \xmapsto{(\Delta_X\otimes 1)\Delta_X}\ & (1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&+(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\\
&+(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\\
&-(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\\
&-(1\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\\
&+(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\\
&+(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\\
&-(1\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\\
&+(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\end{aligned}$$

as an element of $X\otimes_A X\otimes_A X$. We have suppressed all subscripts for brevity; for example, the second summand may be written

$$\big((1_G\otimes_{kG}1_G)\otimes(1_S\otimes_S 1_S)\big)\otimes_A\big((1_G\otimes_{kG}1_G)\otimes(1_S\otimes_S v\otimes_S 1_S)\big)\otimes_A\big((1_G\otimes_{kG}1_G)\otimes(1_S\otimes_S w\otimes_S 1_S)\big)\,.$$

We next apply the map $1_X\otimes\delta\otimes 1_X$; it is nonzero on exactly two summands, the second and the penultimate, and we obtain (with the tensor products over A indicated here)

$$\big((1_G\otimes 1_G)\otimes(1_S\otimes v)\big)\otimes_A\big((1_G\otimes 1_G)\otimes(1_S\otimes w\otimes 1_S)\big)-\big((1_G\otimes 1_G)\otimes(1_S\otimes w\otimes v)\big)\otimes_A\big((1_G\otimes 1_G)\otimes(1_S\otimes 1_S)\big).$$

To apply ϕ_X next, we first rearrange terms with $1_G\otimes\tau^{-1}\otimes 1_S$, producing

$$(1_G\otimes 1_G)\otimes(1_G\otimes 1_G)\otimes(1_S\otimes v)\otimes(1_S\otimes w\otimes 1_S)-(1_G\otimes 1_G)\otimes(1_G\otimes 1_G)\otimes(1_S\otimes w\otimes v)\otimes(1_S\otimes 1_S),$$

and then apply the map $\phi_C\otimes\mu_D\otimes 1_D+1_C\otimes\mu_C\otimes\phi_D$ to obtain

$$(1_G\otimes 1_G\otimes 1_G)\otimes(v\otimes w\otimes 1_S)-(1_G\otimes 1_G)\otimes(1_S\otimes v\wedge w\otimes 1_S).$$

Lastly, we apply κ as the last step of (5.10) and obtain 0 from the first term and $-g$ from the second. Thus

$$(\kappa\circ\delta)\big((1_G\otimes 1_G)\otimes(1_S\otimes v\wedge w\otimes 1_S)\big)=-g=\kappa\big((1_G\otimes 1_G)\otimes(1_S\otimes v\wedge w\otimes 1_S)\big)$$

and $\kappa\circ_X\delta=-\kappa$. We inspect the above calculation with an eye toward switching the order of κ and δ and deduce that $\delta\circ_X\kappa=0$. We conclude, as claimed,

$$[\delta,\kappa]=\delta\circ_X\kappa-\kappa\circ_X\delta=\kappa\,.$$

Zero brackets. We now use Theorem 5.9 to show that $[\lambda,f]=0$ when f is λ or κ. We evaluate composition (5.10) on $X_{1,2}$ with $f'=\lambda$. Other calculations are similar. We first apply $\Delta_X=(1_G\otimes\tau\otimes 1_S)(\Delta_C\otimes\Delta_D)$ to sample input in $X_{1,2}$, noting that

${}^{g^i}w = iv + w$ (with subscripts suppressed again):

$$
\begin{aligned}
&(1\otimes g^i\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&\quad\xmapsto{\Delta_C\otimes\Delta_D}\ (1\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&\qquad +(1\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes w\otimes 1)\\
&\qquad -(1\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes v\otimes 1)\\
&\qquad +(1\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes w\otimes 1)\\
&\qquad -(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes v\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes 1)\\
&\quad\xmapsto{1\otimes\tau\otimes 1}\ (1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&\qquad -(1\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes w\otimes 1)\\
&\qquad +(1\otimes 1)\otimes(1\otimes (iv+w)\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes v\otimes 1)\\
&\qquad +(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes v\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes w\otimes 1)\\
&\qquad -(1\otimes g^i\otimes 1)\otimes(1\otimes w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes v\otimes 1)\\
&\qquad +(1\otimes g^i\otimes 1)\otimes(1\otimes v\wedge w\otimes 1)\otimes(1\otimes 1)\otimes(1\otimes 1)\,,
\end{aligned}
$$

an element of $X\otimes_A X$. Next we apply $\Delta_X\otimes 1_X$: Evaluating $\Delta_C\otimes\Delta_D\otimes 1_X$ on the last expression yields 27 summands; the map $(1_G\otimes\tau\otimes 1_S)\otimes 1_X$ transforms these to 27 summands in $X\otimes_A X\otimes_A X$. A quick check verifies that $1_X\otimes\lambda\otimes 1_X$ vanishes on all but two summands, namely

$$
\begin{aligned}
&-\big((1_G\otimes 1_G)\otimes(1_S\otimes 1_S)\big)\otimes_A\big((1_G\otimes g^i\otimes 1_G)\otimes(1_S\otimes w\otimes 1_S)\big)\otimes_A\big((1_G\otimes 1_G)\otimes(1_S\otimes v\otimes 1_S)\big),\\
&-\big((1_G\otimes 1_G)\otimes(1_S\otimes v\otimes 1_S)\big)\otimes_A\big(1_G\otimes g^i\otimes 1_G)\otimes(1_S\otimes w\otimes 1_S)\big)\otimes_A\big((1_G\otimes 1_G)\otimes(1_S\otimes 1_S)\big),
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
&-\big((1_G\otimes 1_G)\otimes(1_S\otimes 1_S)\big)\otimes_A\big((ig^{i-1}\otimes 1_G)\otimes(1_S\otimes v\otimes 1_S)\big)\\
&-\big((1_G\otimes 1_G)\otimes(1_S\otimes v\otimes 1_S)\big)\otimes_A\big((ig^{i-1}\otimes 1_G)\otimes(1_S\otimes 1_S)\big).
\end{aligned}
$$

Applying ϕ_X followed by $f=\lambda$ or $f=\kappa$ gives 0 as w does not appear in the input.

Remark 6.1. The cocycles λ and κ above were not chosen randomly. These cocycles define a PBW deformation of $S\rtimes G$, and the zero brackets calculated above predict the PBW property. Indeed, in [8], we considered PBW deformations of $S\rtimes G$ given by analogs of Lusztig's graded Hecke algebras and symplectic reflection algebras over fields of positive characteristic. These algebras $\mathcal{H}_{\lambda,\kappa}$ depend on two parameters λ and κ with $\lambda: kG\otimes V\to kG$ and $\kappa: V\otimes V\to kG$. The Hochschild 2-cocycles above of the same name λ and κ are these parameters converted into cocycles on the resolution X; see [8, Example 2.2] and also [10, Section 5]. A sufficient condition for the parameters λ and κ to define a PBW deformation is that

$$[\lambda,\lambda]=0 \text{ and } [\lambda,\kappa]=0$$

when the cochains κ and λ they define are cocycles. (More generally, we require that λ is a cocycle, $[\lambda, \kappa] = 0$, and $[\lambda, \lambda] = 2\partial^* \kappa$.) Thus knowing explicit values for brackets is helpful for finding new deformations. The cocycle δ above is included merely for illustration purposes; it provides an example of a nonzero Gerstenhaber bracket.

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