

HOCHSCHILD COHOMOLOGY AND QUANTUM DRINFELD HECKE ALGEBRAS

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ABSTRACT. Quantum Drinfeld Hecke algebras are generalizations of Drinfeld Hecke algebras in which polynomial rings are replaced by quantum polynomial rings. We identify these algebras as deformations of skew group algebras, giving an explicit connection to Hochschild cohomology. We compute the relevant part of Hochschild cohomology for actions of many reflection groups and we exploit computations from [NSW] for diagonal actions. By combining our work with recent results of Levandovskyy and Shepler [LS], we produce examples of quantum Drinfeld Hecke algebras. These algebras generalize the braided Cherednik algebras of Bazlov and Berenstein [BB].

1. INTRODUCTION

Let G be a finite group acting linearly on a complex vector space V , let $S(V)$ be the symmetric algebra on V , and let $S(V) \rtimes G$ be the corresponding skew group algebra (defined in the next section). Drinfeld (or graded) Hecke algebras manifest themselves as deformations of these skew group algebras. These deformations go by many other names, such as symplectic reflection algebras or rational Cherednik algebras, and have arisen in such diverse areas as representation theory, combinatorics, and orbifold theory [C, D, EG, G, L, SW1].

In this note we replace the symmetric algebra $S(V)$ with a quantum or twisted version: Let

$$S_{\mathbf{q}}(V) := \mathbb{C}\langle v_1, \dots, v_n \mid v_i v_j = q_{ij} v_j v_i \text{ for all } 1 \leq i, j \leq n \rangle,$$

the **quantum symmetric algebra** determined by a basis v_1, \dots, v_n of V and a tuple $\mathbf{q} := (q_{ij})$ of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j . The possible actions of the finite group G on $S_{\mathbf{q}}(V)$ by linear automorphisms are somewhat limited. Alev and Chamarie [AC] gave some results, but not a complete classification, of such actions. Kirrman, Kuzmanovich, and Zhang [KKZ] described actions of some generalized reflection groups and proved a quantum version of the classical Shephard-Todd-Chevalley Theorem; one consequence is that invariants of $S_{\mathbf{q}}(V)$ under these actions again form quantum symmetric algebras.

Bazlov and Berenstein [BB] explored analogs of Cherednik algebras in this context, termed braided Cherednik algebras. More generally: Let $\kappa : V \times V \rightarrow \mathbb{C}G$ be a bilinear map for which $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$. Let $T(V)$ be the tensor algebra on V , and let

$$\mathcal{H}_{\mathbf{q}, \kappa} := T(V) \rtimes G / (v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \mid 1 \leq i, j \leq n),$$

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the quotient of the skew group algebra $T(V) \rtimes G$ by the ideal generated by all elements of the form specified. Giving each v_i degree 1 and each group element g degree 0, $\mathcal{H}_{\mathbf{q},\kappa}$ is a filtered algebra. We call $\mathcal{H}_{\mathbf{q},\kappa}$ a **quantum Drinfeld Hecke algebra** if its associated graded algebra is isomorphic to $S_{\mathbf{q}}(V) \rtimes G$. In case all $q_{ij} = 1$, these are the Drinfeld (or graded) Hecke algebras. Levandovskyy and Shepler [LS] gave necessary and sufficient conditions on the functions κ for $\mathcal{H}_{\mathbf{q},\kappa}$ to be a quantum Drinfeld Hecke algebra.

In this paper, we view these quantum analogs of Drinfeld Hecke algebras as deformations of the skew group algebras $S_{\mathbf{q}}(V) \rtimes G$. We establish the following theorem, which makes explicit a connection to Hochschild cohomology, thus forging another path to understanding these and related deformations. The notation and terminology used below is explained in Section 2.

Theorem 2.2. *The quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ are precisely the deformations of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$ with $\deg \mu_i = -2i$ for all $i \geq 1$.*

With Shroff [NSW] we computed the Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$ in case G acts *diagonally* on the chosen basis $\{v_i\}_{1 \leq i \leq n}$ of V . Here we consider more general actions, focusing on that part of Hochschild cohomology in degree 2 that is relevant to quantum Drinfeld Hecke algebras. We apply the criteria of Levandovskyy and Shepler [LS] to show that when the action of G extends to an action on a quantum exterior algebra, all such Hochschild 2-cocycles do indeed give rise to quantum Drinfeld Hecke algebras:

Theorem 4.4. *Assume that the action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. Then each constant Hochschild 2-cocycle on $S_{\mathbf{q}}(V) \rtimes G$ gives rise to a quantum Drinfeld Hecke algebra.*

Combining the previous two theorems, we obtain the following.

Theorem 4.6. *Assume that the action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. Then each constant Hochschild 2-cocycle on $S_{\mathbf{q}}(V) \rtimes G$ lifts to a deformation of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$.*

We compute the relevant part of the Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$ in degree 2 for several types of complex reflection groups and actions. Our deformations include all of the braided Cherednik algebras of Bazlov and Berenstein [BB], putting them in a larger context: Their vector space V is always the direct sum of a vector space and its dual, and they have some mild additional restrictions on the structure of the corresponding deformations of $S_{\mathbf{q}}(V) \rtimes G$.

We will work over the complex numbers \mathbb{C} , and all tensor products will be taken over \mathbb{C} unless otherwise indicated.

Organization. This paper is organized as follows.

We prove Theorem 2.2 in Section 2. Section 3 develops the homological algebra needed for Section 4, in which we will obtain results on the Hochschild 2-cocycles associated to quantum Drinfeld Hecke algebras. In particular, we prove Theorem 4.4.

In Section 5, using results from [NSW] we classify quantum Drinfeld Hecke algebras for diagonal actions of G on a chosen basis for V .

In Sections 6 and 7, we consider the natural and symplectic representations of several types of complex reflection groups. In each case we classify the corresponding quantum Drinfeld

Hecke algebras by computing the relevant part of the Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$ in degree 2.

2. QUANTUM DRINFELD HECKE ALGEBRAS AND DEFORMATIONS OF $S_{\mathbf{q}}(V) \rtimes G$

Assume the finite group G acts linearly on the complex vector space V , and that there is an induced action on $S_{\mathbf{q}}(V)$ by algebra automorphisms. Then we may form the **skew group algebra** $S_{\mathbf{q}}(V) \rtimes G$, and we recall its definition: Letting $A = S_{\mathbf{q}}(V)$, additively $A \rtimes G$ is the free left A -module with basis G . We write $A \rtimes G = \bigoplus_{g \in G} A_g$, where $A_g = \{ag \mid a \in A\}$, that is for each $a \in A$ and $g \in G$ we denote by $ag \in A_g$ the a -multiple of g . Multiplication on $A \rtimes G$ is determined by

$$(ag)(bh) := a({}^g b)gh$$

for all $a, b \in A$ and $g, h \in G$, where a left superscript denotes the action of the group element. Similarly we define the skew group algebra for any other algebra on which G acts by automorphisms, such as the tensor algebra $T(V)$.

We will show how quantum Drinfeld Hecke algebras may be realized as deformations of $S_{\mathbf{q}}(V) \rtimes G$ by extending the scalars to $\mathbb{C}[t]$: For any algebra R over \mathbb{C} , a **deformation of R over $\mathbb{C}[t]$** is an associative $\mathbb{C}[t]$ -algebra whose underlying vector space is $R[t] = \mathbb{C}[t] \otimes R$, and multiplication is

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \dots$$

for all $r, s \in R$, where rs is the product in R , the $\mu_i : R \otimes R \rightarrow R$ are \mathbb{C} -linear maps extended to be linear over $\mathbb{C}[t]$, and for each r, s the above sum is finite. One consequence of associativity is that μ_1 is a **Hochschild 2-cocycle**, that is

$$(2.1) \quad \mu_1(r \otimes s)u + \mu_1(rs \otimes u) = \mu_1(r \otimes su) + r\mu_1(s \otimes u)$$

for all $r, s, u \in R$.

Let $\kappa : V \times V \rightarrow \mathbb{C}G$ be a function as specified in the introduction. For each $g \in G$, let $\kappa_g : V \times V \rightarrow \mathbb{C}$ be the function determined by the condition

$$\kappa(v, w) = \sum_{g \in G} \kappa_g(v, w)g \quad \text{for all } v, w \in V.$$

(The condition $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ implies that $\kappa_g(v_i, v_j) = -q_{ij}\kappa_g(v_j, v_i)$ for each $g \in G$. It arises when interchanging i and j in the defining relations of $\mathcal{H}_{\mathbf{q}, \kappa}$; in the absence of this condition, $\mathcal{H}_{\mathbf{q}, \kappa}$ is too small in the sense that the group G does not embed in $\mathcal{H}_{\mathbf{q}, \kappa}$ as a subgroup of its group of units.) Let

$$\mathcal{H}_{\mathbf{q}, \kappa, t} := T(V) \rtimes G[t] / (v_i v_j - q_{ij} v_j v_i - \sum_{g \in G} \kappa_g(v_i, v_j)tg \mid 1 \leq i, j \leq n).$$

Giving each v_i degree 1 and each $g \in G$ and t degree 0, we see that $\mathcal{H}_{\mathbf{q}, \kappa, t}$ is a filtered algebra. We are interested in those algebras $\mathcal{H}_{\mathbf{q}, \kappa, t}$ for which the associated graded algebra is isomorphic to $S_{\mathbf{q}}(V) \rtimes G[t]$; call these algebras **quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$** . Specializing to $t = 1$, these are the quantum Drinfeld Hecke algebras as defined in the introduction.

We next prove that quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ are all of the deformations of $S_{\mathbf{q}}(V) \rtimes G$ of a particular form. The proof of Theorem 2.2 below is a straightforward generalization of a special case of [Wi, Theorem 3.2]. We include a proof as we will need some

of the details and wish to highlight the homological meaning of the quantum skew-symmetry of the functions κ_g .

The **quantum exterior algebra** $\Lambda_{\mathbf{q}}(V)$ associated to the tuple $\mathbf{q} = (q_{ij})$ is

$$\Lambda_{\mathbf{q}}(V) := \mathbb{C}\langle v_1, \dots, v_n \mid v_i v_j = -q_{ij} v_j v_i \text{ for all } 1 \leq i, j \leq n \rangle.$$

Since we are working in characteristic 0, the defining relations imply in particular that $v_i^2 = 0$ for each v_i in $\Lambda_{\mathbf{q}}(V)$. This algebra has a basis given by all $v_{i_1} \cdots v_{i_m}$ ($0 \leq m \leq n$, $1 \leq i_1 < \cdots < i_m \leq n$); we will write such a basis element as $v_{i_1} \wedge \cdots \wedge v_{i_m}$ by analogy with the ordinary exterior algebra.

In the theorem below, by the **degree** of μ_i , we mean its degree as a function from the graded algebra $(A \rtimes G)^{\otimes 2}$ to $A \rtimes G$. The theorem gives a one-to-one correspondence between quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ and deformations of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$ satisfying a condition on the degrees of the functions μ_i .

Theorem 2.2. *The quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ are precisely the deformations of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$ with $\deg \mu_i = -2i$ for all $i \geq 1$.*

Proof. Let $\mathcal{H}_{\mathbf{q}, \kappa, t}$ be a quantum Drinfeld Hecke algebra over $\mathbb{C}[t]$. By its definition, this implies that $\mathcal{H}_{\mathbf{q}, \kappa, t}$ is a deformation of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$. Specifically, since the associated graded algebra of $\mathcal{H}_{\mathbf{q}, \kappa, t}$ is isomorphic to $S_{\mathbf{q}}(V) \rtimes G[t]$, and $S_{\mathbf{q}}(V)$ has a basis consisting of all monomials $v_1^{i_1} \cdots v_n^{i_n}$, each element of $\mathcal{H}_{\mathbf{q}, \kappa, t}$ may be written uniquely as a $\mathbb{C}[t]$ -linear combination of elements of the form $v_1^{i_1} \cdots v_n^{i_n} g$. Let $r = v_1^{i_1} \cdots v_n^{i_n} g$ and $s = v_1^{j_1} \cdots v_n^{j_n} h$ be two such basis elements of $\mathcal{H}_{\mathbf{q}, \kappa, t}$. Denoting the product in $\mathcal{H}_{\mathbf{q}, \kappa, t}$ by $*$, since $\mathcal{H}_{\mathbf{q}, \kappa, t}$ is defined as a quotient of $T(V) \rtimes G[t]$, we have

$$r * s = v_1^{i_1} \cdots v_n^{i_n} * {}^g(v_1^{j_1} \cdots v_n^{j_n})gh,$$

and applying the relations defining $\mathcal{H}_{\mathbf{q}, \kappa, t}$ repeatedly, we obtain an expression of the form

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \cdots.$$

The sum will be finite since each time a relation is applied, the degree drops. The product $*$ makes $\mathcal{H}_{\mathbf{q}, \kappa, t}$ an associative algebra by definition, and consequently the functions μ_i will be bilinear. Therefore $\mathcal{H}_{\mathbf{q}, \kappa, t}$ is a deformation of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$. The conditions on the degrees of the μ_i follow from the relations and induction on the degree $\sum_{k=1}^n (i_k + j_k)$ of a product $v_1^{i_1} \cdots v_n^{i_n} * v_1^{j_1} \cdots v_n^{j_n}$.

Conversely, suppose that B is a deformation of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$ satisfying the given degree conditions. Then, as a vector space over $\mathbb{C}[t]$, $B \cong S_{\mathbf{q}}(V) \rtimes G[t]$. Define a map $\phi : T(V) \rtimes G[t] \rightarrow B$ by first requiring

$$\phi(v_i) = v_i \quad \text{and} \quad \phi(g) = g$$

for all i , $1 \leq i \leq n$, and $g \in G$. Since $T(V)$ is free on a basis of V , the map ϕ on the v_i may be extended uniquely to an algebra homomorphism from $T(V)$ to B . By the degree condition on the μ_i , we have $\mu_i(\mathbb{C}G, \mathbb{C}G) = \mu_i(\mathbb{C}G, V) = \mu_i(V, \mathbb{C}G) = 0$, so ϕ may be extended to a $\mathbb{C}[t]$ -algebra homomorphism on all of $T(V) \rtimes G[t]$, as desired. Specifically, $\phi(v_{i_1} \cdots v_{i_m} g) = v_{i_1} * \cdots * v_{i_m} * g$. By the degree requirements, we have for example $g * v_{i_1} * v_{i_2} = (g * v_{i_1}) * v_{i_2} = {}^g v_{i_1} * g * v_{i_2} = {}^g v_{i_1} * {}^g v_{i_2} * g$.

We claim that ϕ is surjective. We will prove that each basis element is in the image of ϕ by induction on the degree of the basis monomial. First note that $\phi(g) = g$ and $\phi(v_i g) = v_i g$ for

all i , $1 \leq i \leq n$, and all $g \in G$. Now let $v_{i_1} \cdots v_{i_m} g$ be an arbitrary basis monomial of B . By induction, $v_{i_2} \cdots v_{i_m} g$ is in the image of ϕ , say $\phi(X) = v_{i_2} \cdots v_{i_m} g$ for some $X \in T(V) \rtimes G[t]$. Then

$$\begin{aligned} \phi(v_{i_1} X) &= v_{i_1} * \phi(X) \\ &= v_{i_1} * (v_{i_2} \cdots v_{i_m} g) \\ &= v_{i_1} \cdots v_{i_m} g + \mu_1(v_{i_1}, v_{i_2} \cdots v_{i_m} g)t + \mu_2(v_{i_1}, v_{i_2} \cdots v_{i_m} g)t^2 + \cdots \end{aligned}$$

By induction, since $\deg(\mu_i) = -2i$, each $\mu_j(v_{i_1}, v_{i_2} \cdots v_{i_m} g)$ is in the image of ϕ . Therefore $v_{i_1} \cdots v_{i_m} g$ is in the image of ϕ , which implies ϕ is surjective.

Finally we determine the kernel of ϕ . Note that

$$\begin{aligned} \phi(v_i v_j) &= v_i * v_j = v_i v_j + \mu_1(v_i \otimes v_j)t \\ \phi(v_j v_i) &= v_j * v_i = v_j v_i + \mu_1(v_j \otimes v_i)t \end{aligned}$$

since $\deg(\mu_i) = -2i$ for all i . As $v_j v_i = q_{ji} v_i v_j$ in $S_{\mathbf{q}}(V)$, we find

$$\phi(v_j v_i - q_{ji} v_i v_j) = (\mu_1(v_j \otimes v_i) - q_{ji} \mu_1(v_i \otimes v_j))t.$$

Since $\deg(\mu_1) = -2$ and $\phi(g) = g$ for all $g \in G$, this implies that

$$(2.3) \quad v_j v_i - q_{ji} v_i v_j - (\mu_1(v_j \otimes v_i) - q_{ji} \mu_1(v_i \otimes v_j))t$$

is in the kernel of ϕ for all i, j , and that

$$\mu_1(v_j \otimes v_i) - q_{ji} \mu_1(v_i \otimes v_j) = \sum_{g \in G} \kappa_g(v_j, v_i)g$$

for some functions κ_g . By interchanging i, j we find that $\kappa_g(v_i, v_j) = -q_{ij} \kappa_g(v_j, v_i)$, and by definition each κ_g is linear, so we may view each κ_g as a linear function on $\bigwedge_{\mathbf{q}}^2(V)$, or equivalently as a bilinear function on $V \times V$ that satisfies $\kappa_g(v_i, v_j) = -q_{ij} \kappa_g(v_j, v_i)$ for all i, j . Let $I[t]$ be the ideal of $T(V) \rtimes G[t]$ generated by all such expressions (2.3), so that $I[t] \subset \text{Ker } \phi$. We claim that $I[t] = \text{Ker } \phi$: By the form of the relations, as a vector space $T(V) \rtimes G[t]/I[t]$ is a quotient of $S_{\mathbf{q}}(V) \rtimes G[t]$, so has dimension in each degree no greater than that of $S_{\mathbf{q}}(V) \rtimes G[t]$. Since ϕ induces a map from $T(V) \rtimes G[t]/I[t]$ onto the vector space $B \cong S_{\mathbf{q}}(V) \rtimes G[t]$, this forces $I[t] = \text{Ker } \phi$. Therefore B is a quantum Drinfeld Hecke algebra. \square

In particular, if we search for quantum Drinfeld Hecke algebras, Theorem 2.2 shows that we might first determine the Hochschild two-cocycles μ_1 of degree -2 as maps from $(A \rtimes G)^{\otimes 2}$ to $A \rtimes G$. We will call these **constant** Hochschild 2-cocycles; this choice of terminology will be justified by results of the next section, where we recall and develop the needed tools from homological algebra. As a consequence we will show in Theorem 4.4 that in fact *all* constant Hochschild 2-cocycles give rise to quantum Drinfeld Hecke algebras under the condition that the action of G extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms.

One outcome of the above proof is an explicit relationship between the functions κ_g and the Hochschild 2-cocycles μ_1 :

$$\sum_{g \in G} \kappa_g(v_j, v_i)g = \mu_1(v_j \otimes v_i) - q_{ji} \mu_1(v_i \otimes v_j).$$

3. TWO RESOLUTIONS

In this section we develop the homological algebra needed for Section 4, in which we will obtain results on the Hochschild 2-cocycles associated to quantum Drinfeld Hecke algebras.

The Hochschild cohomology of an algebra R is $\mathrm{HH}^*(R) := \mathrm{Ext}_{R^e}^*(R, R)$, where the enveloping algebra $R^e := R \otimes R^{op}$ acts on R by left and right multiplication. When $R = A \rtimes G$ is a skew group algebra in a characteristic not dividing the order of the finite group G , it is well-known that there is an action of G on $\mathrm{HH}^*(A, A \rtimes G) := \mathrm{Ext}_{A^e}^*(A, A \rtimes G)$ for which $\mathrm{HH}^*(A \rtimes G) \cong \mathrm{HH}^*(A, A \rtimes G)^G$, the elements of $\mathrm{HH}^*(A, A \rtimes G)$ that are invariant under G . (See, for example, Ştefan [S, Corollary 3.4].)

For the purpose of computing Hochschild cohomology, we first recall the quantum Koszul resolution. Our goal is to understand the Hochschild cohomology of $S_{\mathbf{q}}(V)$ and of $S_{\mathbf{q}}(V) \rtimes G$, and to use this knowledge to give explicitly any corresponding deformations of $S_{\mathbf{q}}(V) \rtimes G$.

Set $A = S_{\mathbf{q}}(V)$. For each $g \in G$, A_g is a (left) A^e -module via the action

$$(a \otimes b) \cdot (cg) := acgb = ac({}^g b)g$$

for all $a, b, c \in A$, $g \in G$. According to Wambst [W, Proposition 4.1(c)], the following is a free A^e -resolution of A :

$$(3.1) \quad \cdots \rightarrow A^e \otimes \Lambda_{\mathbf{q}}^2(V) \xrightarrow{d_2} A^e \otimes \Lambda_{\mathbf{q}}^1(V) \xrightarrow{d_1} A^e \xrightarrow{\mathrm{mult}} A \rightarrow 0,$$

that is, for $1 \leq m \leq n$, the degree m term is $A^e \otimes \Lambda_{\mathbf{q}}^m(V)$; the differential d_m is defined by

$$\begin{aligned} & d_m(1^{\otimes 2} \otimes v_{j_1} \wedge \cdots \wedge v_{j_m}) \\ &= \sum_{i=1}^m (-1)^{i+1} \left[\left(\prod_{s=1}^i q_{j_s, j_i} \right) v_{j_i} \otimes 1 - \left(\prod_{s=i}^m q_{j_i, j_s} \right) \otimes v_{j_i} \right] \otimes v_{j_1} \wedge \cdots \wedge \hat{v}_{j_i} \wedge \cdots \wedge v_{j_m} \end{aligned}$$

whenever $1 \leq j_1 < \cdots < j_m \leq n$, and mult denotes the multiplication map. (While $\Lambda_{\mathbf{q}}^m(V)$ is isomorphic to $\Lambda^m(V)$ as a vector space, we retain the \mathbf{q} in the notation as a reminder to apply the relation $v_i \wedge v_j = -q_{ij}v_j \wedge v_i$ whenever we wish to rewrite elements in this way, such as after having applied a group action.) The complex (3.1) is a twisted version of the usual Koszul resolution for a polynomial ring. See also Bergh and Oppermann [BO] for construction of more general twisted products of resolutions.

Let us write the above formula for d_m in a more convenient form. We first introduce some notation following Wambst [W]. Let \mathbb{N}^n denote the set of all n -tuples of elements from \mathbb{N} . For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the **length** of α , denoted $|\alpha|$, is the sum $\sum_{i=1}^n \alpha_i$. For all $\alpha \in \mathbb{N}^n$, define $v^\alpha := v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_n^{\alpha_n}$. For all $i \in \{1, \dots, n\}$, define $[i] \in \mathbb{N}^n$ by $[i]_j = \delta_{i,j}$, for all $j \in \{1, \dots, n\}$. For any $\beta = (\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, let $v^{\wedge \beta}$ denote the vector $v_{j_1} \wedge \cdots \wedge v_{j_m} \in \Lambda_{\mathbf{q}}^m(V)$ which is defined by $m = |\beta|$, $\beta_{j_k} = 1$ for all $k \in \{1, \dots, m\}$, and $j_1 < \cdots < j_m$. Then, for any $\beta \in \{0, 1\}^n$ with $|\beta| = m$ we have

$$d_m(1^{\otimes 2} \otimes v^{\wedge \beta}) = \sum_{i=1}^n \delta_{\beta_i, 1} (-1)^{\sum_{s=1}^{i-1} \beta_s} \left[\left(\prod_{s=1}^i q_{s,i}^{\beta_s} \right) v_i \otimes 1 - \left(\prod_{s=i}^n q_{i,s}^{\beta_s} \right) \otimes v_i \right] \otimes v^{\wedge(\beta - [i])}.$$

Applying the functor $\text{Hom}_{A^e}(\cdot, A_g)$ to the A^e -resolution of A in (3.1), and making appropriate identifications, we obtain

$$(3.2) \quad 0 \rightarrow A_g \xrightarrow{d_1^*} A_g \otimes \Lambda_{\mathbf{q}-1}^1(V^*) \xrightarrow{d_2^*} A_g \otimes \Lambda_{\mathbf{q}-1}^2(V^*) \rightarrow \cdots,$$

where $d_m^*(ag \otimes (v^*)^{\wedge \beta})$ is equal to

$$(3.3) \quad \sum_{i=1}^n \delta_{\beta_i, 0} (-1)^{\sum_{s=1}^i \beta_s} \left[\left(\left(\prod_{s=1}^i q_{s,i}^{\beta_s} \right) v_i a - \left(\prod_{s=i}^n q_{i,s}^{\beta_s} \right) a^{(g} v_i) \right) g \right] \otimes (v^*)^{\wedge (\beta + [i])},$$

for all $a \in A$ and $\beta \in \{0, 1\}^n$ with $|\beta| = m - 1$. (Note that the relations on dual functions are indeed $v_i^* \wedge v_j^* = -q_{ij}^{-1} v_j^* \wedge v_i^*$, for all i, j , as may be determined by applying each side of this equation to $v_i \wedge v_j$ and using the defining relations in $\Lambda_{\mathbf{q}}(V)$.)

We may compute $\text{HH}^m(A \rtimes G)$ as follows:

$$\text{HH}^m(A, A_g) \cong \text{Ker } d_{m+1}^* / \text{Im } d_m^* \quad \text{and} \quad \text{HH}^m(A \rtimes G) \cong \left(\bigoplus_{g \in G} \text{HH}^m(A, A_g) \right)^G.$$

Hochschild 2-cocycles give rise to elements of $\text{Hom}_{\mathbb{C}}((A \rtimes G)^{\otimes 2}, A \rtimes G)$ that satisfy the 2-cocycle condition (2.1). We wish to describe this correspondence explicitly. To this end, we will next introduce maps that translate between the complex (3.2) and the bar complex for $A \rtimes G$.

First we consider chain maps between the quantum Koszul resolution (3.1) and the bar resolution of A :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{\otimes 4} & \xrightarrow{\delta_2} & A^{\otimes 3} & \xrightarrow{\delta_1} & A^e \xrightarrow{\text{mult}} A \longrightarrow 0 \\ & & \Psi_2 \updownarrow \Phi_2 & & \Psi_1 \updownarrow \Phi_1 & & = \updownarrow = \\ \cdots & \longrightarrow & A^e \otimes \Lambda_{\mathbf{q}}^2 V & \xrightarrow{d_2} & A^e \otimes \Lambda_{\mathbf{q}}^1 V & \xrightarrow{d_1} & A^e \xrightarrow{\text{mult}} A \longrightarrow 0. \end{array}$$

Here the differentials δ_i in the bar resolution are defined as

$$\delta_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^i (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for all $a_0, \dots, a_{i+1} \in A$. We will only need to know the values of Ψ_2 on elements of the form $1 \otimes v_i \otimes v_j \otimes 1$, and to find these values, we will need to know the values of Ψ_1 on $1 \otimes v_i \otimes 1$ and on $1 \otimes v_i v_j \otimes 1$. Since the bar resolution consists of free modules, we may choose these values to be any inverse images, under d_1 , of $\delta_1(1 \otimes v_i \otimes 1)$ and of $\delta_1(1 \otimes v_i v_j \otimes 1)$. We choose $\Psi_1(1 \otimes v_i \otimes 1) = 1 \otimes 1 \otimes v_i$ and $\Psi_1(1 \otimes v_i v_j \otimes 1) = q_{ij} \otimes v_i \otimes v_j + q_{ij} v_j \otimes 1 \otimes v_i$. It follows that

$$\begin{aligned} \Psi_1 \delta_2(1 \otimes v_i \otimes v_j \otimes 1) &= \Psi_1(v_i \otimes v_j \otimes 1 - 1 \otimes v_i v_j \otimes 1 + 1 \otimes v_i \otimes v_j) \\ &= (v_i \otimes 1 - q_{ij} \otimes v_i) \otimes v_j - (q_{ij} v_j \otimes 1 - 1 \otimes v_j) \otimes v_i. \end{aligned}$$

When $i < j$, this is precisely $d_2(1 \otimes 1 \otimes v_i \wedge v_j)$, so we may let

$$(3.4) \quad \Psi_2(1 \otimes v_i \otimes v_j \otimes 1) = 1 \otimes 1 \otimes v_i \wedge v_j \quad (1 \leq i < j \leq n).$$

A similar analysis shows that we may let $\Psi_2(1 \otimes v_i \otimes v_j \otimes 1) = 0$ whenever $i \geq j$. (This asymmetric choice can make hand computations less onerous; alternatively, a more elegant, quantum symmetric choice is possible.)

Chain maps Φ_i are defined in [NSW], and more generally in [W], that embed the quantum Koszul resolution as a subcomplex of the bar resolution. We will not need these maps here.

Two more maps are defined as in [SW2]: We define the Reynold's operator, or averaging map, which ensures G -invariance of the image, compensating for the possibility that Ψ_2 may not preserve the action of G :

$$\begin{aligned} \mathcal{R}_2 : \text{Hom}_{\mathbb{C}}(A^{\otimes 2}, A \rtimes G) &\rightarrow \text{Hom}_{\mathbb{C}}(A^{\otimes 2}, A \rtimes G)^G \\ \mathcal{R}_2(\gamma) &:= \frac{1}{|G|} \sum_{g \in G} g\gamma. \end{aligned}$$

A map that tells how to extend a function defined on $A^{\otimes 2}$ to a function defined on $(A \rtimes G)^{\otimes 2}$ is also from [SW2]:

$$\begin{aligned} \Theta_2^* : \text{Hom}_{\mathbb{C}}(A^{\otimes 2}, A \rtimes G)^G &\rightarrow \text{Hom}_{\mathbb{C}}((A \rtimes G)^{\otimes 2}, A \rtimes G) \\ \Theta_2^*(\kappa)(a_1 g_1 \otimes a_2 g_2) &:= \kappa(a_1 \otimes g_1 a_2) g_1 g_2. \end{aligned}$$

We will identify $(A \rtimes G) \otimes \Lambda_{\mathbf{q}^{-1}}^2(V^*)$ with $\text{Hom}_{A^e}(A^e \otimes \Lambda_{\mathbf{q}}^2(V), A \rtimes G)$ by sending $b \otimes v_i^* \wedge v_j^*$ ($i < j$) to the A^e -homomorphism taking $1 \otimes 1 \otimes v_k \wedge v_l$ ($k < l$) to b if $v_i \wedge v_j = v_k \wedge v_l$ and to 0 otherwise. We will further identify $\text{Hom}_{A^e}(A^e \otimes U, W)$ with $\text{Hom}_{\mathbb{C}}(U, W)$, for any vector space U and A^e -module W , where convenient.

We will use the following, which is Theorem 4.3 of [SW2]. The hypothesis on the group action always holds in the case $q_{ij} = 1$ for all i, j , and as well in the case $q_{ij} = -1$ for all $i \neq j$. For other choices of \mathbf{q} , the hypothesis is equivalent to conditions on the entries of the matrices by which group elements act on V .

Theorem 3.5 ([SW2]). *Assume there is an action of G on the quantum Koszul complex (3.1), that is, an action of G on each $A^e \otimes \Lambda_{\mathbf{q}}^i(V)$ that commutes with the differentials. The composition $\Theta_2^* \mathcal{R}_2 \Psi_2^*$ induces an isomorphism*

$$\left(\bigoplus_{g \in G} \text{HH}^2(A, A_g) \right)^G \xrightarrow{\sim} \text{HH}^2(A \rtimes G).$$

Moreover, $\Theta_2^* \mathcal{R}_2 \Psi_2^*$ maps $\bigoplus_{g \in G} \text{HH}^2(A, A_g)$ onto $\text{HH}^2(A \rtimes G)$.

For later use, we record a much simpler consequence. Let $\alpha \in (A \rtimes G) \otimes \Lambda_{\mathbf{q}^{-1}}^2(V^*)$. Then

$$(3.6) \quad [\Theta_2^* \mathcal{R}_2 \Psi_2^*(\alpha)](v_i \otimes v_j) = \frac{1}{|G|} \sum_{g \in G} g(\alpha(\Psi_2(1 \otimes g^{-1} v_i \otimes g^{-1} v_j \otimes 1))).$$

Note that those elements of $\text{Hom}_{A^e}(A^e \otimes \Lambda_{\mathbf{q}}^2(V), A \rtimes G)^G \cong ((A \rtimes G) \otimes \Lambda_{\mathbf{q}^{-1}}^2(V^*))^G$ that correspond to *constant* Hochschild two-cocycles, that is, those of degree -2 as maps from $(A \rtimes G) \otimes (A \rtimes G)$ to $A \rtimes G$, are precisely those in $(\mathbb{C}G \otimes \Lambda_{\mathbf{q}^{-1}}^2(V^*))^G$, due to the form of the chain map Ψ_2 . Thus we wish first to find those elements of $\mathbb{C}G \otimes \Lambda_{\mathbf{q}^{-1}}^2(V^*)$ that are in the kernel of d_3^* , and then to restrict to G -invariants. Note that the intersection of the image

of d_2^* with $\mathbb{C}G \otimes \bigwedge_{\mathbf{q}-1}^2(V^*)$ is 0. Applying our earlier formula, letting $\beta = [j] + [k]$,

$$\begin{aligned} & d_3^*(g \otimes v_j^* \wedge v_k^*) \\ &= \sum_{i \notin \{j,k\}} (-1)^{\sum_{s=1}^i \beta_s} \left[\left(\left(\prod_{s=1}^i q_{s,i}^{\beta_s} \right) v_i - \left(\prod_{s=i}^n q_{i,s}^{\beta_s} \right) g v_i \right) g \right] \otimes (v^*)^{\wedge(\beta+[i])}. \end{aligned}$$

4. QUANTUM DRINFELD HECKE ALGEBRAS AND CONSTANT HOCHSCHILD 2-COCYCLES

Levandovskyy and Shepler [LS] gave necessary and sufficient conditions on the functions κ_g for $\mathcal{H}_{\mathbf{q},\kappa}$ to be a quantum Drinfeld Hecke algebra, and we restate their result as Theorem 4.1 below. Note that our formulation of their result is a little different as we assume from the outset that G acts by automorphisms on $S_{\mathbf{q}}(V)$, our indices on \mathbf{q} are reversed, and our functions κ_g are also reversed in the defining relations. Theorem 4.1 will allow us to give explicitly the quantum Drinfeld Hecke algebras as deformations of $S_{\mathbf{q}}(V) \rtimes G$ corresponding to Hochschild 2-cocycles found via the quantum Koszul resolution. We will apply this theorem to particular types of group actions in the next few sections.

For each group element $g \in G$, let g_i^j be the scalars for which

$$g v_j = \sum_{i=1}^n g_i^j v_i.$$

Define the **quantum** (i, j, k, l) -**minor determinant** of g as

$$\det_{ijkl}(g) := g_l^j g_k^i - q_{ji} g_l^i g_k^j.$$

It may be checked directly that for each i, j , if $q_{ij} \neq 1$, then $g_k^i g_k^j = 0$ for all k , since G acts as automorphisms on $S_{\mathbf{q}}(V)$. (Apply g to both sides of the equation $v_i v_j = q_{ij} v_j v_i$ and equate coefficients of basis elements.) The following is a restatement of [LS, Theorem 7.6].

Theorem 4.1 ([LS]). *The algebra $\mathcal{H}_{\mathbf{q},\kappa}$ (defined in the introduction) is a quantum Drinfeld Hecke algebra if and only if*

(i) *for all $g \in G$ and $1 \leq i < j < k \leq n$,*

$$(q_{ki} q_{kj} g v_k - v_k) \kappa_g(v_j, v_i) + (q_{kj} v_j - q_{ji} g v_j) \kappa_g(v_k, v_i) + (g v_i - q_{ji} q_{ki} v_i) \kappa_g(v_k, v_j) = 0,$$

(ii) *for all $i < j$ and all $g, h \in G$, $\kappa_{h^{-1}gh}(v_j, v_i) = \sum_{k < l} \det_{ijkl}(h) \kappa_g(v_l, v_k)$.*

We use Theorem 4.1 to show in the next theorem that all constant Hochschild 2-cocycles give rise to quantum Drinfeld Hecke algebras under an additional assumption. First we need two lemmas, whose proofs are straightforward.

Lemma 4.2. *The action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by automorphisms if, and only if, for all $g \in G$, $i \neq j$ and $k < l$,*

$$(1 - q_{ij} q_{lk}) g_k^i g_l^j + (q_{ij} - q_{lk}) g_l^i g_k^j = 0.$$

Note that if all $q_{ij} = 1$, or if $q_{ij} = -1$ for all $i \neq j$, the condition in the lemma clearly holds. In general it imposes strong conditions on the matrix entries g_i^j :

Lemma 4.3. *Assume that the action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. Then for all $g \in G$ and i, j, k, l ($i < j$, $k < l$), if $g_l^i g_k^j \neq 0$ then $q_{lk} = q_{ij}$, and if $g_k^i g_l^j \neq 0$, then $q_{lk} = q_{ij}^{-1}$.*

Theorem 4.4. *Assume that the action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. Then each constant Hochschild 2-cocycle on $S_{\mathbf{q}}(V) \rtimes G$ gives rise to a quantum Drinfeld Hecke algebra.*

Proof. Let α be any constant Hochschild 2-cocycle on $S_{\mathbf{q}}(V) \rtimes G$, so that α may be expressed in terms of the quantum Koszul complex as

$$\alpha = \sum_{g \in G} \sum_{1 \leq r < s \leq n} \alpha_{rs}^g g \otimes v_r^* \wedge v_s^*$$

for some scalars α_{rs}^g . For each $i < j$ and $g \in G$, let $\kappa_g(v_i, v_j) = \alpha_{ij}^g$, so that

$$\sum_{g \in G} \kappa_g(v_i, v_j) g = \sum_{g \in G} \alpha_{ij}^g g = \alpha(v_i \otimes v_j - q_{ij} v_j \otimes v_i).$$

We will check that the conditions of Theorem 4.1 hold, and the conclusion will follow.

For all $i < j < k$, $d_3^*(\alpha)$ applied to $1 \otimes 1 \otimes v_i \wedge v_j \wedge v_k$ yields

$$\begin{aligned} & \alpha((v_i \otimes 1 - q_{ij} q_{ik} \otimes v_i) \otimes v_j \wedge v_k - (q_{ij} v_j \otimes 1 - q_{jk} \otimes v_j) \otimes v_i \wedge v_k \\ & \quad + (q_{ik} q_{jk} v_k \otimes 1 - 1 \otimes v_k) \otimes v_i \wedge v_j) \\ &= v_i \alpha(v_j \wedge v_k) - q_{ij} q_{ik} \alpha(v_j \wedge v_k) v_i - q_{ij} v_j \alpha(v_i \wedge v_k) + q_{jk} \alpha(v_i \wedge v_k) v_j \\ & \quad + q_{ik} q_{jk} v_k \alpha(v_i \wedge v_j) - \alpha(v_i \wedge v_j) v_k \\ &= \sum_{g \in G} (\alpha_{jk}^g v_i g - q_{ij} q_{ik} \alpha_{jk}^g g v_i - q_{ij} v_j \alpha_{ik}^g g + q_{jk} \alpha_{ik}^g g v_j + q_{ik} q_{jk} v_k \alpha_{ij}^g g - \alpha_{ij}^g g v_k). \end{aligned}$$

So $d_3^*(\alpha) = 0$ if and only if

$$(4.5) \quad \alpha_{jk}^g (v_i - q_{ij} q_{ik}^g v_i) - \alpha_{ik}^g (q_{ij} v_j - q_{jk}^g v_j) + \alpha_{ij}^g (q_{ik} q_{jk} v_k - {}^g v_k) = 0$$

for all $g \in G$, and i, j, k . This is indeed condition (i) of Theorem 4.1: Replace $\kappa_g(v_j, v_i)$ by $-q_{ji} \kappa_g(v_i, v_j)$, and similarly for the others. Then multiply the equation by $q_{ij} q_{ik} q_{jk}$.

We claim that G -invariance of α is equivalent to condition (ii) of Theorem 4.1: α is G -invariant if, and only if, $\alpha({}^h v_i \wedge {}^h v_j) = {}^h(\alpha(v_i \wedge v_j))$ for all i, j , and all $h \in G$. Using the notation $h(v_i) = \sum_k h_k^i v_k$, we have

$$\begin{aligned} \alpha({}^h v_i \wedge {}^h v_j) &= \sum_{k, l} h_k^i h_l^j \alpha(v_k \wedge v_l) \\ &= \sum_{k < l} h_k^i h_l^j \alpha(v_k \wedge v_l) - \sum_{k < l} q_{lk} h_l^i h_k^j \alpha(v_k \wedge v_l) \\ &= \sum_{k < l, g \in G} (h_k^i h_l^j - q_{lk} h_l^i h_k^j) \alpha_{kl}^g g \end{aligned}$$

and

$${}^h(\alpha(v_i \wedge v_j)) = {}^h \left(\sum_{g \in G} \alpha_{ij}^g g \right) = \sum_{g \in G} \alpha_{ij}^g h g h^{-1} = \sum_{g \in G} \alpha_{ij}^{h^{-1} g h} g.$$

Equating the two, we find that

$$\alpha_{ij}^{h^{-1}gh} = \sum_{k < l} (h_k^i h_l^j - q_{lk} h_l^i h_k^j) \alpha_{kl}^g.$$

By the proof of Lemma 4.3, $q_{lk} h_l^i h_k^j = q_{ij} h_l^i h_k^j$, so we may rewrite this as

$$\alpha_{ij}^{h^{-1}gh} = \sum_{k < l} (h_k^i h_l^j - q_{ij} h_l^i h_k^j) \alpha_{kl}^g.$$

Substituting $\alpha_{ij}^{h^{-1}gh} = -q_{ij} \alpha_{ji}^{h^{-1}gh}$, $\alpha_{kl}^g = -q_{kl} \alpha_{lk}^g$, $q_{kl} h_k^i h_l^j = q_{ij} h_k^i h_l^j$, and $q_{kl} h_k^j h_l^i = q_{ji} h_k^j h_l^i$, this is precisely Theorem 4.1(ii). \square

Note that the hypothesis in Theorem 4.4 is necessary for the statement to make sense: It is the action of G on $\bigwedge_{\mathbf{q}}(V)$ that allows cocycles to be expressed in terms of the quantum Koszul complex, a *constant* cocycle being defined via such an expression.

Combining Theorems 2.2 and 4.4, we obtain the following.

Theorem 4.6. *Assume that the action of G on V extends to an action on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. Then each constant Hochschild 2-cocycle on $S_{\mathbf{q}}(V) \rtimes G$ lifts to a deformation of $S_{\mathbf{q}}(V) \rtimes G$ over $\mathbb{C}[t]$.*

5. DIAGONAL ACTIONS

As before, let G denote a finite group acting linearly on a vector space V . In case v_1, \dots, v_n is a basis of common eigenvectors for G , there is always an induced action of G on $S_{\mathbf{q}}(V)$ and on $\bigwedge_{\mathbf{q}}(V)$ by algebra automorphisms. In this case, the Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$ was computed in [NSW], and we apply those results in this section: We give an explicit description of those elements of $\mathrm{HH}^2(S_{\mathbf{q}}(V) \rtimes G)$ corresponding to maps of degree -2 . Let $\lambda_{g,i} \in \mathbb{C}$ be the scalars for which ${}^g v_i = \lambda_{g,i} v_i$ for all $g \in G, i = 1, \dots, n$.

For each $g \in G$, define

$$(5.1) \quad C_g := \left\{ \gamma \in (\mathbb{N} \cup \{-1\})^n \mid \text{for each } i \in \{1, \dots, n\}, \prod_{s=1}^n q_{is}^{\gamma_s} = \lambda_{g,i} \text{ or } \gamma_i = -1 \right\}.$$

We recall the following from [NSW].

Theorem 5.2 ([NSW]). *If G acts diagonally on V , then $\mathrm{HH}^*(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V) \rtimes G)$ is the graded vector subspace of $(S_{\mathbf{q}}(V) \rtimes G) \otimes \bigwedge_{\mathbf{q}^{-1}}(V^*)$ given by:*

$$\mathrm{HH}^m(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V) \rtimes G) \cong \bigoplus_{g \in G} \bigoplus_{\substack{\beta \in \{0,1\}^n \\ |\beta|=m}} \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ \alpha - \beta \in C_g}} \mathrm{span}_{\mathbb{C}}\{(v^\alpha g) \otimes (v^*)^{\wedge \beta}\},$$

for all $m \in \mathbb{N}$, and $\mathrm{HH}^m(S_{\mathbf{q}}(V) \rtimes G)$ is its G -invariant subspace.

We immediately obtain the following.

Corollary 5.3. *The subspace of $\mathrm{HH}^2(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V) \rtimes G)$ consisting of constant Hochschild 2-cocycles is isomorphic to*

$$\bigoplus_{g \in G} \bigoplus_{\substack{r < s \text{ s.t.} \\ \forall r' \neq r, s \\ q_{rr'} q_{sr'} = \lambda_{g,r'}} \mathrm{span}_{\mathbb{C}}\{g \otimes v_r^* \wedge v_s^*\}.$$

Let \mathcal{R} denote a complete set of representatives of conjugacy classes in G , let $C_G(a)$ denote the centralizer of a in G , and let $[G/C_G(a)]$ denote a set of representatives of the cosets of $C_G(a)$ in G . Combining Theorems 3.5, 4.4, and Corollary 5.3, we obtain the following theorem. The notation $\delta_{i,j}$ is the Kronecker delta. We note that the scalar $\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}$ in the sum below is independent of choice of representative g of a coset of $C_G(a)$ under the given assumption that $\lambda_{h,r} = \lambda_{h,s}^{-1}$ for all $h \in C_G(a)$.

Theorem 5.4. *The maps $\kappa : V \times V \rightarrow \mathbb{C}G$ for which $\mathcal{H}_{\mathbf{q},\kappa}$ is a quantum Drinfeld Hecke algebra form a vector space with basis consisting of maps*

$$f_{r,s,a} : V \times V \rightarrow \mathbb{C}G : (v_i, v_j) \mapsto (\delta_{i,r}\delta_{j,s} - q_{sr}\delta_{i,s}\delta_{j,r}) \sum_{g \in [G/C_G(a)]} \lambda_{g,r}^{-1}\lambda_{g,s}^{-1} gag^{-1}$$

where $r < s$ and $a \in \mathcal{R}$ satisfy $q_{rr'}q_{sr'} = \lambda_{a,r'}$ for all $r' \neq r, s$ and $\lambda_{h,r} = \lambda_{h,s}^{-1}$ for all $h \in C_G(a)$.

Proof. Let $\alpha = a \otimes v_r^* \wedge v_s^*$ where $a \in G$ and $r < s$ satisfy $q_{rr'}q_{sr'} = \lambda_{a,r'}$ for all $r' \neq r, s$. Thus, α represents a class in $\mathrm{HH}^2(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V) \rtimes G)$ that is *constant*, and by Corollary 5.3(ii), all constant Hochschild 2-cocycles have this form. Applying (3.6), we get

$$[\Theta_2^* \mathcal{R}_2 \Psi_2^*(\alpha)](v_i \otimes v_j) = \frac{1}{|G|} \delta_{i,r} \delta_{j,s} \left(\sum_{h \in C_G(a)} \lambda_{h,r} \lambda_{h,s} \right) \sum_{g \in [G/C_G(a)]} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} gag^{-1}$$

The sum $\sum_{h \in C_G(a)} \lambda_{h,r} \lambda_{h,s}$ is nonzero if and only if $\lambda_{h,r} = \lambda_{h,s}^{-1}$ for all $h \in C_G(a)$. To see this, define a homomorphism $\varphi : C_G(a) \rightarrow \mathbb{C}^\times : h \mapsto \lambda_{h,r} \lambda_{h,s}$. Since $\mathrm{Im} \varphi$ is finite, we have $\mathrm{Im} \varphi = \mathcal{C}_n$ for some n , where \mathcal{C}_n is the group of n th roots of unity. Thus,

$$\sum_{h \in C_G(a)} \lambda_{h,r} \lambda_{h,s} = \sum_{h \in C_G(a)} \varphi(h) = |\ker \varphi| \sum_{\xi \in \mathcal{C}_n} \xi.$$

The last expression is nonzero if and only if the group \mathcal{C}_n is trivial, equivalently $\lambda_{h,r} = \lambda_{h,s}^{-1}$ for all $h \in C_G(a)$. The stated result now follows from Theorems 3.5 and 4.4. \square

One readily finds many examples of quantum Drinfeld Hecke algebras by applying this theorem. We illustrate next with one example in which the scalars q_{ij} are not all 1 or -1 .

Example 5.5. Let G be a cyclic group of order 3 generated by g . Let q be a primitive third root of 1. Let $V = \mathbb{C}^3$, with basis v_1, v_2, v_3 , and let $q_{21} = q$, $q_{32} = q$, and $q_{13} = q$. Let G act on V as follows:

$${}^g v_1 = qv_1, \quad {}^g v_2 = q^2 v_2, \quad {}^g v_3 = v_3.$$

By the previous theorem, the map $\kappa := f_{1,2,g}$ gives rise to a quantum Drinfeld Hecke algebra, which is

$$\mathcal{H}_{\mathbf{q},\kappa} = T(V) \rtimes G / (v_2 v_1 - qv_1 v_2 + qg, v_3 v_2 - qv_2 v_3, v_1 v_3 - qv_3 v_1).$$

6. COMPLEX REFLECTION GROUPS: NATURAL REPRESENTATIONS

In this section, we classify the quantum Drinfeld Hecke algebras for the complex reflection groups $G(m, p, n)$, $n \geq 4$, acting naturally on a vector space of dimension n . Specifically, we show that, with the exception of certain special cases, there do not exist any nontrivial quantum Drinfeld Hecke algebras.

Let m, p, n be positive integers with p dividing m . Let ζ denote a primitive m th root of unity and let

$$\zeta_i = \text{diag}(1, \dots, \zeta, \dots, 1)$$

denote the diagonal matrix having ζ in the i th position.

Each element of $G(m, 1, n)$ can be written uniquely in the form

$$\zeta^\lambda \sigma \quad \text{where } \lambda \in (\mathbb{Z}/m\mathbb{Z})^n, \sigma \in G(1, 1, n) \cong S_n,$$

and

$$\zeta^\lambda = \zeta_1^{\lambda_1} \zeta_2^{\lambda_2} \cdots \zeta_n^{\lambda_n}.$$

The multiplication is determined by

$$(\zeta^\lambda \sigma)(\zeta^\nu \tau) = \zeta^{\lambda+\sigma \cdot \nu} \sigma \tau,$$

where S_n acts on $(\mathbb{Z}/m\mathbb{Z})^n$ by permuting the coordinates, that is, $(\sigma \cdot \lambda)_i = \lambda_{\sigma^{-1}(i)}$.

The group $G(m, p, n)$ is defined to be the following subgroup of $G(m, 1, n)$:

$$G(m, p, n) = \{\zeta^\lambda \sigma \in G(m, 1, n) \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n \equiv_p 0\},$$

where \equiv_p denotes equivalence modulo p .

Let V be a vector space with an ordered basis v_1, \dots, v_n . Let $S_{-1}(V)$ be the algebra generated by v_1, \dots, v_n subject to the relations $v_i v_j = -v_j v_i$ for all $i \neq j$. In this case the corresponding quantum exterior algebra $\bigwedge_{-1}(V)$ is commutative: $v_i \wedge v_j = v_j \wedge v_i$ for all i, j . Set $G := G(m, p, n)$ and consider the natural action of G on V . This action extends to an action of G on $S_{-1}(V)$ by algebra automorphisms.

We begin with the following:

Theorem 6.1. *Assume that $n \geq 4$. The constant Hochschild 2-cocycles representing elements in $\text{HH}^2(S_{-1}(V), S_{-1}(V) \rtimes G(m, p, n))$ form a vector space with basis all*

$$\begin{aligned} & \zeta^\lambda \otimes v_r^* \wedge v_s^* \quad (\lambda_{r'} \equiv_m 0 \text{ for all } r' \notin \{r, s\} \text{ and } \lambda_r + \lambda_s \equiv_p 0), \\ & \zeta^\lambda(rs) \otimes v_r^* \wedge v_s^* \quad (\lambda_{r'} \equiv_m 0 \text{ for all } r' \notin \{r, s\} \text{ and } \lambda_r + \lambda_s \equiv_p 0), \\ & \zeta^{\lambda_s} [\zeta^\lambda(rs)] \otimes v_r^* \wedge v_{r'}^* + \zeta^\lambda(rs) \otimes v_s^* \wedge v_{r'}^* \quad (\lambda_{s'} \equiv_m 0 \text{ for all } s' \notin \{r, s\} \text{ and } \lambda_r + \lambda_s \equiv_m 0), \\ & \zeta^{\lambda_s} [\zeta^\lambda(rs)(r's')] \otimes v_r^* \wedge v_{r'}^* + \zeta^{\lambda_s + \lambda_{r'}} [\zeta^\lambda(rs)(r's')] \otimes v_r^* \wedge v_{s'}^* \\ & \quad + \zeta^\lambda(rs)(r's') \otimes v_s^* \wedge v_{r'}^* + \zeta^{\lambda_{r'}} [\zeta^\lambda(rs)(r's')] \otimes v_s^* \wedge v_{s'}^* \\ & \quad (\lambda_t \equiv_m 0 \text{ for all } t \notin \{r, r', s, s'\} \text{ and } \lambda_r + \lambda_s \equiv_m 0 \equiv_m \lambda_{r'} + \lambda_{s'}), \\ & \zeta^{\lambda_s + \lambda_{r'}} [\zeta^\lambda(rsr')] \otimes v_r^* \wedge v_s^* + \zeta^\lambda(rsr') \otimes v_s^* \wedge v_{r'}^* + \zeta^{\lambda_s} [\zeta^\lambda(rsr')] \otimes v_r^* \wedge v_{r'}^* \\ & \quad (\lambda_{s'} \equiv_m 0 \text{ for all } s' \notin \{r, r', s\} \text{ and } \lambda_r + \lambda_{r'} + \lambda_s \equiv_m 0). \end{aligned}$$

Proof. For each $g \in G$ and $1 \leq r < s \leq n$, let $\alpha_{rs}^g \in \mathbb{C}$ and

$$(6.2) \quad \alpha = \sum_{g \in G} \sum_{1 \leq r < s \leq n} \alpha_{rs}^g g \otimes v_r^* \wedge v_s^*$$

be an arbitrary constant cochain. (That is, α is a cochain whose polynomial part has degree 0 in the factor $S_{\mathbf{q}}(V) \rtimes G$ of the term $(S_{\mathbf{q}}(V) \rtimes G) \otimes \bigwedge_{\mathbf{q}-1}^2(V^*)$ of the complex (3.2).) By the analysis in the proof of Theorem 4.4, $d_3^*(\alpha) = 0$ if, and only if, equation (4.5) holds, that is, since $q_{rs} = -1$ for all $r \neq s$,

$$(6.3) \quad \alpha_{jk}^g (v_i - {}^g v_i) + \alpha_{ik}^g (v_j - {}^g v_j) + \alpha_{ij}^g (v_k - {}^g v_k) = 0$$

for all $g \in G$ and all $i < j < k$. By symmetry, (6.3) holds for all triples i, j, k if and only if it holds for all $i < j < k$. If α is any of the types of cochains in the statement of the theorem, these equations do indeed hold.

Conversely, assume that the equations (6.3) hold for a cochain α of the form (6.2). Fix a group element $g = \zeta^\lambda \sigma$. Suppose $\alpha_{ij}^g \neq 0$ for some i, j . If $\sigma = 1$ or $\sigma = (ij)$, after reindexing, we obtain the first and second listed cocycles. Now suppose there is some other k for which $\sigma(k) \neq k$. As a consequence of (6.3) we must have either $\sigma(k) = i$ or $\sigma(k) = j$. Suppose $\sigma(k) = i$ (the other possibility is similar). Then $\zeta^{\lambda_i} \alpha_{ij}^g = \alpha_{jk}^g$ and either

- (1) $\sigma(i) = k$, $\zeta^{\lambda_k} \alpha_{jk}^g = \alpha_{ij}^g$, and $\alpha_{ik}^g(v_j - {}^g v_j) = 0$ or
- (2) $\sigma(i) = j$, $\sigma(j) = k$, $\zeta^{\lambda_j} \alpha_{jk}^g = \alpha_{ik}^g$, and $\zeta^{\lambda_k} \alpha_{ik}^g = \alpha_{ij}^g$.

In the second case, σ contains the 3-cycle (ijk) in its disjoint cycle decomposition. We claim that $\sigma = (ijk)$: If not then there is some other l with $\sigma(l) \neq l$. Apply (6.3) to i, j, l to obtain

$$\alpha_{jl}^g(v_i - \zeta^{\lambda_i} v_j) + \alpha_{il}^g(v_j - \zeta^{\lambda_j} v_k) + \alpha_{ij}^g(v_l - {}^g v_l) = 0.$$

This is not possible since $\alpha_{ij}^g(v_l - {}^g v_l) \neq 0$. Putting all the conditions together and reindexing we obtain the fifth listed cocycle.

In the first case, σ contains the transposition (ik) in its disjoint cycle decomposition. If $\sigma = (ik)$, then, after reindexing, we obtain a third type or a second type plus a third type of the cocycles listed. Now suppose there is some other l for which $\sigma(l) \neq l$. The equation (6.3) applied to i, j, l becomes

$$\alpha_{jl}^g(v_i - \zeta^{\lambda_k} v_k) + \alpha_{il}^g(v_j - {}^g v_j) + \alpha_{ij}^g(v_l - {}^g v_l) = 0.$$

This forces $\sigma(j) = l$, $\sigma(l) = j$, $\alpha_{jl}^g = 0$, $\alpha_{ik}^g = 0$, $\zeta^{\lambda_i} \alpha_{il}^g = \alpha_{ij}^g$, and $\zeta^{\lambda_j} \alpha_{ij}^g = \alpha_{il}^g$. We now have the product of transpositions $(ik)(jl)$ as part of the disjoint cycle decomposition of σ . We claim that $\sigma = (ik)(jl)$: If there were another m for which $\sigma(m) \neq m$, applying the above analysis to the triple i, j, m would force $\sigma(j) = m$ as well, a contradiction. Applying (6.3) to i, k, l gives

$$\alpha_{kl}^g(v_j - \zeta^{\lambda_l} v_l) + \alpha_{jk}^g(v_l - \zeta^{\lambda_j} v_j) = 0.$$

This forces $\zeta^{\lambda_j} \alpha_{jk}^g = \alpha_{kl}^g$ and $\zeta^{\lambda_l} \alpha_{kl}^g = \alpha_{jk}^g$. Putting all the conditions together and reindexing we obtain the fourth listed cocycle. \square

Lemma 6.4. *Assume that $n \geq 4$. Let α be a 2-cocycle from the list in Theorem 6.1. The image of $v_i \otimes v_j$ ($i \neq j$) under $(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\alpha)$ is zero whenever*

- (i) α is of the first, third, or fourth type and $m \geq 2$, or
- (ii) α is of the second or fifth type and $m \geq 3$.

Proof. Let $\alpha = \zeta^{\lambda_s} [\zeta^\lambda(rs)] \otimes v_r^* \wedge v_s^* + \zeta^\lambda(rs) \otimes v_s^* \wedge v_r^*$, the third 2-cocycle from the list in Theorem 6.1. By (3.6), $|G|$ times $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\alpha)](v_i \otimes v_j)$ is equal to

$$\sum_{\tau \in S_n} \sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv p0}} \zeta^{\nu\tau} \left(\alpha(\Psi_2(1 \otimes (\zeta^{\nu\tau})^{-1} v_i \otimes (\zeta^{\nu\tau})^{-1} v_j \otimes 1)) \right).$$

Applying (3.4) and then evaluating α , the above expression becomes

$$\sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv_p 0}} \left(\sum_{\substack{\tau \in S_n \\ \tau(r)=i \\ \tau(r')=j}} \zeta^{\lambda_s - \nu_i - \nu_j} \left[\zeta^{\nu \tau} (\zeta^\lambda(rs)) \right] + \sum_{\substack{\tau \in S_n \\ \tau(s)=i \\ \tau(r')=j}} \zeta^{-\nu_i - \nu_j} \left[\zeta^{\nu \tau} (\zeta^\lambda(rs)) \right] \right).$$

Note that the map $\tau \mapsto \tau(rs)$ is a bijection between the two sets over which the two inner summations are taken. Taking this fact and the conjugation action into account, the above expression can be rewritten as

$$\begin{aligned} & \sum_{\substack{\tau \in S_n \\ \tau(r)=i \\ \tau(r')=j}} \sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv_p 0}} \left(\zeta^{\lambda_s - \nu_i - \nu_j} \left[\zeta^{\nu + \tau \cdot \lambda - (i\tau(s)) \cdot \nu} (i\tau(s)) \right] + \zeta^{-\nu_i - \nu_j} \left[\zeta^{\nu + \tau(rs) \cdot \lambda - (i\tau(s)) \cdot \nu} (i\tau(s)) \right] \right) \\ &= \sum_{k \neq i, j} \sum_{\substack{\tau \in S_n \\ \tau(r)=i \\ \tau(r')=j \\ \tau(s)=k}} \sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv_p 0}} \left(\zeta^{\lambda_s - \nu_i - \nu_j} \left[\zeta^{\nu + \tau \cdot \lambda - (ik) \cdot \nu} (ik) \right] + \zeta^{-\nu_i - \nu_j} \left[\zeta^{\nu + \tau(rs) \cdot \lambda - (ik) \cdot \nu} (ik) \right] \right) \\ &= \sum_{k \neq i, j} \sum_{\substack{\tau \in S_n \\ \tau(r)=i \\ \tau(r')=j \\ \tau(s)=k}} \sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv_p 0}} \left(\zeta^{\lambda_s - \nu_i - \nu_j} \left[\zeta_i^{\lambda_r + \nu_i - \nu_k} \zeta_k^{\lambda_s + \nu_k - \nu_i} (ik) \right] + \zeta^{-\nu_i - \nu_j} \left[\zeta_i^{\lambda_s + \nu_i - \nu_k} \zeta_k^{\lambda_r + \nu_k - \nu_i} (ik) \right] \right). \end{aligned}$$

Since the inner-most summand does not depend on τ , the above expression is proportional to

$$(*) \quad \sum_{k \neq i, j} \sum_{\substack{\nu \in (\mathbb{Z}/m\mathbb{Z})^n \\ \nu_1 + \dots + \nu_n \equiv_p 0}} \left(\zeta^{\lambda_s - \nu_i - \nu_j} \left[\zeta_i^{\lambda_r + \nu_i - \nu_k} \zeta_k^{\lambda_s + \nu_k - \nu_i} (ik) \right] + \zeta^{-\nu_i - \nu_j} \left[\zeta_i^{\lambda_s + \nu_i - \nu_k} \zeta_k^{\lambda_r + \nu_k - \nu_i} (ik) \right] \right).$$

Rewriting the second summation in (*) as $\sum_{\nu_1, \dots, \nu_l, \dots, \nu_n \in \mathbb{Z}/m\mathbb{Z}} \sum_{\substack{\nu_l \in \mathbb{Z}/m\mathbb{Z} \\ \nu_1 + \dots + \nu_n \equiv_p 0}}$, where l is chosen so

that $l \neq i, j, k$, and observing that the inner-most summand of (*) depends only on ν_i , ν_j , and ν_k , we see that the expression in (*) is proportional to

$$\begin{aligned} & \sum_{k \neq i, j} \sum_{l_1, l_2, l_3 \in \mathbb{Z}/m\mathbb{Z}} \left(\zeta^{\lambda_s - l_1 - l_2} \left[\zeta_i^{\lambda_r + l_1 - l_3} \zeta_k^{\lambda_s + l_3 - l_1} (ik) \right] + \zeta^{-l_1 - l_2} \left[\zeta_i^{\lambda_s + l_1 - l_3} \zeta_k^{\lambda_r + l_3 - l_1} (ik) \right] \right) \\ &= \sum_{k \neq i, j} \sum_{l_1, l_2 \in \mathbb{Z}/m\mathbb{Z}} \left(\zeta^{\lambda_s - l_1 - l_2} \sum_{l_3 \in \mathbb{Z}/m\mathbb{Z}} \left[\zeta_i^{\lambda_r + l_3} \zeta_k^{\lambda_s - l_3} (ik) \right] + \zeta^{-l_1 - l_2} \sum_{l_3 \in \mathbb{Z}/m\mathbb{Z}} \left[\zeta_i^{\lambda_s + l_3} \zeta_k^{\lambda_r - l_3} (ik) \right] \right) \\ &= \sum_{k \neq i, j} \left(T_1 \sum_{l_3 \in \mathbb{Z}/m\mathbb{Z}} \left[\zeta_i^{\lambda_r + l_3} \zeta_k^{\lambda_s - l_3} (ik) \right] + T_2 \sum_{l_3 \in \mathbb{Z}/m\mathbb{Z}} \left[\zeta_i^{\lambda_s + l_3} \zeta_k^{\lambda_r - l_3} (ik) \right] \right), \end{aligned}$$

where $T_1 = \sum_{l_1, l_2 \in \mathbb{Z}/m\mathbb{Z}} \zeta^{\lambda_s - l_1 - l_2}$ and $T_2 = \sum_{l_1, l_2 \in \mathbb{Z}/m\mathbb{Z}} \zeta^{-l_1 - l_2}$. The above expression is zero since both T_1 and T_2 are zero whenever $m \geq 2$.

The proofs for the other 2-cocycles are similar. \square

Remark 6.5. The calculations corresponding to the second and fifth 2-cocycles involve the sum $\sum_{l \in \mathbb{Z}/m\mathbb{Z}} \zeta^{2l}$, which is zero if and only if $m \geq 3$, explaining the assumption $m \geq 3$ in the second part of the previous lemma.

Combining Theorems 3.5, 4.4, 6.1, and Lemma 6.4, we obtain:

Theorem 6.6. *If $n \geq 4$ and $m \geq 3$, then the vector space of maps $\kappa : V \times V \rightarrow \mathbb{C}G(m, p, n)$ for which $\mathcal{H}_{\mathbf{q}, \kappa}$ is a quantum Drinfeld Hecke algebra is trivial, and hence there are no non-trivial quantum Drinfeld Hecke algebras.*

Remark 6.7. The analogue of the above theorem for the case $\mathbf{q} = \mathbf{1}$ (that is, $q_{ij} = 1$ for all i, j) was proved by Ram and Shepler [RS].

Next, we consider the special cases $G(1, 1, n)$, $G(2, 1, n)$, and $G(2, 2, n)$. In all three cases we assume that $n \geq 4$. In these cases, there do exist nontrivial quantum Drinfeld Hecke algebras and in what follows we classify them.

The symmetric group $G(1, 1, n) = S_n$

In this case, Theorem 6.1 reduces to the following:

Theorem 6.8. *Assume that $n \geq 4$. The constant Hochschild 2-cocycles representing elements in $\text{HH}^2(S_{-1}(V), S_{-1}(V) \rtimes S_n)$ form a vector space with basis all*

$$\begin{aligned} & 1 \otimes v_r^* \wedge v_s^*, \\ & (rs) \otimes v_r^* \wedge v_s^*, \\ & (rs) \otimes (v_r^* \wedge v_{r'}^* + v_s^* \wedge v_{r'}^*), \\ & (rs)(r's') \otimes (v_r^* \wedge v_{r'}^* + v_r^* \wedge v_{s'}^* + v_s^* \wedge v_{r'}^* + v_s^* \wedge v_{s'}^*), \\ & (rsr') \otimes (v_r^* \wedge v_s^* + v_s^* \wedge v_{r'}^* + v_r^* \wedge v_{r'}^*). \end{aligned}$$

Combining Theorems 3.5, 4.4, and 6.8, we obtain:

Theorem 6.9. *Assume that $n \geq 4$. The maps $\kappa : V \times V \rightarrow \mathbb{C}S_n$ for which $\mathcal{H}_{\mathbf{q}, \kappa}$ is a quantum Drinfeld Hecke algebra form a five-dimensional vector space with basis consisting of maps $V \times V \rightarrow \mathbb{C}S_n$ determined by their effect on pairs (v_i, v_j) , $i \neq j$, given by*

$$\begin{aligned} (v_i, v_j) & \mapsto 1, \\ (v_i, v_j) & \mapsto (ij), \\ (v_i, v_j) & \mapsto \sum_{k \neq i, j} ((ik) + (jk)), \\ (v_i, v_j) & \mapsto \sum_{k, l \notin \{i, j\}} (ik)(jl), \\ (v_i, v_j) & \mapsto \sum_{k \neq i, j} ((ijk) + (ikj)). \end{aligned}$$

The hyperoctahedral group $G(2, 1, n) = WB_n$

We know from Lemma 6.4 that if α is a first, third, or fourth type of 2-cocycle in the list in Theorem 6.1, then the image of $v_i \otimes v_j$ ($i \neq j$) under $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\alpha)]$ is zero. For the second and fifth type of 2-cocycles, we have the following:

Lemma 6.10. *If α is a second or fifth type of 2-cocycle, then the corresponding image of $v_i \otimes v_j$ ($i \neq j$) under $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\alpha)]$ is, respectively, proportional to*

$$\begin{cases} (ij) - \zeta_i \zeta_j (ij) & \text{if } \lambda_r, \lambda_s \text{ have the same parity} \\ \zeta_i (ij) - \zeta_j (ij) & \text{if } \lambda_r, \lambda_s \text{ have different parity,} \end{cases}$$

$$\sum_{k \neq i, j} (2(ijk) - 2\zeta_i \zeta_j (ijk) - 2\zeta_j \zeta_k (ijk) + 2\zeta_i \zeta_k (ijk) + (ikj) - \zeta_i \zeta_j (ikj) + \zeta_j \zeta_k (ikj) - \zeta_i \zeta_k (ikj)).$$

Combining Theorems 3.5, 4.4, 6.1, and Lemma 6.10, we obtain:

Theorem 6.11. *Assume that $n \geq 4$. The maps $\kappa : V \times V \rightarrow \mathbb{C}G(2, 1, n)$ for which $\mathcal{H}_{\mathbf{q}, \kappa}$ is a quantum Drinfeld Hecke algebra form a two-dimensional vector space with basis consisting of maps $V \times V \rightarrow \mathbb{C}G(2, 1, n)$ determined by their effect on pairs (v_i, v_j) , $i \neq j$, given by*

$$\begin{aligned} (v_i, v_j) &\mapsto (ij) - \zeta_i \zeta_j (ij), \\ (v_i, v_j) &\mapsto \sum_{k \neq i, j} ((ijk) - \zeta_i \zeta_j (ijk) - \zeta_j \zeta_k (ijk) + \zeta_i \zeta_k (ijk) \\ &\quad + (ikj) - \zeta_i \zeta_j (ikj) + \zeta_j \zeta_k (ikj) - \zeta_i \zeta_k (ikj)). \end{aligned}$$

The type D_n Weyl group $G(2, 2, n) = WD_n$

As was the case for $G(2, 1, n)$, we do not get nontrivial quantum Drinfeld Hecke algebras from the first, third, or fourth type of 2-cocycles, but there do exist nontrivial quantum Drinfeld Hecke algebras for the second and fifth type of 2-cocycles. The corresponding result for this case is simply Theorem 6.11 with $G(2, 1, n)$ replaced by $G(2, 2, n)$.

7. COMPLEX REFLECTION GROUPS: SYMPLECTIC REPRESENTATIONS

In this section, we classify the quantum Drinfeld Hecke algebras for certain symplectic representations of the complex reflection groups $G(m, 1, n)$, m even, $n \geq 3$. We also explain how our classification includes all of the braided Cherednik algebras of Bazlov and Berenstein [BB].

Let m be an *even* positive integer, let G denote the complex reflection group $G(m, 1, n)$, and let ζ denote a primitive m th root of unity. Let U be a vector space of dimension n , and let x_1, \dots, x_n be an ordered basis for U . Denote by y_1, \dots, y_n the basis for U^* dual to $\{x_i\}$, so that $U^* = \text{span}_{\mathbb{C}}\{y_1, \dots, y_n\}$. Set $v_i := x_i, v_{n+i} := y_i, 1 \leq i \leq n$, and $V := U \oplus U^*$, so that $\dim V = 2n$ and v_1, \dots, v_{2n} is an ordered basis for V .

Consider the natural action of G on U . The dual action of G on U^* is determined by

$${}^g y_i = \varepsilon^{-1} y_j \text{ if and only if } {}^g x_i = \varepsilon x_j.$$

Thus, we have an action of G on $V := U \oplus U^*$, and this action induces an action of G on $S_{-\mathbf{1}}(V)$ by algebra automorphisms, where $-\mathbf{1}$ is defined below.

For each pair i, j of elements in $\{1, \dots, 2n\}$, let q_{ij} be a scalar defined as follows:

$$q_{ij} := \begin{cases} 1, & \text{if } i \equiv j \pmod{n} \\ -1, & \text{otherwise.} \end{cases}$$

Denote by $-\mathbf{1}$ the tuple of scalars (q_{ij}) .

Let $\mathcal{C} := \{\zeta^l \mid l \in \mathbb{Z}/m\mathbb{Z}\}$. Since $|\mathcal{C}| = m$ is even, -1 is an element of \mathcal{C} . For each $\varepsilon \in \mathcal{C}$ and $i, j \in \{1, \dots, n\}, i \neq j$, define $\sigma_{ij}^{(\varepsilon)} \in \text{GL}(U)$ by

$$\sigma_{ij}^{(\varepsilon)}(x_k) := \begin{cases} x_k & \text{if } k \notin \{i, j\} \\ \varepsilon x_j & \text{if } k = i \\ -\varepsilon^{-1} x_i & \text{if } k = j. \end{cases}$$

That is, $\sigma_{ij}^{(\varepsilon)} = (-\zeta_i^{-l})\zeta_j^l(ij) \in G(m, 1, n)$, where $\varepsilon = \zeta^l$.

For each $\varepsilon \in \mathcal{C}$ and $i \in \{1, \dots, n\}$, define $t_i^{(\varepsilon)} \in \text{GL}(U)$ by

$$t_i^{(\varepsilon)}(x_k) := \begin{cases} x_k & \text{if } k \neq i \\ \varepsilon x_i & \text{if } k = i. \end{cases}$$

That is, $t_i^{(\varepsilon)} = \zeta_i^l \in G(m, 1, n)$, where $\varepsilon = \zeta^l$.

Theorem 7.1. *Suppose m is even and $n \geq 3$. The constant Hochschild 2-cocycles representing elements in $\text{HH}^2(S_{-1}(V), S_{-1}(V) \rtimes G)$ form a vector space with basis all*

$$\begin{aligned} & t_r^{(-1)} t_s^{(-1)} \otimes x_r^* \wedge x_s^*, \\ & t_r^{(-1)} t_s^{(-1)} \otimes y_r^* \wedge y_s^*, \\ & t_r^{(-1)} t_s^{(-1)} \otimes x_r^* \wedge y_s^* \quad (r \neq s), \\ & t_r^{(\varepsilon)} \otimes x_r^* \wedge y_r^*, \\ & \sigma_{rs}^{(\xi)} \otimes x_r^* \wedge y_r^* + \sigma_{rs}^{(\xi)} \otimes x_s^* \wedge y_s^* + \xi \sigma_{rs}^{(\xi)} \otimes x_r^* \wedge y_s^* - \xi^{-1} \sigma_{rs}^{(\xi)} \otimes x_s^* \wedge y_r^* \quad (r \neq s). \end{aligned}$$

Proof. For each $g \in G(m, 1, n)$ and $1 \leq r < s \leq 2n$, let $\alpha_{rs}^g \in \mathbb{C}$ and

$$\alpha = \sum_{g \in G(m, 1, n)} \sum_{1 \leq r < s \leq 2n} \alpha_{rs}^g g \otimes v_r^* \wedge v_s^*$$

be an arbitrary constant cochain. Then $d_3^*(\alpha) = 0$ if, and only if, equation (4.5) holds for all g, i, j, k . Fix $g = \zeta^\lambda \sigma$ and suppose $\alpha_{ij}^g \neq 0$ for some i, j .

First we consider the case $\sigma = 1$, so that $g = \zeta^\lambda$ for some λ , and g acts diagonally on the basis $x_1, \dots, x_n, y_1, \dots, y_n$. Assume that $1 \leq i < j \leq n$. In this case, since $\alpha_{ij}^g \neq 0$, as a consequence of (4.5) we have $\zeta^{\lambda k} = q_{ik} q_{jk}$ for each $k \notin \{i, j\}$, $1 \leq k \leq n$, and $\zeta^{-\lambda k} = q_{ik} q_{jk}$ when $n+1 \leq k \leq 2n$. By choosing $k = i+n$ or $k = j+n$ we obtain $\zeta^{-\lambda i} = -1$, $\zeta^{-\lambda j} = -1$. Other choices yield $\zeta^{\lambda k} = 1$ when $k \notin \{i, j\}$. Thus we obtain the first cocycle in the list. A similar analysis yields the second cocycle. If we instead assume $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$, $j \neq i+n$, we obtain the third. Finally assume $1 \leq i \leq n$ and $j = i+n$. Then for all $k \neq i$ ($1 \leq k \leq n$),

$$\zeta^{\lambda k} = q_{ik} q_{i+n, k} = 1$$

and for all $k \neq i+n$ ($n+1 \leq k \leq 2n$),

$$\zeta^{-\lambda k} = q_{ik} q_{i+n, k} = 1,$$

while there is no restriction on λ_i . Thus we obtain the fourth cocycle in the list, completing the analysis of the case $\sigma = 1$.

Next consider the case $g = \zeta^\lambda \sigma$, $\sigma \neq 1$, and $\alpha_{ij}^g \neq 0$ for some i, j . Assume $1 \leq i < j \leq n$ and $\sigma = (ij)$. Consider equation (4.5) in case $k = i + n$:

$$\alpha_{j,i+n}^g(v_i - q_{ij}q_{i,i+n}\zeta^{\lambda_j}v_j) - \alpha_{i,i+n}^g(q_{ij}v_j - q_{j,i+n}\zeta^{\lambda_i}v_i) + \alpha_{ij}^g(q_{i,i+n}q_{j,i+n}v_{i+n} - \zeta^{-\lambda_j}v_{j+n}) = 0.$$

Since $\alpha_{ij}^g \neq 0$, we must have

$$q_{i,i+n}q_{j,i+n}v_{i+n} - \zeta^{-\lambda_j}v_{j+n} = 0,$$

which is not possible. Therefore this case cannot occur. Similarly if $n + 1 \leq i < j \leq 2n$ and $\sigma = (i - n, j - n)$, we arrive at a contradiction. Assume now that $1 \leq i \leq n$, $n + 1 \leq j \leq 2n$, and $\sigma = (i, j - n)$. A similar analysis forces $\alpha_{j-n,i+n}^g$, $\alpha_{j-n,j}^g$, and $\alpha_{i,i+n}^g$ all to be nonzero and one obtains the fifth cocycle in the list.

Finally assume that $\sigma \neq 1$, $\alpha_{ij}^g \neq 0$ for some i, j , and σ is *not* a 2-cycle moving only i, j . Then there is some $k \notin \{i, j\}$, $1 \leq k \leq n$, for which $\sigma(k) \neq k$. Assume first that $1 \leq i < j \leq n$. Applying equation (4.5) to i, j, k , we see that $\sigma(k)$ is forced to be either i or j . Without loss of generality assume $\sigma(k) = i$. Then $\zeta^{\lambda_i}\alpha_{ij}^g = \alpha_{jk}^g$, and again equation (4.5) forces either one of the following two possibilities:

- (1) $\sigma(i) = k$, $\alpha_{jk}^g q_{ij} \zeta^{\lambda_k} = \alpha_{ij}^g q_{jk}$, and $\alpha_{ik}^g (q_{ij}v_j - q_{jk}v_i) = 0$, or
- (2) $\sigma(i) = j$, $\sigma(j) = k$, $\alpha_{jk}^g q_{ik} \zeta^{\lambda_j} = \alpha_{ik}^g$, and $\alpha_{ik}^g \zeta^{\lambda_k} = \alpha_{ij}^g q_{ik}$.

In case (1), further consider equation (4.5) with k replaced by $i + n$:

$$\alpha_{j,i+n}^g(v_i - q_{ij}q_{i,i+n}\zeta^{\lambda_k}v_k) - \alpha_{i,i+n}^g(q_{ij}v_j - q_{j,i+n}v_i) + \alpha_{ij}^g(q_{i,i+n}q_{j,i+n}v_{i+n} - \zeta^{-\lambda_k}v_{k+n}) = 0.$$

This forces $\alpha_{j,i+n}^g = 0$ since v_i, v_k cannot both occur in other terms of this equation. However, since also $j \notin \{i + n, k + n\}$ and $\alpha_{ij}^g \neq 0$, the equation cannot hold. Therefore this case does not occur. A similar analysis applies if we assume $n + 1 \leq i < j \leq 2n$ or if $1 \leq i \leq n$ and $n + 1 \leq j \leq 2n$.

In case (2), consider equation (4.5) applied to indices $i, j, i + n$:

$$\alpha_{j,i+n}^g(v_i - q_{ij}q_{i,i+n}\zeta^{\lambda_j}v_j) - \alpha_{i,i+n}^g(q_{ij}v_j - q_{j,i+n}\zeta^{\lambda_k}v_k) + \alpha_{ij}^g(q_{i,i+n}q_{j,i+n}v_{i+n} - \zeta^{-\lambda_j}v_{j+n}) = 0.$$

Since $\alpha_{ij}^g \neq 0$ and v_{i+n}, v_{j+n} do not occur in other terms in this equation, the equation in fact does not hold. Therefore there is no such cocycle. A similar analysis applies if we assume $n + 1 \leq i < j \leq 2n$ or if $1 \leq i \leq n$ and $n + 1 \leq j \leq 2n$. \square

Lemma 7.2. For each 2-cocycle α from the list in Theorem 7.1, the image of $v_i \otimes v_j$ ($i \neq j$) under $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\alpha)]$ is, respectively, given by

$$\begin{array}{l} 0 \\ 0 \\ 0 \\ \left\{ \begin{array}{l} x_i \otimes x_j \mapsto 0 \\ y_i \otimes y_j \mapsto 0 \\ y_j \otimes x_i \mapsto 0 \\ x_i \otimes y_j \mapsto 0 \quad (i \neq j) \\ x_i \otimes y_i \mapsto \frac{1}{n} t_i^{(\varepsilon)} \end{array} \right. \\ \left\{ \begin{array}{l} x_i \otimes x_j \mapsto 0 \\ y_i \otimes y_j \mapsto 0 \\ y_j \otimes x_i \mapsto 0 \\ x_i \otimes y_j \mapsto \frac{2}{mn(n-1)} \sum_{\eta \in \mathcal{C}} \eta \sigma_{ij}^{(\eta)} \quad (i \neq j) \\ x_i \otimes y_i \mapsto \frac{2}{mn(n-1)} \sum_{\substack{\eta \in \mathcal{C} \\ j \neq i}} \sigma_{ij}^{(\eta)} \end{array} \right. \end{array}$$

Combining Theorems 3.5, 4.4, 7.1, and Lemma 7.2, we obtain:

Theorem 7.3. Assume that m is even and $n \geq 3$. The maps $\kappa : V \times V \rightarrow \mathbb{C}G(m, 1, n)$ for which $\mathcal{H}_{\mathbf{q}, \kappa}$ is a quantum Drinfeld Hecke algebra form an $(m+1)$ -dimensional vector space with basis consisting of maps $V \times V \rightarrow \mathbb{C}G(m, 1, n)$ determined by their effect on pairs $(x_i, x_j), (y_i, y_j), (x_i, y_j), (x_i, y_i), i \neq j$, given by

$$\left\{ \begin{array}{l} (x_i, x_j) \mapsto 0 \\ (y_i, y_j) \mapsto 0 \\ (x_i, y_j) \mapsto 0 \\ (x_i, y_i) \mapsto t_i^{(\varepsilon)} \end{array} \right. \quad \left\{ \begin{array}{l} (x_i, x_j) \mapsto 0 \\ (y_i, y_j) \mapsto 0 \\ (x_i, y_j) \mapsto \sum_{\eta \in \mathcal{C}} \eta \sigma_{ij}^{(\eta)} \\ (x_i, y_i) \mapsto \sum_{\substack{\eta \in \mathcal{C} \\ j \neq i}} \sigma_{ij}^{(\eta)}. \end{array} \right.$$

Remark 7.4. Let \mathcal{C}' be a subgroup of \mathcal{C} , the group of m th roots of unity (m even), and let $W_{\mathcal{C}, \mathcal{C}'}$ denote the subgroup of $G(m, 1, n)$ generated by all $\sigma_{ij}^{(\varepsilon)}$, $\varepsilon \in \mathcal{C}$ and $t_i^{(\varepsilon')}$, $\varepsilon' \in \mathcal{C}'$. Let $c : \mathcal{C}' \rightarrow \mathbb{C}$ be any function, and $c_{\varepsilon'} := c(\varepsilon')$. In [BB], Bazlov and Berenstein defined algebras $\mathcal{H}_c(W_{\mathcal{C}, \mathcal{C}'})$ with generators $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ subject to the following relations

- (i) $x_i x_j + x_j x_i = y_i y_j + y_j y_i = 0$ for all $i \neq j$;

- (ii) $gx_i g^{-1} = {}^g x_i$, $gy_i g^{-1} = {}^g y_i$ for all $g \in W_{\mathcal{C}, \mathcal{C}'}$, $i = 1, \dots, n$;
- (iii) $y_j x_i + x_i y_j = c_1 \sum_{\varepsilon \in \mathcal{C}} \varepsilon \sigma_{ij}^{(\varepsilon)}$ for all $i \neq j$, and

$$y_i x_i - x_i y_i = 1 + c_1 \sum_{\substack{\varepsilon \in \mathcal{C} \\ j \neq i}} \sigma_{ij}^{(\varepsilon)} + \sum_{\varepsilon' \in \mathcal{C}' \setminus \{1\}} c_{\varepsilon'} t_i^{(\varepsilon')} \text{ for } i = 1, \dots, n.$$

Let f_ε , $\varepsilon \in \mathcal{C}$, denote the maps of the first type in Theorem 7.3 and let \tilde{f} denote the map of the second type in Theorem 7.3. The algebras $\mathcal{H}_c(W_{\mathcal{C}, \mathcal{C}'})$ are precisely the quantum Drinfeld Hecke algebras arising from the map $\left(f_1 + \sum_{\varepsilon' \in \mathcal{C}' \setminus \{1\}} \left(c_{\varepsilon'} f_{\varepsilon'} + c_1 \tilde{f} \right) \right) : V \times V \rightarrow \mathbb{C}W_{\mathcal{C}, \mathcal{C}'}$.

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