

POINCARÉ-BIRKHOFF-WITT DEFORMATIONS OF SMASH PRODUCT ALGEBRAS FROM HOPF ACTIONS ON KOSZUL ALGEBRAS

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ABSTRACT. Let H be a Hopf algebra and let B be a Koszul H -module algebra. We provide necessary and sufficient conditions for a filtered algebra to be a Poincaré-Birkhoff-Witt (PBW) deformation of the smash product algebra $B\#H$. Many examples of these deformations are given.

0. INTRODUCTION

Given a Hopf algebra (H -) action on a Koszul algebra B , the aim of this work is to provide necessary and sufficient conditions for a certain filtered algebra, namely $\mathcal{D}_{B,\kappa}$ in Notation 0.3 below, to be a *Poincaré-Birkhoff-Witt (PBW) deformation* of the smash product algebra $B\#H$, i.e. $\text{gr}\mathcal{D}_{B,\kappa} \cong B\#H$ [Definition 0.1]. One well-studied case is that of group actions on polynomial rings, where many algebras of interest arise as such deformations, see for example [5], [7], [9], [20], [29], [31]. For group actions on other Koszul algebras, see [18], [24], [30], [32]. There are some results involving Hopf algebra actions, such as those of Khare [16] when H is cocommutative and B is a polynomial algebra. More specifically, the case when $H = U(\mathfrak{g})$, with \mathfrak{g} the Lie algebra of a (not necessarily connected) reductive algebraic group, was studied by Etingof, Gan, and Ginzburg [8] and by Khare and Tikaradze [17] where $\mathfrak{g} = \mathfrak{sl}_2$. Results for an action of the quantized enveloping algebra $H = U_q(\mathfrak{sl}_2)$ on the quantum plane are provided by Gan and Khare [10].

The goal of this paper is to provide a general theorem encompassing all of the above known classes of examples from the literature. Specifically, Theorem 0.4 gives PBW deformation conditions for $B\#H$, and it only requires the following of H and B : (1) the antipode of the Hopf algebra H is bijective, (2) the Koszul H -module algebra B is connected ($B_0 = k$), and (3) the H -action on B preserves the grading of B . We then apply our theorem to several different choices of Hopf algebras acting on Koszul algebras to obtain nontrivial PBW deformations, both known and new. Our work indicates that such examples abound.

Many ring theoretic properties are preserved under PBW deformation. To discuss this, let us consider the following definition.

Definition 0.1. Let $D = \bigcup_{i \geq 0} F_i$ be a filtered algebra with $\{0\} \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq D$. We say that D is a *Poincaré-Birkhoff-Witt (PBW) deformation* of an \mathbb{N} -graded algebra A if A is isomorphic to the associated graded algebra $\text{gr}_F D = \bigoplus_{i \geq 0} F_i/F_{i-1}$, as \mathbb{N} -graded algebras.

Now if $\text{gr}_F D$ is an integral domain, prime, or (right) noetherian, then so is D . Moreover, if D is affine with the standard filtration F' , then the Gelfand-Kirillov (GK) dimensions of D and of $\text{gr}_{F'} D$ are equal; $\text{GKdim}(\text{gr}_F D) \leq \text{GKdim}(D)$ for a general filtration F of D . The Krull dimension and global dimension of $\text{gr}_F D$ serve as upper bounds for the corresponding dimensions of D . These ring theoretic results can all be found in [22]. Homological properties preserved under PBW deformation have also been investigated; see [1] and [36] regarding the Calabi-Yau property, for instance. The representation theory of some classes of PBW deformations of smash product algebras has been thoroughly studied in the literature and still remains an active area of research. Some examples of PBW deformations whose representation theory is of interest include rational Cherednik algebras, symplectic reflection algebras, and various types of Hecke algebras (see, for example, [7], [8], [9], [20], [29], and for more recent work, see [6], [19], [33], [34]).

In order to state the main result, we need the following notation and terminology. Let k be a field of arbitrary characteristic and let an unadorned \otimes mean \otimes_k . Let \mathbb{N} denote the natural numbers, including 0. Recall that an \mathbb{N} -graded algebra is *Koszul* if its trivial module k admits a linear minimal graded free resolution; see [27, Chapter 2] for more details.

Notation 0.2. $[H, B, I, \kappa, \kappa^C, \kappa^L]$ First, let V be a finite dimensional vector space over k .

(i) Let H be a Hopf algebra with the standard structure notation: $(H, m, \Delta, u, \epsilon, S)$. Here, we assume that the antipode S of H is bijective.

(ii) Let $B = T_k(V)/(I)$ be an \mathbb{N} -graded, Koszul, left H -module algebra $B = \bigoplus_{j \geq 0} B_j$ with $B_0 = k$ and $I \subseteq V \otimes V$. We assume that the action of H preserves the grading and the subspace I of $V \otimes V$. So in this case, V is an H -module.

(iii) Take $\kappa : I \rightarrow H \oplus (V \otimes H)$ to be a k -bilinear map, where κ is the sum of its *constant* and *linear* parts $\kappa^C : I \rightarrow H$ and $\kappa^L : I \rightarrow V \otimes H$, respectively.

Notation 0.3. $[\mathcal{D}_{B,\kappa}]$ Let $\mathcal{D}_{B,\kappa}$ be the filtered k -algebra given by

$$\mathcal{D}_{B,\kappa} = \frac{T_k(V) \# H}{(r - \kappa(r))_{r \in I}}.$$

Here, we assign the elements of H degree 0.

Our main result is given as follows.

Theorem 0.4 (Theorem 3.1). *The algebra $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B \# H$ if and only if the following conditions hold:*

(a) κ is H -invariant [Definition 1.4]; and

If $\dim_k V \geq 3$, then the following equations hold for the maps $\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C$ and $\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L$, which are defined on the intersection $(I \otimes V) \cap (V \otimes I)$:

- (b) $\text{Im}(\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) \subseteq I$;
- (c) $\kappa^L \circ (\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) = -(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)$;
- (d) $\kappa^C \circ (\text{id} \otimes \kappa^L - \kappa^L \otimes \text{id}) \equiv 0$.

In the case that H is cocommutative and B is the symmetric algebra $S(V)$, this result was proven by Khare [16, Theorem 2.1], via the Diamond Lemma. Our proof is a generalization of that of Braverman and Gaitsgory [2, Lemma 0.4, Theorem 0.5] (where $H = k$) and of Shepler and the second author [30, Theorem 5.4] (where H is a group algebra).

Background information on Hopf algebra (co)actions, Hochschild cohomology, and deformations of algebras are provided in Section 1. In Section 2, we produce a free resolution of the smash product algebra $B \# H$; see Construction 2.5 and Theorem 2.10. This resolution is adapted from Guccione and Guccione [12]; Negron independently constructed a similar resolution [25]. Our resolution is used in the proof of Theorem 0.4, which is given in Section 3. Many examples of PBW deformations of $B \# H$ are provided in Section 4, including/involving:

- [Example 4.1] the *Crawley-Boevey-Holland algebras*;
- [Examples 4.2 and 4.4] some actions of semisimple, noncommutative, noncocommutative, Hopf algebras on skew polynomial rings;
- [Examples 4.13 and 4.16] actions of the Sweedler and the Taft algebras on the polynomial ring $k[u, v]$;
- [Example 4.18] the *quantized symplectic oscillator algebras of rank 1*.

All of the examples of $B \# H$ above have nontrivial PBW deformations.

1. BACKGROUND MATERIAL

We begin by discussing Hopf (co)actions on algebras and (co)modules and end with a discussion on deformations of algebras. For further background on these topics, we refer the reader to [23] and [2, 11], respectively.

1.1. Hopf algebra (co) actions.

Definition 1.1. (i) For a left H -module M , we denote the H -action by $\cdot : H \otimes M \rightarrow M$, that is by $h \cdot m \in M$ for all $h \in H, m \in M$. Similarly for all $h \in H$ and $m \in M$, we denote the right h -action on m by $m \cdot h$.

(ii) Given a Hopf algebra H and an algebra A , we say that H acts on A (from the left, as a Hopf algebra) if A is a left H -module and

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A$$

for all $h \in H, a, b \in A$, where the comultiplication is given by $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler's notation). In this case, we say that A is a *left H -module algebra*.

(iii) For any left H -comodule M , we denote the left H -coaction by $\rho(M) \subseteq H \otimes M$, where $\rho(m) = \sum m_{-1} \otimes m_0$ for $m_{-1} \in H$ and $m, m_0 \in M$. Likewise, the right H -coaction on a right H -comodule M is given by $\rho(m) = \sum m_0 \otimes m_1$ for $m, m_0 \in M$ and $m_1 \in H$.

Note that H is naturally an H -bimodule via left and right multiplication. This yields a *left H -adjoint action on H* given by

$$(1.2) \quad h \cdot \ell := \sum h_1 \ell S(h_2)$$

for $h, \ell \in H$. Moreover, if V is a left H -module, we give $V \otimes H$ an H -bimodule structure as follows: $h(v \otimes \ell) = \sum (h_1 \cdot v) \otimes h_2 \ell$ and $(v \otimes \ell)h = v \otimes \ell h$, for all $h, \ell \in H$ and $v \in V$. A *left H -adjoint action on $V \otimes H$* arises by combining these:

$$(1.3) \quad h \cdot (v \otimes \ell) := \sum (h_1 \cdot v) \otimes h_2 \ell S(h_3).$$

The left H -adjoint actions in (1.2) and (1.3) extend to the standard left H -adjoint action on $A = B \# H$ (where $B = T_k(V)/(I)$ as in Notation 0.2(ii)), via Definition 1.1, since the action of H preserves I .

Now we discuss the H -invariance of the map κ [Notation 0.2(iii)], which is one of the necessary conditions for the filtered algebra $\mathcal{D}_{B,\kappa}$ [Notation 0.3] to be a PBW deformation of $B \# H$.

Definition 1.4. Recall Notation 0.2. We say that the map κ is *H -invariant* if $h \cdot (\kappa(r)) = \kappa(h \cdot r)$ in $H \oplus (V \otimes H)$, for any relation $r \in I$ and $h \in H$.

1.2. Deformations of algebras and Hochschild cohomology. In this part, we remind the reader of the notion of a deformation of a k -algebra A and how Hochschild cohomology plays a role in its construction. This is seminal work of Gerstenhaber [11], adapted to our graded setting as in Braverman and Gaitsgory [2].

Definition 1.5. $[A_t, A_{(j)}]$ Let A be an associative algebra and let t be an indeterminate. A *deformation* of A over $k[t]$ is an associative $k[t]$ -algebra A_t over $k[t]$, which is isomorphic to $A[t]$ as k -vector spaces, with multiplication given by

$$a_1 * a_2 = \mu_0(a_1 \otimes a_2) + \mu_1(a_1 \otimes a_2)t + \mu_2(a_1 \otimes a_2)t^2 + \cdots,$$

for all $a_1, a_2 \in A$. Here, $\mu_i : A \otimes A \rightarrow A$ is a k -linear map, referred to as the *i -th multiplication map*. Moreover, $\mu_0(a_1 \otimes a_2) = a_1 a_2$ is the usual product in A .

Now assume that A is graded by \mathbb{N} . A *graded deformation* of A over $k[t]$ is an algebra A_t as above, which is itself graded by \mathbb{N} , setting $\deg(t) = 1$. The map μ_i is homogeneous of degree $-i$ in this case. A *j -th level graded deformation* of A is a graded associative algebra $A_{(j)}$ over $k[t]/(t^{j+1})$, that is isomorphic to $A[t]/(t^{j+1})$ as k -vector spaces, with multiplication given by

$$a_1 * a_2 = \mu_0(a_1 \otimes a_2) + \mu_1(a_1 \otimes a_2)t + \cdots + \mu_j(a_1 \otimes a_2)t^j.$$

The maps $\mu_i : A \otimes A \rightarrow A$ are extended to be linear over $k[t]/(t^{j+1})$.

The associativity of $*$ for the deformation A_t imposes conditions on the maps μ_i . Specifically, for each degree i , the following equation must hold for all $a_1, a_2, a_3 \in A$:

$$(1.6) \quad \sum_{j=0}^i \mu_j(\mu_{i-j}(a_1 \otimes a_2) \otimes a_3) = \sum_{j=0}^i \mu_j(a_1 \otimes \mu_{i-j}(a_2 \otimes a_3)).$$

We use Hochschild cohomology to study these equations.

Definition 1.7. $[\mathbf{B}\cdot(A)]$ Let A be a k -algebra and let M be an A -bimodule, or equivalently, an A^e -module. Here, $A^e := A \otimes A^{op}$. The *Hochschild cohomology* of M is $\mathrm{HH}^n(A, M) := \mathrm{Ext}_{A^e}^n(A, M)$. Moreover, this cohomology may be derived from the *bar resolution* $\mathbf{B}\cdot(A)$ of the A^e -module A :

$$\mathbf{B}\cdot(A) : \quad \cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A \otimes A \xrightarrow{\delta_0} A \longrightarrow 0$$

where

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $n \geq 0$ and $a_0, \dots, a_{n+1} \in A$. When $M = A$, write $\mathrm{HH}^n(A)$ for $\mathrm{HH}^n(A, A)$. Moreover, if A is (\mathbb{N}) -graded, then $\mathrm{HH}^n(A)$ inherits the grading of A : If $A = \bigoplus_i A_i$, then $\mathrm{HH}^n(A) = \bigoplus_i \mathrm{HH}^{n,i}(A)$.

Note that $\mathrm{Hom}_k(A^{\otimes n}, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A)$ since the A^e -module $A^{\otimes(n+2)}$ is induced from the k -module $A^{\otimes n}$. We will identify these two Hom spaces often without further comment. Now we make some remarks about the multiplication maps μ_i .

Remark 1.8. Using (1.6) for $i = 1$, we see that

$$(1.9) \quad \mu_1(a_1 \otimes a_2)a_3 + \mu_1(a_1 a_2 \otimes a_3) = a_1 \mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2 a_3),$$

for all $a_1, a_2, a_3 \in A$. In other words, μ_1 is a Hochschild 2-cocycle on the bar resolution of A , that is, $\delta_3^*(\mu_1) := \mu_1 \circ \delta_3$ vanishes. (Here we have identified the input $a_1 \otimes a_2 \otimes a_3$ with $1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1$ to apply δ_3 , under the identification of Hom spaces described above.)

Next, using (1.6) for $i = 2$, we see that

$$\begin{aligned} \mu_2(a_1 \otimes a_2)a_3 + \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3) + \mu_2(a_1 a_2 \otimes a_3) \\ = a_1 \mu_2(a_2 \otimes a_3) + \mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3)) + \mu_2(a_1 \otimes a_2 a_3). \end{aligned}$$

Therefore

$$(1.10) \quad \delta_3^*(\mu_2)(a_1 \otimes a_2 \otimes a_3) = \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3) - \mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3)),$$

for all $a_1, a_2, a_3 \in A$. In other words, μ_2 is a cochain on the bar resolution of A whose coboundary is given by the right-hand side of (1.10).

For all $i \geq 1$, (1.6) is equivalent to

$$(1.11) \quad \delta_3^*(\mu_i)(a_1 \otimes a_2 \otimes a_3) = \sum_{j=1}^{i-1} \mu_j(\mu_{i-j}(a_1 \otimes a_2) \otimes a_3) - \mu_j(a_1 \otimes \mu_{i-j}(a_2 \otimes a_3)).$$

That is to say, μ_i is a cochain on the bar resolution of A whose coboundary is given by the right-hand side of (1.11).

Definition 1.12. The right-hand side of (1.11) is the $(i-1)$ -th *obstruction* of the deformation A_t of A from Definition 1.5. An $(i-1)$ -th level graded deformation (defined by maps μ_1, \dots, μ_{i-1}) *lifts* to an i -th level graded deformation if there exists a map μ_i for which $\mu_1, \dots, \mu_{i-1}, \mu_i$ define an i -th level graded deformation.

The next proposition makes clear the choice of terminology in the above definition. Ultimately one is interested in a deformation of A over $k[t]$, and its specializations at particular values of t . The i -th level graded deformations are steps in this direction.

Proposition 1.13. [2, Proposition 1.5] *All obstructions to lifting an $(i-1)$ -th level graded deformation to the next level lie in $\mathrm{HH}^{3,-i}(A)$. An $(i-1)$ -th level deformation lifts to the i -th level if and only if its $(i-1)$ -th obstruction cocycle is zero in cohomology, i.e., there is a map μ_i such that (1.11) holds for all a_1, a_2, a_3 in A .*

The connection between graded deformations and PBW deformations is well known; the following statement is a consequence of the canonical embedding of A as a k -linear direct summand of $A[t]$, with splitting map given by specialization at $t = 0$.

Proposition 1.14. [2, Remark 1.4] *Given a graded algebra A and a graded deformation A_t of A , then A_t specialized at $t = 1$ is a PBW deformation of A . \square*

Now we explain that the two notions of deformation of $B\#H$ coincide; recall Notations 0.2 and 0.3. The following result is well known in related contexts, but we include some details for the readers' convenience.

Proposition 1.15. *The following statements are equivalent.*

- *The algebra $\mathcal{D}_{B,\kappa} := (T_k(V)\#H) / (r - \kappa(r))_{r \in I}$ is a PBW deformation of $B\#H$.*
- *The algebra $\mathcal{D}_{B,\kappa,t} := (T_k(V)\#H)[t] / (r - \kappa^L(r)t - \kappa^C(r)t^2)_{r \in I}$ is a graded deformation of $B\#H$ over $k[t]$.*

Proof. Assume that $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$. By its definition, $\mathcal{D}_{B,\kappa,t}$ is an associative algebra, and so we need only see that it is isomorphic to $B\#H[t]$ as a vector space. To this end, use the PBW property to define a k -linear map $\pi : B\#H \rightarrow T_k(V)\#H$ whose composition with the quotient map onto $\mathcal{D}_{B,\kappa}$ is an isomorphism of filtered vector spaces. Extend π to a $k[t]$ -linear map from $B\#H[t]$ to $T_k(V)\#H[t]$. Its composition with the quotient map to $\mathcal{D}_{B,\kappa,t}$ is an isomorphism of k -vector spaces; one sees this by a degree argument.

Conversely, assume that $\mathcal{D}_{B,\kappa,t}$ is a graded deformation of $B\#H$ over $k[t]$. We may specialize to $t = 1$ to obtain $\mathcal{D}_{B,\kappa}$. Now apply Proposition 1.14 to conclude that $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$. \square

2. RESOLUTIONS FOR SMASH PRODUCT ALGEBRAS

In this section, let A denote the smash product $B\#H$, which is an \mathbb{N} -graded algebra: $A = \bigoplus_{j \geq 0} (B_j \otimes H)$. Thus $A_0 \cong H$. The aim is to construct a free A^e -resolution X_\bullet of the A^e -module A from resolutions of H and of B (denoted by C_\bullet and D_\bullet , respectively). This construction simultaneously generalizes results of Guccione and Guccione [12] and of Shepler and the second author [30, Section 4]. A similar resolution was constructed independently by Negron [25].

Definition 2.1. [C_\bullet, C_i, C'_i] For $i \geq 0$, let C_i denote the H^e -module $H^{\otimes(i+2)}$. The left H -comodule structure $\rho : C_i \rightarrow H \otimes C_i$ is given by

$$\rho(h^0 \otimes h^1 \otimes \cdots \otimes h^{i+1}) := \sum h_1^0 \cdots h_1^{i+1} \otimes h_2^0 \otimes \cdots \otimes h_2^{i+1} \in H \otimes C_i$$

for all $h^0, \dots, h^{i+1} \in H$. For $h \in H$, the left (resp., right) h -action on an element $x \in C_i$ is given by left (resp., right) multiplication by h in the leftmost (resp., rightmost) factor of x . Now let

$$C_\bullet : \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow H \rightarrow 0$$

be the bar resolution $\mathbf{B}_\bullet(H)$ of H [Definition 1.7], which is an H^e -free resolution of H .

There is an isomorphism of free H^e -modules $C_i \cong H \otimes C'_i \otimes H$ where $C'_i = H^{\otimes i}$ if $i \geq 1$ and $C'_0 = k$. We give each C'_i the H -comodule structure inducing that on C_i under the usual tensor product of comodules.

Remark 2.2. The resolution C_\bullet satisfies the following conditions.

(i) The right H -action and left H -coaction on C_i commute in the sense that for all $x \in C_i$ and $h \in H$,

$$\sum (x \cdot h)_{-1} \otimes (x \cdot h)_0 = \sum x_{-1} h_1 \otimes (x_0 \cdot h_2).$$

That is, each C_i is a Hopf module (for which the action is a left action and the coaction is a right coaction).

(ii) The differentials are left H -comodule homomorphisms.

Definition 2.3. [D_\bullet, D_i, D'_i] Recall that B is a Koszul algebra. Let

$$\cdots \rightarrow D_1 \rightarrow D_0 \rightarrow B \rightarrow 0$$

be the *Koszul resolution* of B as a B^e -module: $D_0 = B \otimes B$, $D_1 = B \otimes V \otimes B$, $D_2 = B \otimes I \otimes B$, and for each $n \geq 3$, $D_i = B \otimes D'_i \otimes B$ where

$$D'_i = \bigcap_{j=0}^{i-2} (V^{\otimes j} \otimes I \otimes V^{\otimes(i-2-j)}).$$

Each D_i is a subspace of $B^{\otimes(i+2)}$, and the differential on the Koszul resolution is the one induced by the canonical embedding of the Koszul resolution into the bar resolution of B .

Remark 2.4. The resolution D_\bullet satisfies the following conditions.

- (i) Each B^e -module D_i is a left H -module and the differentials are H -module homomorphisms.
- (ii) The left actions of B and of H on D_i are compatible in the sense that they induce a left action of $A = B\#H$ on D_i .
- (iii) In addition, the right B -action on D_i is compatible with the left H -action on D_i in the sense that for all $h \in H$, $y = b^0 \otimes y' \otimes b^1 \in D_i$, where $y' \in D'_i$, b^0, b^1 , and $b \in B$,

$$\begin{aligned} h \cdot (y \cdot b) &= \sum (h_1 \cdot y) \cdot (h_2 \cdot b) = \left[\sum (h_1 \cdot b^0) \otimes (h_2 \cdot y') \otimes (h_3 \cdot b^1) \right] \cdot (h_4 \cdot b) \\ &= \sum (h_1 \cdot b^0) \otimes (h_2 \cdot y') \otimes (h_3 \cdot b^1) b = (h \cdot y) \cdot b. \end{aligned}$$

- (iv) Each D_i is considered to be a left H -comodule in a trivial way by requiring that it be H -coinvariant, that is, the comodule structure is given by maps $\rho_i : D_i \rightarrow H \otimes D_i$, where $\rho_i(y) = 1 \otimes y$ for all $y \in D_i$. The maps ρ_i are maps of left H -modules, if we give $H \otimes D_i$ the tensor product H -module structure, where the factor H has the adjoint H -module structure. See Section 1.1.

Construction 2.5. [X_\bullet] We wish to combine the two resolutions, C_\bullet and D_\bullet from Definitions 2.1 and 2.3, to form a resolution X_\bullet of $A = B\#H$ by A -bimodules, via a tensor product. To that end, we first apply $(A \otimes_H -)$ to C_\bullet . Note that A is free as a right H -module (under multiplication), and that $A \otimes_H H \cong A$. The following sequence of $A \otimes H^{op}$ -modules is therefore exact:

$$\cdots \rightarrow A \otimes_H C_1 \rightarrow A \otimes_H C_0 \rightarrow A \rightarrow 0.$$

Similarly, we apply $(- \otimes_B A)$ to D_\bullet . Note that A is free as a left B -module, and that $B \otimes_B A \cong A$. The following sequence of $B \otimes A^{op}$ -modules is therefore exact:

$$\cdots \rightarrow D_1 \otimes_B A \rightarrow D_0 \otimes_B A \rightarrow A \rightarrow 0.$$

We will next extend the actions on the modules in each of these two sequences so that they become A^e -modules. Then, we will take their tensor product over A .

We extend the right H -module structure on $A \otimes_H C_i$ to a right A -module structure by defining a right-action of B on $A \otimes_H C_i$ as follows. For all $a \in A$, $x \in C_i$, $b \in B$, we set

$$(2.6) \quad (a \otimes_H x) \cdot b := \sum a(x_{-1} \cdot b) \otimes_H x_0.$$

This does indeed make $A \otimes_H C_i$ into a right B -module, and by combining with the right action of H , gives a right action of A on $A \otimes_H C_i$. Note that for $x = x^0 \otimes \cdots \otimes x^{i+1} \in C_i$ (with $x^0, \dots, x^{i+1} \in H$),

$$\rho(hx) = \sum (hx)_{-1} \otimes (hx)_0 = \sum h_1 x_1^0 \cdots x_1^{i+1} \otimes h_2 x_2^0 \otimes \cdots \otimes x_2^{i+1} = \sum h_1 x_{-1} \otimes h_2 x_0.$$

The action is well-defined: If $h \in H$, then

$$(ah \otimes_H x) \cdot b \stackrel{(2.6)}{=} \sum ah(x_{-1} \cdot b) \otimes_H x_0 = \sum a(h_1 x_{-1} \cdot b) \otimes_H h_2 x_0 \stackrel{(2.6)}{=} (a \otimes_H hx) \cdot b.$$

Since the differentials on C_\bullet are H -comodule homomorphisms [Remark 2.2(ii)], this action commutes with the differentials.

We extend the left B -module structure on $D_i \otimes_B A$ to a left A -module structure by defining a left action of H by

$$(2.7) \quad h \cdot (y \otimes_B a) := \sum (h_1 \cdot y) \otimes_B h_2 a$$

for all $h \in H$, $y \in D_i$, $a \in A$. It is well-defined since for all $h \in H$, $b \in B$, we have by the definitions in Section 1.1 that:

$$\begin{aligned} h \cdot (yb \otimes_B a) &\stackrel{(2.7)}{=} \sum (h_1 \cdot (yb)) \otimes_B (h_2 a S(h_3)) = \sum (h_1 \cdot y)(h_2 \cdot b) \otimes_B h_3 a S(h_4) \\ &= \sum (h_1 \cdot y) \otimes_B (h_2 \cdot b) h_3 a S(h_4) = \sum (h_1 \cdot y) \otimes_B (h_2 \cdot (ba)) \stackrel{(2.7)}{=} h \cdot (y \otimes_B ba). \end{aligned}$$

The left H -action on D_i is compatible with the right B -action on D_i by Remark 2.4(iii). Again this action commutes with the differentials since the differentials on D_\bullet are H -module homomorphisms [Remark 2.4(i)].

We may now consider $A \otimes_H C_\bullet$ and $D_\bullet \otimes_B A$ to be complexes of A^e -modules via the A -bimodule structure defined above. We take their tensor product over A , letting $X_{\bullet,\bullet} := (A \otimes_H C_\bullet) \otimes_A (D_\bullet \otimes_B A)$, that is, for all $i, j \geq 0$,

$$(2.8) \quad X_{i,j} := (A \otimes_H C_i) \otimes_A (D_j \otimes_B A),$$

with horizontal and vertical differentials

$$d_{i,j}^h : X_{i,j} \rightarrow X_{i-1,j} \quad \text{and} \quad d_{i,j}^v : X_{i,j} \rightarrow X_{i,j-1}$$

given by $d_{i,j}^h := d_i^{C_\bullet} \otimes \text{id}$ and $d_{i,j}^v := (-1)^i \text{id} \otimes d_j^{D_\bullet}$.

Finally, let X_\bullet be the total complex of $X_{\bullet,\bullet}$:

$$(2.9) \quad \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0,$$

with $X_n = \bigoplus_{i+j=n} X_{i,j}$, where $X_{i,j}$ is defined in (2.8).

Theorem 2.10. *We have the following statements.*

- (a) *For each i, j , the A^e -module $X_{i,j}$ is isomorphic to $A \otimes C'_i \otimes D'_j \otimes A$.*
- (b) *The complex X_\bullet given in (2.9) is a free resolution of the A^e -module A .*

Proof. (a) Write $C_i \cong H \otimes C'_i \otimes H$ and $D_j \cong B \otimes D'_j \otimes B$ for vector spaces C'_i and D'_j , as in Definitions 2.1 and 2.3. Then

$$\begin{aligned} X_{i,j} &\cong (A \otimes_H H \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes B \otimes_B A) \\ &\cong (A \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes A). \end{aligned}$$

We will show that this is isomorphic to $A \otimes C'_i \otimes D'_j \otimes A$ as an A^e -module. First define a map as follows:

$$(2.11) \quad \begin{aligned} (A \otimes C'_i \otimes H) \times (B \otimes D'_j \otimes A) &\rightarrow A \otimes C'_i \otimes D'_j \otimes A \\ (a \otimes x \otimes h, b \otimes y \otimes a') &\mapsto \sum a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes (h_2 \cdot y) \otimes h_3a' \end{aligned}$$

for all $a, a' \in A$, $x \in C'_i$, $y \in D'_j$, $h \in H$, $b \in B$. This map is k -bilinear by its definition, and we will check that it is A -balanced. First let $b' \in B$. We rewrite $(a \otimes x \otimes h) \cdot b'$ as follows. First, using $A \otimes C'_i \otimes H \cong A \otimes_H C_i$, identify this element with $a \otimes_H (1 \otimes x \otimes h) \in A \otimes_H C_i$. By (2.6),

$$(a \otimes_H (1 \otimes x \otimes h)) \cdot b' = \sum a((1 \otimes x \otimes h)_{-1} \cdot b') \otimes_H (1 \otimes x \otimes h)_0.$$

By Definition 2.1, and by identifying $x \in C'_i$ with $x^1 \otimes x^2 \otimes \cdots \otimes x^i$, we have that

$$\begin{aligned} \rho(1 \otimes x \otimes h) &= \sum (1 \otimes x \otimes h)_{-1} \otimes (1 \otimes x \otimes h)_0 \\ &= \sum (x_1^1 x_1^2 \cdots x_1^i h_1) \otimes (1 \otimes x_2^1 \otimes x_2^2 \otimes \cdots \otimes x_2^i \otimes h_2). \end{aligned}$$

So, $(1 \otimes x \otimes h)_{-1} = x_{-1}h_1$ and $(1 \otimes x \otimes h)_0 = x_0 \otimes h_2$. Now $C_i \cong H \otimes C'_i \otimes H$ as an H -comodule, so

$$\begin{aligned} ((a \otimes x \otimes h) \cdot b', b \otimes y \otimes a') &= \sum (a(x_{-1}h_1 \cdot b') \otimes x_0 \otimes h_2, b \otimes y \otimes a') \\ &\mapsto \sum a(x_{-2}h_1 \cdot b')(x_{-1}h_2 \cdot b) \otimes x_0 \otimes (h_3 \cdot y) \otimes h_4a'. \end{aligned}$$

On the other hand,

$$\begin{aligned} ((a \otimes x \otimes h, b' \cdot (b \otimes y \otimes a'))) &= (a \otimes x \otimes h, b'b \otimes y \otimes a') \\ &\mapsto \sum a(x_{-1}h_1 \cdot (b'b)) \otimes x_0 \otimes (h_2 \cdot y) \otimes h_3a', \end{aligned}$$

which is the same as the previous image since B is an H -module algebra. Now let $\ell \in H$. Then

$$\begin{aligned} ((a \otimes x \otimes h) \cdot \ell, b \otimes y \otimes a') &= (a \otimes x \otimes h\ell, b \otimes y \otimes a') \\ &\mapsto \sum a(x_{-1}h_1\ell_1 \cdot b) \otimes x_0 \otimes (h_2\ell_2 \cdot y) \otimes h_3\ell_3a'. \end{aligned}$$

On the other hand,

$$\begin{aligned} (a \otimes x \otimes h, \ell \cdot (b \otimes y \otimes a')) &= \sum (a \otimes x \otimes h, (\ell_1 \cdot b) \otimes (\ell_2 \cdot y) \otimes \ell_3 a') \\ &\mapsto \sum a(x_{-1}h_1\ell_1 \cdot b) \otimes x_0 \otimes (h_2\ell_2 \cdot y) \otimes h_3\ell_3a', \end{aligned}$$

which is the same as the previous image. Therefore, there is an induced map

$$(A \otimes C'_i \otimes H) \otimes_A (B \otimes D'_j \otimes A) \rightarrow A \otimes C'_i \otimes D'_j \otimes A.$$

Now, we verify that the map below is an inverse map of (2.11):

$$(2.12) \quad a \otimes x \otimes y \otimes a' \mapsto (a \otimes x \otimes 1) \otimes_A (1 \otimes y \otimes a').$$

It is clear that first applying (2.12) then (2.11) yields the identity map on $A \otimes C'_i \otimes D'_j \otimes A$. On the other hand, the image of first applying (2.11), then (2.12) to $(a \otimes x \otimes h, b \otimes y \otimes a')$ is:

$$\begin{aligned} &\sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (1 \otimes (h_2 \cdot y) \otimes h_3a') \\ &= \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (\epsilon(h_2) \otimes (h_3 \cdot y) \otimes h_4a') \\ &= \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes 1) \otimes_A (h_2 \cdot (1 \otimes y \otimes a')) \\ &= \sum (a(x_{-1}h_1 \cdot b) \otimes x_0 \otimes h_2) \otimes_A (1 \otimes y \otimes a') \\ &\stackrel{(2.6)}{=} ((a \otimes x \otimes h) \cdot b) \otimes_A (1 \otimes y \otimes a') \\ &= (a \otimes x \otimes h) \otimes_A (b \otimes y \otimes a'). \end{aligned}$$

Therefore the two A^e -modules, X_{ij} and $A \otimes C'_i \otimes D'_j \otimes A$, are isomorphic as claimed.

(b) We wish to apply the Künneth Theorem to show that the complex X_\bullet is a free resolution of the A^e -module A . To that end, we check that each term in the complex $D_\bullet \otimes_B A$ is a free left A -module, and that the image of each differential in the complex is also projective as a left A -module. First write each $D_i \otimes_B A \cong (B \otimes D'_i \otimes B) \otimes_B A \cong B \otimes D'_i \otimes A$. Define a k -linear map $f : A \otimes D'_i \otimes B \rightarrow B \otimes D'_i \otimes A$ by

$$f(rh \otimes y \otimes b) = \sum r \otimes (h_1 \cdot y) \otimes h_2b,$$

for $h \in H$, $y \in D'_i$, and $r, b \in B$. Give $A \otimes D'_i \otimes B$ the structure of a left A -module by requiring A to act by left multiplication on the leftmost factor. Clearly this is a free left A -module. The map f is an A -module homomorphism by the definition of the left A -action on $B \otimes D'_i \otimes A$; see (2.7). We claim that the following map is an inverse map, so that f is an isomorphism of A -modules: Let S^{-1} denote the (composition) inverse of the antipode S of H . Let $g : B \otimes D'_i \otimes A \rightarrow A \otimes D'_i \otimes B$ be the k -linear map defined by

$$g(r \otimes y \otimes hb) = \sum rh_2 \otimes (S^{-1}(h_1) \cdot y) \otimes b.$$

Since for each $h \in H$, we have $\sum h_2S^{-1}(h_1) = \epsilon(h) = \sum S^{-1}(h_2)h_1$ (see e.g. [28, Proposition 7.1.10]), the function g is indeed the inverse of f . Thus, each term in the complex $D_\bullet \otimes_B A$ is a free left A -module.

That the image of each differential is projective as a left A -module may be proved inductively, starting on one end of the complex

$$\cdots \longrightarrow D_1 \otimes_B A \xrightarrow{d_1 \otimes \text{id}} D_0 \otimes_B A \xrightarrow{d_0 \otimes \text{id}} A \longrightarrow 0,$$

as follows. Since A is a projective left A -module and $d_0 \otimes \text{id}$ is surjective, the map splits, implying that $\text{Ker}(d_0 \otimes \text{id}) = \text{Im}(d_1 \otimes \text{id})$ is a direct summand of the free left A -module $D_0 \otimes_B A$. Therefore it is projective. Again, since $\text{Im}(d_1 \otimes \text{id})$ is projective, the map $d_1 \otimes \text{id}$ from $D_1 \otimes_B A$ to its image splits so that $\text{Ker}(d_1 \otimes \text{id}) = \text{Im}(d_2 \otimes \text{id})$ is a direct summand of the free left A -module $D_1 \otimes_B A$. Continuing in this way, we see that $\text{Im}(d_i \otimes \text{id})$ is a free left A -module for each i .

The Künneth Theorem [35, Theorem 3.6.3] then gives for each n an exact sequence:

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i+j=n} \text{H}_i(A \otimes_H C_\bullet) \otimes_A \text{H}_j(D_\bullet \otimes_B A) &\longrightarrow \text{H}_n((A \otimes_H C_\bullet) \otimes_A (D_\bullet \otimes_B A)) \\ &\longrightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^A(\text{H}_i(A \otimes_H C_\bullet), \text{H}_j(D_\bullet \otimes_B A)) \longrightarrow 0. \end{aligned}$$

Now $A \otimes_H C_\bullet$ and $D_\bullet \otimes_B A$ are exact other than in degree 0, where their homologies are each A : That is, $H_0(A \otimes_H C_\bullet) = A$ and $H_0(D_\bullet \otimes_B A) = A$. Therefore the only potentially nonzero Tor term is when $i = 0 = j$, or $\text{Tor}_1^A(A, A)$, yet this equals 0 since A is flat over A . So for each n we have

$$H_n((A \otimes_H C_\bullet) \otimes_A (D_\bullet \otimes_B A)) \cong \bigoplus_{i+j=n} H_j(A \otimes_H C_\bullet) \otimes_A H_i(D_\bullet \otimes_B A).$$

Again the right side is only nonzero when $i = 0 = j$, so that we have

$$H_0((A \otimes_H C_\bullet) \otimes_A (D_\bullet \otimes_B A)) \cong H_0(A \otimes_H C_\bullet) \otimes_A H_0(D_\bullet \otimes_B A) \cong A \otimes_A A \cong A,$$

and $H_n((A \otimes_H C_\bullet) \otimes_A (D_\bullet \otimes_B A)) = 0$ for all $n > 0$. Thus we have proven that X_\bullet is an A^e -free resolution of A . \square

We next relate the resolution X_\bullet of A (from Construction 2.5) to the bar resolution $\mathbf{B}_\bullet(A)$ of A .

Lemma 2.13. *There exist degree-preserving chain maps between X_\bullet and the bar resolution $\mathbf{B}_\bullet(A)$ of A :*

$$\phi_\bullet : X_\bullet \longrightarrow \mathbf{B}_\bullet(A) \quad \text{and} \quad \psi_\bullet : \mathbf{B}_\bullet(A) \longrightarrow X_\bullet,$$

such that $\psi_n \phi_n$ is the identity map on the A^e -submodule $X_{0,n}$ of X_n , for each $n \geq 0$.

Proof. Recall by Notation 0.2 that B is generated by the vector space V , with quadratic relations $I \subseteq V \otimes V$. First we prove by induction on n that there are degree-preserving maps $\phi_n : X_n \rightarrow A^{\otimes(n+2)}$ and $\psi_n : A^{\otimes(n+2)} \rightarrow X_n$ commuting with the differentials. For clarity, we denote the differential on the bar resolution of A by δ . We have the following diagram:

$$\begin{array}{ccccccccccc} X_\bullet : & \cdots & \longrightarrow & X_3 & \xrightarrow{d_3} & X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & X_0 & \xrightarrow{d_0} & A & \longrightarrow & 0 \\ & & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow & & & & \\ & & & \psi_3 \phi_3 & & \psi_2 \phi_2 & & \psi_1 \phi_1 & & \psi_0 \phi_0 & & & & \\ \mathbf{B}_\bullet(A) : & \cdots & \longrightarrow & A^{\otimes 5} & \xrightarrow{\delta_3} & A^{\otimes 4} & \xrightarrow{\delta_2} & A^{\otimes 3} & \xrightarrow{\delta_1} & A \otimes A & \xrightarrow{\delta_0} & A & \longrightarrow & 0, \end{array}$$

where $\mathbf{B}_n(A) = A^{\otimes(n+2)}$ and $X_n = \bigoplus_{i+j=n} X_{i,j}$, with $X_{i,j}$ defined in (2.8); see also Theorem 2.10(a).

Define $\phi_0 = \text{id} \otimes \text{id} = \psi_0$, the identity map from $A \otimes A$ to itself. We wish to define ϕ_\bullet so that when restricted to $X_{0,\bullet}$, it corresponds to the standard embedding of the Koszul complex into the bar complex: For $n = 1$, this is the embedding of $A \otimes V \otimes A$ into $A \otimes A \otimes A$ via the containment of V in A . We may define ϕ_1 on $X_1 = X_{0,1} \oplus X_{1,0} \cong (A \otimes V \otimes A) \oplus (A \otimes H \otimes A)$ by $\phi_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$ and $\phi_1(1 \otimes h \otimes 1) = 1 \otimes h \otimes 1$ for all $v \in V$, $h \in H$. Note that for $n \geq 2$,

$$(2.14) \quad X_{0,n} \cong A \otimes \left(\bigcap_{i=0}^{n-2} V^{\otimes i} \otimes I \otimes V^{\otimes(n-i-2)} \right) \otimes A,$$

which is a free A^e -submodule of $A^{\otimes(n+2)}$. For each i, j with $i + j = n$, choose a basis of the vector space $C'_i \otimes D'_j$ (whose elements are homogeneous of degree j , as H is declared to have degree 0). By hypothesis, ϕ_{n-1} is degree-preserving, and d_n is degree-preserving by construction. So, applying $\phi_{n-1} d_n$ to these basis elements of $C'_i \otimes D'_j$ produces elements of degree j in the kernel of δ_{n-1} , that is, the image of δ_n . We define ϕ_n by choosing (arbitrary) corresponding elements in the inverse image of $\text{Im}(\delta_n)$. If we start with an element in $X_{0,n}$, we may choose its image in $A^{\otimes(n+2)}$ under the canonical embedding of $X_{0,n}$ into $A^{\otimes(n+2)}$ (see (2.14)). Given $X_{i,j}$ and $X_{i',j'}$ with $i + j = i' + j' = n$ and $i \neq 0$, $i' \neq 0$, elements of $X_{i,j}$ have degree j and elements of $X_{i',j'}$ have degree j' . So their images under ϕ_n may be chosen independently, and in particular, independently of those of $X_{0,n}$. Thus, we have the maps ϕ_n as desired.

Now we show that ψ_n may be chosen so that $\psi_n \phi_n$ is the identity map on $X_{0,n}$. In degree 1, we have summands $X_{0,1} \cong A \otimes V \otimes A$ and $X_{1,0} \cong A \otimes H \otimes A$. Note that $V \oplus H$ is a direct summand of A as a vector space. We may therefore define $\psi_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$ in $X_{0,1}$ for all $v \in V$, and $\psi_1(1 \otimes h \otimes 1) = 1 \otimes h \otimes 1$ in $X_{1,0}$ for all $h \in H$. We also have that ψ_1 is the identity map on elements of the form $1 \otimes z \otimes 1$, for z ranging over a basis of a chosen complement of $V \oplus H$ as a vector subspace of A . This complement may be chosen arbitrarily subject to the condition that $d_1 \psi_1(1 \otimes z \otimes 1) = \psi_0 \delta_1(1 \otimes z \otimes 1)$. Since ψ_0, d_1, δ_1 all have

degree 0 as maps, one may also choose ψ_1 to have degree 0. In particular, note that $\psi_1\phi_1$ is the identity map on $X_{0,1}$. Now let $n \geq 2$ and assume that ψ_{n-2} and ψ_{n-1} have been defined to be degree 0 maps for which $d_{n-1}\psi_{n-1} = \psi_{n-2}\delta_{n-1}$ and $\psi_{n-1}\phi_{n-1}$ is the identity map on $X_{0,n-1}$. To define ψ_n , first note that $A^{\otimes(n+2)}$ contains the space $X_{0,n}$ as an A^e -submodule (see (2.14)) and the image of each $X_{i,j}$ under ϕ_n ($n = i + j$, $i \geq 1$). By construction, their images intersect in 0, the image of $X_{0,n}$ under ϕ_n is free, and moreover ϕ_n is injective on restriction to $X_{0,n}$. Choose a set of free generators of $\phi_n(X_{0,n})$, and choose a set of free generators of its complement in $A^{\otimes(n+2)}$. For each chosen generator x of $X_{0,n}$, we define $\psi_n(\phi_n(x))$ to be x . On the complement of $\phi_n(X_{0,n})$, define ψ_n arbitrarily subject to being a chain map of degree 0. Thus, $\psi_n\phi_n$ is the identity map on $X_{0,n}$. Now for all $x \in X_{0,n}$, since $d_n(x) \in X_{0,n-1}$, we have that $\psi_{n-1}\phi_{n-1}d_n(x) = d_n(x)$, by induction. As $\delta_n\phi_n(x) = \phi_{n-1}d_n(x)$, it follows that $d_n\psi_n\phi_n(x) = \psi_{n-1}\delta_n\phi_n(x)$. So ψ_n also extends the chain map from degree $n-1$ to degree n , as desired. \square

3. POINCARÉ-BIRKHOFF-WITT THEOREM FOR HOPF ALGEBRA ACTIONS

Consider the algebra $\mathcal{D}_{B,\kappa}$ from Notation 0.3. The goal of this section is to prove our main result, Theorem 0.4. We provide necessary and sufficient conditions for $\mathcal{D}_{B,\kappa}$ to be a PBW deformation of $B\#H$ [Definition 0.1] as follows.

Theorem 3.1. *The algebra $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$ if and only if the following conditions hold:*

(a) κ is H -invariant [Definition 1.4];

If $\dim_k V \geq 3$, then the following equations hold for the maps $\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C$ and $\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L$, which are defined on the intersection $(I \otimes V) \cap (V \otimes I)$:

(b) $\text{Im}(\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) \subseteq I$;

(c) $\kappa^L \circ (\kappa^L \otimes \text{id} - \text{id} \otimes \kappa^L) = -(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)$;

(d) $\kappa^C \circ (\text{id} \otimes \kappa^L - \kappa^L \otimes \text{id}) \equiv 0$.

Recall Notation 0.2: B is generated by the k -vector space V with quadratic relations $I \subset V \otimes V$, so that $B = T_k(V)/(I)$. Moreover, consider:

Notation 3.2. $[U, T_H(U), R, P]$ Let $U := V \otimes H$, which is an H -bimodule under the actions defined in Section 1.1. Set $R = I \otimes H$, similarly an H -bimodule, and an H -subbimodule of $U \otimes_H U$. Let $P = \{r \otimes 1 - \kappa(r) \mid r \in I\}$ be the relation space of $\mathcal{D}_{B,\kappa}$, generating an H -submodule of $H \oplus U \oplus (U \otimes_H U)$ in the tensor algebra:

$$T_H(U) = H \oplus U \oplus (U \otimes_H U) \oplus (U \otimes_H U \otimes_H U) \oplus \cdots.$$

Note that $U^{\otimes n} \cong V^{\otimes n} \otimes H$ as k -vector spaces. We see that $\pi(P) = R$, where the map π is the projection onto the homogeneous quadratic part of P .

Consider the following preliminary results.

Lemma 3.3. *Since $T_H(U)$ is canonically isomorphic to $T_k(V)\#H$, we have that*

$$T_H(U)/(P) \cong \mathcal{D}_{B,\kappa} \quad \text{and} \quad T_H(U)/(R) \cong (T_k(V)\#H)/(I) \cong B\#H,$$

where (I) is identified as an ideal of $T_k(V)\#H$, generated by I .

Hence, $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$ if and only if $T_H(U)/(P)$ is a PBW deformation of $T_H(U)/(R)$. \square

Lemma 3.4. [30, Lemma 5.2] *If $T_H(U)/(P)$ is a PBW deformation of $T_H(U)/(R)$, then the following conditions hold for maps $\alpha : R \rightarrow U$ and $\beta : R \rightarrow H$ for which $P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$:*

(i) $\text{Im}(\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha) \subseteq R$;

(ii) $\alpha \circ (\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha) = -(\beta \otimes_H \text{id} - \text{id} \otimes_H \beta)$;

(iii) $\beta \circ (\text{id} \otimes_H \alpha - \alpha \otimes_H \text{id}) \equiv 0$.

Here, the maps $\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha$ and $\beta \otimes_H \text{id} - \text{id} \otimes_H \beta$ are defined on the subspace $(R \otimes_H U) \cap (U \otimes_H R)$ of $T_H(U)$. \square

Remark 3.5. Given the maps $\kappa^L : I \rightarrow V \otimes H$ and $\kappa^C : I \rightarrow H$ as in Notation 0.2, we see that $\alpha = \kappa^L \otimes \text{id}_H$ and $\beta = \kappa^C \otimes \text{id}_H$.

Lemma 3.6. *Consider the algebra*

$$(T_H(U)/(P))_t := \frac{T_H(U)[t]}{(x - \alpha(x)t - \beta(x)t^2)_{x \in R}}.$$

We have that $(T_H(U)/(P))_t$ is a PBW deformation of $T_H(U)/(R)$ over $k[t]$ if and only if $\mathcal{D}_{B,\kappa,t}$ (of Proposition 1.15) is a PBW deformation of $B\#H$ over $k[t]$. \square

Proof. This follows from Lemma 3.3 and Remark 3.5. \square

Now we provide the proof of Theorem 3.1. A somewhat shorter proof would suffice in case H is semisimple: The first proof of [31, Theorem 3.1] may be generalized from semisimple group algebras to semisimple Hopf algebras. In that context, one has on hand a much smaller resolution than that which we will use below.

Proof of Theorem 3.1. Note that we will employ the identifications given in the lemmas and remark above, sometimes without comment. Namely, results from Section 2 will be used here where, for instance, I is identified so that $R = I \otimes H$, and $B\#H$ is identified with $T_H(U)/(R)$.

Necessity of conditions (a)–(d). Let us first show that conditions (a)–(d) are necessary. Assume that $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$ and take Q to be the relation space of $\mathcal{D}_{B,\kappa}$. Then, for all $h \in H$ and $r \in I$, we have that $h \cdot r - h \cdot (\kappa(r)) \in Q$. (Refer to Section 1.1 for the definition of these actions.) We also have that $h \cdot r - \kappa(h \cdot r) \in Q$, so $h \cdot (\kappa(r)) - \kappa(h \cdot r) \in Q$. This implies that $h \cdot (\kappa(r)) = \kappa(h \cdot r)$ in $\mathcal{D}_{B,\kappa}$ since Q cannot contain nonzero elements in degree less than two. Thus, condition (a) holds. Moreover, by Lemma 3.3, $T_H(U)/(P)$ satisfies the PBW property.

Now by applying Lemma 3.4, we see that conditions (i),(ii),(iii) hold for $T_H(U)/(P)$. These conditions are equivalent to conditions (b),(c),(d) in Theorem 3.1 for the algebra $\mathcal{D}_{B,\kappa}$ by Notation 3.2 and Remark 3.5. Thus, if $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$, then conditions (a)–(d) of Theorem 3.1 must hold.

Sufficiency of conditions (a)–(d). Conversely, let us assume that conditions (a)–(d) of Theorem 3.1 hold for the algebra $\mathcal{D}_{B,\kappa}$. Equivalently by Notation 3.2, Lemma 3.3, and Remark 3.5, we assume the following statements for the algebra $T_H(U)/(P)$:

- * the maps α and β are H -invariant; and
- * conditions (i),(ii),(iii) of Lemma 3.4 hold.

The goal is to show that $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H$ which, by Proposition 1.15, is equivalent to showing that $\mathcal{D}_{B,\kappa,t}$ (of Proposition 1.15) is a graded deformation of $B\#H$ over $k[t]$. Hence, by Lemma 3.6, the goal is then equivalent to verifying that the algebra $(T_H(U)/(P))_t$ is a graded deformation of $T_H(U)/(R)$ over $k[t]$. We thus have the following strategy:

- Let A denote $T_H(U)/(R)$.
- Construct multiplication maps, $\mu_i : A \otimes A \rightarrow A$, as in Definition 1.5, subject to the restraints listed in Remark 1.8.
- Form the graded deformation A_t of A as in Definition 1.5.
- Conclude that $A' := T_H(U)/(P) \cong (A_t)|_{t=1}$ is a PBW deformation of A by Proposition 1.15.

We generalize the proof in [30] from group actions to Hopf algebra actions. Namely, we use the free resolution X_\bullet of the A^e -module A in Construction 2.5 to define the maps μ_i . Recall that X_\bullet is constructed from $C_\bullet = \mathbf{B}_\bullet(A)$, the bar resolution of H , and from D_\bullet , the Koszul resolution of B .

Extending α and β to be maps on X_\bullet .

We first extend α and β to be maps on X_\bullet , as follows. In degree 2, X_2 contains as a direct summand $X_{0,2} \cong A \otimes I \otimes A$; see (2.14). As α, β are H -bilinear by H -invariance, we may extend them to A^e -module maps from $A \otimes R \otimes_H A \cong A \otimes I \otimes A$ to A by composing with the multiplication map. By *abuse of notation*, denote these extended maps by α, β as well. Extend α and β yet further by setting them equal to 0 on the summands $X_{2,0}$ and $X_{1,1}$ of X_2 so that they become maps $\alpha, \beta : X_2 \rightarrow A$. More precisely, $\alpha, \beta \in \text{Hom}_{A^e}(X_2, A) \cong \text{Hom}_k(X'_2, A)$ for $X_2 \cong A \otimes X'_2 \otimes A$.

Construction of the multiplication map μ_1 .

To build $\mu_1 \in \text{Hom}_k(A \otimes A, A) \cong \text{Hom}_{A^e}(A^{\otimes 4}, A)$, recall that it must be a Hochschild 2-cocycle as in (1.9). We will show that $\alpha : X_2 \rightarrow A$ is a Hochschild 2-cocycle on X_\bullet , that is, $d_3^*(\alpha) = 0$. Recall the chain maps of Lemma 2.13. We set $\mu_1 = \psi_2^*(\alpha)$, which will be a Hochschild 2-cocycle on $B_\bullet(A)$, that is, $\delta_3^*(\mu_1) = 0$.

To show that $d_3^*(\alpha) : X_3 \rightarrow A$ is the zero map, first note that $X_3 = X_{0,3} \oplus X_{1,2} \oplus X_{2,1} \oplus X_{3,0}$ from (2.9) and that the images of $X_{2,1}$ and $X_{3,0}$ under d_3 lie in $X_{1,1} \oplus X_{2,0}$. Since $\alpha|_{X_{1,1} \oplus X_{2,0}} \equiv 0$ by the extension above, it suffices to show that $d_3^*(\alpha)|_{X_{0,3}}$ and $d_3^*(\alpha)|_{X_{1,2}}$ are zero maps.

Rewriting condition (i) of Lemma 3.4, we see that it implies that α is 0 on the image of the differential on $X_{0,3}$ as follows. Let $\sum_i 1 \otimes x_i \otimes y_i \otimes z_i \otimes 1 \in A \otimes ((I \otimes V) \cap (V \otimes I)) \otimes A = X_{0,3}$; see (2.14). Then

$$\begin{aligned} & \alpha \left(d_3 \left(\sum_i 1 \otimes x_i \otimes y_i \otimes z_i \otimes 1 \right) \right) \\ &= \alpha \left(\sum_i x_i \otimes y_i \otimes z_i \otimes 1 - \sum_i 1 \otimes x_i y_i \otimes z_i \otimes 1 + \sum_i 1 \otimes x_i \otimes y_i z_i \otimes 1 - \sum_i 1 \otimes x_i \otimes y_i \otimes z_i \right) \\ &= \sum_i (x_i \alpha(y_i \otimes z_i) - 0 + 0 - \alpha(x_i \otimes y_i) z_i). \end{aligned}$$

(To see this, note that applying the multiplication map of A to elements in I yields 0.) Thus $d_3^*(\alpha) = \text{id} \otimes \alpha - \alpha \otimes \text{id}$ on $X_{0,3}$; here, we identify $\text{id} \otimes \alpha - \alpha \otimes \text{id}$ with $m \circ (\text{id} \otimes \alpha - \alpha \otimes \text{id})$ where m is the multiplication map on A . We see that condition (i) indeed implies (in fact, is equivalent to) $d_3^*(\alpha)|_{X_{0,3}} \equiv 0$.

Next, we claim that α being H -invariant implies that α is also 0 on the image of the differential on $X_{1,2}$. Let $a, b \in A$, $h \in H$, and $r \in I$, and consider $a \otimes h \otimes r \otimes b$ as an element of $X_{1,2} \cong A \otimes H \otimes I \otimes A$ by Theorem 2.10(a). By the definition of the differential on $X_{1,2}$,

$$\begin{aligned} d(a \otimes h \otimes r \otimes b) &= d((a \otimes h \otimes 1) \otimes (1 \otimes r \otimes b)) \\ &= d(a \otimes h \otimes 1) \otimes (1 \otimes r \otimes b) - (a \otimes h \otimes 1) \otimes d(1 \otimes r \otimes b). \end{aligned}$$

The second term lies in $X_{1,1}$, but α is 0 on $X_{1,1}$ by definition. Therefore

$$\begin{aligned} (3.7) \quad \alpha(d(a \otimes h \otimes r \otimes b)) &= \alpha((ah \otimes 1 - a \otimes h) \otimes (1 \otimes r \otimes b)) \\ &= \alpha(ah \otimes r \otimes b - \sum a \otimes (h_1 \cdot r) \otimes h_2 b) \\ &= ah\alpha(r)b - \sum a\alpha(h_1 \cdot r)h_2 b. \end{aligned}$$

Since α is H -invariant (*), we have that

$$h\alpha(r) = \sum h_1 \epsilon(h_2) \alpha(r) = \sum h_1 \alpha(r) \epsilon(h_2) = \sum h_1 \alpha(r) S(h_2) h_3 \stackrel{(*)}{=} \sum \alpha(h_1 \cdot r) h_2.$$

Thus, α is zero on the image of $d = d_3$ on $X_{1,2}$ by (3.7). It follows that α is a Hochschild 2-cocycle on X_\bullet .

Now, let $\mu_1 = \psi_2^*(\alpha)$, where ψ_\bullet is a chain map satisfying the conditions of Lemma 2.13. We conclude that

$$\delta_3^*(\mu_1) = \delta_3^*(\psi_2^*(\alpha)) = \psi_3^*(d_3^*(\alpha)) \equiv 0,$$

as desired. So, we have a first level graded deformation $A_{(1)}$ of A with first multiplication map $\mu_1 : A \otimes A \rightarrow A$.

As an aside, we also get that $\phi_2^*(\mu_1) = \alpha$ as cochains. To see this, first note that since α is homogeneous of degree -1 by its definition, so is μ_1 . Let $x \in X_{0,2}$. By Lemma 2.13, $\psi_2 \phi_2(x) = x$, and thus

$$\mu_1 \phi_2(x) = \alpha \psi_2 \phi_2(x) = \alpha(x).$$

Now let y be a free generator of $X_{1,1}$ or of $X_{2,0}$, which may always be chosen to have degree 1 or 0, respectively. Then $\psi_2 \phi_2(y)$ has respectively degree 1 or 0, implying that its component in $X_{0,2}$ is 0. It follows that $\mu_1 \phi_2(y) = \alpha \psi_2 \phi_2(y) = 0 = \alpha(y)$; the last equation follows from the extension of α to X_\bullet . Therefore $\phi_2^*(\mu_1) = \alpha$.

Construction of the multiplication map μ_2 .

Given μ_1 as above, note that the map μ_2 must satisfy (1.10), that is, $\delta_3^*(\mu_2) = \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ as cochains on the bar resolution $\mathbf{B}_\bullet(A)$ of A . We will show that a modification of $\psi_2^*(\beta)$ is such a map as follows.

First, note that $\beta = \phi_2^*(\psi_2^*(\beta))$ as cochains by a similar argument to that above for α . Moreover, condition (ii) implies that $d_3^*(\beta) = \alpha \circ (\alpha \otimes_H \text{id} - \text{id} \otimes_H \alpha)$ as cochains on $X_{0,3}$. Let

$$(3.8) \quad \gamma = \delta_3^* \psi_2^*(\beta) - \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1).$$

Then $\phi_3^*(\gamma)$ is zero on $X_{0,3}$: $\phi_3^* \delta_3^* \psi_2^*(\beta) = d_3^*(\beta)$ and $\phi_3^*(\mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)) = \alpha \circ (\alpha \otimes \text{id} - \text{id} \otimes \alpha)$ by Lemma 3.4(ii). To see the last statement, note that the image of ϕ_3 on $X_{0,3}$ is contained in $A \otimes ((I \otimes V) \cap (V \otimes I)) \otimes A$ with $\phi^*(\mu_1) = \alpha$. We also see that $\phi_3^*(\gamma)$ is 0 on $X_{2,1}$ and on $X_{3,0}$ since it is a map of degree -2 . We claim it is also 0 on $X_{1,2}$ as follows. As an A^e -module, the image of $X_{1,2}$ under ϕ_3 is generated by elements of degree 2. Since $\mu_1 = \psi_2^*(\alpha)$, it is zero on elements of degree less than two, and so the map $\mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ must be 0 on the image of $X_{1,2}$ under ϕ_3 . Since β is H -invariant so that β is a cocycle (see the argument following (3.7)) and $\phi_2^* \psi_2^*(\beta) = \beta$, we have that $\phi_3^* \delta_3^* \psi_2^*(\beta) = d_3^* \phi_2^* \psi_2^*(\beta) = d_3^*(\beta)$ is 0 on $X_{1,2}$. Therefore $\phi_3^*(\gamma)$ is 0 on $X_{1,2}$.

We have shown that $\phi_3^*(\gamma)$ is 0 on all of X_3 , and so γ must be a coboundary on the bar resolution $\mathbf{B}_\bullet(A)$ of A . Thus there is a 2-cochain μ of degree -2 on the bar resolution with

$$(3.9) \quad \delta_3^*(\mu) = \gamma.$$

Consider $\psi_2^*(\beta) - \mu$, yet note that $\phi_2^*(\psi_2^*(\beta) - \mu)$ may not agree with β on X_2 . We need such a statement for the next step of constructing μ_3 . Now

$$d_3^* \phi_2^*(\mu) = \phi_3^* \delta_3^*(\mu) = \phi_3^*(\gamma) = 0,$$

so the 2-cochain $\phi_2^*(\mu)$ is a *cocycle* on the complex X_\bullet . Thus, $\phi_2^*(\mu)$ lifts to a *cocycle* μ' of degree -2 on the bar complex $\mathbf{B}_\bullet(A)$. In other words, $\phi_2^*(\psi_2^*(\beta) - \mu + \mu')$ agrees with β on X_2 .

Moreover, $\delta_3^*(\mu') = 0$, and by (3.8) and (3.9), we have that $\delta_3^*(\psi_2^*(\beta) - \mu) = \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$. So,

$$(3.10) \quad \delta_3^*(\psi_2^*(\beta) - \mu + \mu') - \mu_1 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1) = 0$$

on the bar resolution $\mathbf{B}_\bullet(A)$ of A .

Thus, set μ_2 equal to $\psi_2^*(\beta) - \mu + \mu'$ and we have maps μ_1, μ_2 to obtain a second level graded deformation $A_{(2)}$ of A extending $A_{(1)}$.

Construction of the multiplication map μ_3 .

Recall the restraint on μ_3 given in (1.11), that is, μ_3 is a cochain on $\mathbf{B}_\bullet(A)$ whose coboundary is given by $\mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2) + \mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$. We construct μ_3 as follows.

By (3.10) and condition (iii) of Lemma 3.4, we have that $\mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1)$ is 0 on the image of ϕ . By degree considerations, $\mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2)$ is always 0 on the image of ϕ . Therefore, the obstruction

$$\mu_2 \circ (\mu_1 \otimes \text{id} - \text{id} \otimes \mu_1) + \mu_1 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2)$$

is a coboundary. Thus there exists a 2-cochain μ_3 , necessarily having degree -3 , satisfying the restraint given above, and the deformation lifts to the third level.

Construction of the multiplication maps μ_i for $i \geq 4$.

The obstruction for a third level graded deformation $A_{(3)}$ of A to lift to the fourth level lies in $\text{HH}^{3,-4}(A)$ by Proposition 1.13. We apply ϕ_3^* to this obstruction to obtain a cochain on X_3 . Since there are no cochains of degree -4 on X_3 by definition (as it is generated by elements of degree 3 or less), ϕ_3^* applied to the obstruction is automatically zero. Therefore, the deformation may be continued to the fourth level. Similar arguments show that it can be continued to the fifth level, and so on.

Construction of A_t .

Let A_t be the graded deformation of A that we obtain in this manner [Definition 1.5]. Then, A_t is the k -vector space $A[t]$ with multiplication, for all $a_1, a_2 \in A$,

$$a_1 * a_2 = a_1 a_2 + \mu_1(a_1 \otimes a_2)t + \mu_2(a_1 \otimes a_2)t^2 + \mu_3(a_1 \otimes a_2)t^3 + \dots,$$

where $a_1 a_2$ is the product in A and each $\mu_i : A \otimes A \rightarrow A$ is a k -linear map of homogeneous degree $-i$. Now for any r in R , $\mu_1(r) = (\psi_2^* \alpha)(r)$ and $\mu_2(r) = (\psi_2^*(\beta) - \mu + \mu')(r)$ by construction, and $\mu_i(r) = 0$ for $i \geq 3$ since $\deg(r) = 2$.

Conclusion that $A' := T_H(U)/(P)$ is a PBW deformation of $A = T_H(U)/(R)$.

Now we show that $A' := T_H(U)/(P)$ is isomorphic, as a filtered algebra, to the fiber of the deformation A_t at $t = 1$ as follows. Let $A'' = (A_t)|_{t=1}$. Then A'' is generated by V and H and one thus obtains a surjective algebra homomorphism $T_H(U) \cong T_k(V) \# H \rightarrow A''$. The elements of P lie in the kernel (by definition of A''), and thus this map induces a surjective algebra homomorphism $A' = T_H(U)/(P) \twoheadrightarrow A''$. This map is in fact an isomorphism of filtered algebras by a dimension argument in each degree. Therefore A' is a PBW deformation of A , since A'' is as such [Proposition 1.14]. \square

4. EXAMPLES

For our examples, we restrict k to be an algebraically closed field of characteristic zero. There are many interesting examples, both known and new, in this setting. Less is known about Hopf actions on Koszul algebras and corresponding deformations in positive characteristic.

As an application of Theorem 3.1, we provide various examples of PBW deformations $\mathcal{D}_{B,\kappa}$ of smash products $B \# H$; recall Notations 0.2 and 0.3. We do this by describing deformation parameter(s) $\kappa = \kappa^C + \kappa^L$ below. In particular, Examples 4.1, 4.2, and 4.4 involve semisimple Hopf actions, and Examples 4.13, 4.16, and 4.18 involve non-semisimple Hopf actions on (skew) polynomial rings. Recall that skew polynomial rings are Koszul by [27, Example 4.2.1 and Theorem 4.3.1].

4.1. Semisimple Hopf actions. We begin by revisiting the well-known PBW deformations of Crawley-Boevey and Holland [5].

Example 4.1. Take $H = k\Gamma$, for Γ a finite subgroup of $SL_2(k)$, and $B = k[u, v]$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, let the action of g on B be given by $g \cdot u = au + cv$ and $g \cdot v = bu + dv$.

By [5], the deformation parameter κ of the PBW deformation $\mathcal{D}_{B,\kappa}$ of $B \# H$ must be in the center of Γ , which we verify again with Theorem 3.1. We assume here that $\kappa^L \equiv 0$ as in [5].

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. So we have that for all $g \in \Gamma$: $g \cdot (\kappa(uv - vu)) = \kappa(g \cdot (uv - vu))$. Now since the determinant of g is 1, $g \cdot (\kappa(uv - vu)) = \kappa(uv - vu)$, and the image of κ lies in the center of $k\Gamma$. That is,

$$\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# k\Gamma}{(uv - vu - \lambda)}$$

is a PBW deformation of $k[u, v] \# k\Gamma$ if and only if $\lambda \in Z(k\Gamma)$.

It is worth pointing out that there are analogues of Crawley-Boevey-Holland algebras when working in positive characteristic; see the work of Emily Norton [26] for some examples that are quite different from those in characteristic zero.

The following two Hopf actions were produced by the first author in joint work with Kenneth Chan, Ellen Kirkman, and James Zhang [4]. We thank Chan, Kirkman, and Zhang for permitting us to use these results.

Example 4.2. Let $H := H_8$ be the unique noncommutative noncocommutative semisimple 8-dimensional Hopf algebra [13, 21], and let $B = k\langle u, v \rangle / (u^2 + v^2)$ (which is isomorphic to the skew polynomial ring $k\langle u, v \rangle / (uv + vu)$). The Hopf algebra H_8 is generated by x, y, z and the relations are

$$x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz, \quad zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy).$$

The rest of the structure of H_8 and left H_8 -action on B are given by

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \Delta(z) &= \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \\ \epsilon(x) &= \epsilon(y) = \epsilon(z) = 1, & S(x) &= x, & S(y) &= y, & S(z) &= z, \\ x \cdot u &= -u, & x \cdot v &= v, & y \cdot u &= u, & y \cdot v &= -v, & z \cdot u &= v, & z \cdot v &= u. \end{aligned}$$

Let $r := u^2 + v^2$ and note that

$$x \cdot r = r, \quad y \cdot r = r, \quad z \cdot r = r,$$

so the ideal of relations of B , $I = \langle r \rangle$, is H -stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. We begin by computing κ^C . Let $\kappa^C(r) = \gamma_0 + \gamma_1 x + \gamma_2 y + \gamma_3 xy + \gamma_4 z + \gamma_5 xz + \gamma_6 yz + \gamma_7 xyz$ for some scalars $\gamma_i \in k$. Since $h \cdot (\kappa^C(r)) = \sum h_1(\kappa^C(r))S(h_2)$ (see Section 1.1), both $x \cdot (\kappa^C(r)) = \kappa^C(r)$ and $y \cdot (\kappa^C(r)) = \kappa^C(r)$ imply that $\gamma_7 = \gamma_4$ and $\gamma_6 = \gamma_5$. Moreover,

$$z \cdot (\kappa^C(r)) = \gamma_0 + \gamma_2 x + \gamma_1 y + \gamma_3 xy + \gamma_4 z + \gamma_5 xz + \gamma_5 yz + \gamma_4 xyz = \kappa^C(r),$$

which implies that $\gamma_2 = \gamma_1$. Thus,

$$(4.3) \quad \kappa^C(r) =: g(\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_5) = \gamma_0 + \gamma_1(x + y) + \gamma_3 xy + \gamma_4(z + xyz) + \gamma_5(xz + yz).$$

On the other hand, let $\kappa^L(r) = u \otimes f + v \otimes f' \in V \otimes H$ with $f = \delta_0 + \delta_1 x + \delta_2 y + \delta_3 xy + \delta_4 z + \delta_5 xz + \delta_6 yz + \delta_7 xyz$ and $f' = \delta'_0 + \delta'_1 x + \delta'_2 y + \delta'_3 xy + \delta'_4 z + \delta'_5 xz + \delta'_6 yz + \delta'_7 xyz$ for some scalars $\delta_i, \delta'_i \in k$. Note that $h \cdot (\kappa^L(r)) = \sum h_1 \cdot u \otimes h_2 f S(h_3) + \sum h_1 \cdot v \otimes h_2 f' S(h_3)$ (see Section 1.1). Since $x \cdot (\kappa^L(r)) = \kappa^L(r)$, it follows that:

$$\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0, \quad \delta_4 = -\delta_7, \quad \delta_5 = -\delta_6 \quad \text{and} \quad \delta'_4 = \delta'_7, \quad \delta'_5 = \delta'_6.$$

By considering the coefficient of u in the equation $y \cdot (\kappa^L(r)) = \kappa^L(r)$, we now find that $f = 0$. Similarly, by considering the coefficient of v in the equation $y \cdot (\kappa^L(r)) = \kappa^L(r)$, we find that $f' = 0$. Hence, $\kappa^L(r) = 0$.

Thus the deformation parameter κ of $\mathcal{D}_{B,\kappa}$ equals its constant part κ^C , which depends on five scalar parameters as described above. In short,

$$\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# H_8}{(u^2 + v^2 - \kappa(u^2 + v^2))}$$

is a PBW deformation of $(k\langle u, v \rangle / (u^2 + v^2)) \# H_8$ if and only if $\kappa(u^2 + v^2) = g(\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_5)$ as given in (4.3). This yields a five parameter family of PBW deformations of $B \# H_8$.

Example 4.4. Let H be $H_{a,1}$, one of the 16-dimensional semisimple Hopf algebras classified by Kashina in [14] and let B be the skew polynomial ring:

$$B = \frac{k\langle t, u, v, w \rangle}{\left(\begin{array}{lll} r_{tu} := tu - ut, & r_{tv} := tv - vt, & r_{tw} := tw + wt \\ r_{uv} := uv - vu, & r_{uw} := uw + wu, & r_{vw} := vw - wv \end{array} \right)}.$$

The Hopf algebra $H_{a,1}$ is generated by x, y, z subject to relations

$$x^4 = y^2 = z^2 = 1, \quad yx = xy, \quad zx = xyz, \quad zy = yz.$$

The rest of the structure of $H_{a,1}$ and the left $H_{a,1}$ -action on B are given by

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \Delta(z) &= \frac{1}{2}(1 \otimes 1 + 1 \otimes x^2 + y \otimes 1 - y \otimes x^2)(z \otimes z), \\ \epsilon(x) &= \epsilon(y) = \epsilon(z) = 1, & S(x) &= x^3, & S(y) &= y, & S(z) &= \frac{1}{2}(1 + x^2 + y - x^2 y)z, \end{aligned}$$

$$\begin{array}{llllll} x \cdot t = it, & y \cdot t = -t, & z \cdot t = u, & x \cdot u = -iu, & y \cdot u = -u, & z \cdot u = t, \\ x \cdot v = v, & y \cdot v = -v, & z \cdot v = w, & x \cdot w = -w, & y \cdot w = -w, & z \cdot w = v, \end{array}$$

where i is a primitive fourth root of unity in k . Note that

$$\begin{array}{llllll} x \cdot r_{tu} = r_{tu}, & x \cdot r_{tv} = ir_{tv}, & x \cdot r_{tw} = -ir_{tw}, & x \cdot r_{uv} = -ir_{uv}, & x \cdot r_{uw} = ir_{uw}, & x \cdot r_{vw} = -r_{vw}, \\ y \cdot r_{tu} = r_{tu}, & y \cdot r_{tv} = r_{tv}, & y \cdot r_{tw} = r_{tw}, & y \cdot r_{uv} = r_{uv}, & y \cdot r_{uw} = r_{uw}, & y \cdot r_{vw} = r_{vw}, \\ z \cdot r_{tu} = r_{tu}, & z \cdot r_{tv} = r_{uv}, & z \cdot r_{tw} = r_{uv}, & z \cdot r_{uv} = r_{tw}, & z \cdot r_{uw} = r_{tv}, & z \cdot r_{vw} = -r_{vw}. \end{array}$$

So, the ideal of relations $I = \langle r_{tu}, r_{tv}, r_{tw}, r_{uv}, r_{uw}, r_{vw} \rangle$ of B is H -stable.

Now we compute the possible values $\kappa^C(r) \in H$ for all $r \in I$, under condition (a) of Theorem 3.1. Take $\kappa^C(r) = g(\underline{\gamma}) \in H$ given as follows:

$$\begin{aligned} g(\underline{\gamma}) &= \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 y + \gamma_5 xy + \gamma_6 x^2 y + \gamma_7 x^3 y \\ &\quad + \gamma_8 z + \gamma_9 xz + \gamma_{10} x^2 z + \gamma_{11} x^3 z + \gamma_{12} yz + \gamma_{13} xyz + \gamma_{14} x^2 yz + \gamma_{15} x^3 yz, \end{aligned}$$

where $\gamma_i \in k$. Note that $h \cdot g(\underline{\gamma}) = \sum h_1 g(\underline{\gamma}) S(h_2)$. With the assistance of Affine, a subpackage of Maxima, we have the following computations:

$$(4.5) \quad \begin{aligned} x \cdot g(\underline{\gamma}) &= x g(\underline{\gamma}) x^3 \\ &= \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 y + \gamma_5 xy + \gamma_6 x^2 y + \gamma_7 x^3 y \\ &\quad + \gamma_8 yz + \gamma_9 xyz + \gamma_{10} x^2 yz + \gamma_{11} x^3 yz + \gamma_{12} z + \gamma_{13} xz + \gamma_{14} x^2 z + \gamma_{15} x^3 z; \end{aligned}$$

$$(4.6) \quad y \cdot g(\underline{\gamma}) = y g(\underline{\gamma}) y = g(\underline{\gamma});$$

$$(4.7) \quad \begin{aligned} z \cdot g(\underline{\gamma}) &= \frac{1}{2} (z g(\underline{\gamma}) S(z) + z g(\underline{\gamma}) S(x^2 z) + yz g(\underline{\gamma}) S(z) - yz g(\underline{\gamma}) S(x^2 z)) \\ &= \gamma_0 + \gamma_1 xy + \gamma_2 x^2 + \gamma_3 x^3 y + \gamma_4 y + \gamma_5 x + \gamma_6 x^2 y + \gamma_7 x^3 \\ &\quad + \gamma_8 z + \gamma_9 xyz + \gamma_{10} x^2 z + \gamma_{11} x^3 yz + \gamma_{12} yz + \gamma_{13} xz + \gamma_{14} x^2 yz + \gamma_{15} x^3 z. \end{aligned}$$

For r_{tu} , let $\kappa^C(r_{tu}) = g(\underline{\gamma})$. We have that $x \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu})$ and $y \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu})$ implies that $\gamma_8 = \gamma_{12}$, $\gamma_9 = \gamma_{13}$, $\gamma_{10} = \gamma_{14}$, $\gamma_{11} = \gamma_{15}$. Moreover, $z \cdot \kappa^C(r_{tu}) = \kappa^C(r_{tu})$ implies that $\gamma_1 = \gamma_5$, $\gamma_3 = \gamma_7$, $\gamma_9 = \gamma_{13}$, $\gamma_{11} = \gamma_{15}$. Therefore,

$$(4.8) \quad \begin{aligned} \kappa^C(r_{tu}) &= g(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_6, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}) \\ &= \gamma_0 + \gamma_1(x + xy) + \gamma_2 x^2 + \gamma_3(x^3 + x^3 y) + \gamma_4 y + \gamma_6 x^2 y \\ &\quad + \gamma_8(z + yz) + \gamma_9(xz + xyz) + \gamma_{10}(x^2 z + x^2 yz) + \gamma_{11}(x^3 z + x^3 yz). \end{aligned}$$

For r_{vw} , let $\kappa^C(r_{vw}) = g(\underline{\gamma}')$. We have that $x \cdot \kappa^C(r_{vw}) = -\kappa^C(r_{vw})$ and $y \cdot \kappa^C(r_{vw}) = \kappa^C(r_{vw})$ implies that $\gamma'_0 = \dots = \gamma'_7 = 0$, $\gamma'_8 = -\gamma'_{12}$, $\gamma'_9 = -\gamma'_{13}$, $\gamma'_{10} = -\gamma'_{14}$, $\gamma'_{11} = -\gamma'_{15}$. Moreover, we have that $z \cdot \kappa^C(r_{vw}) = -\kappa^C(r_{vw})$. So the conditions on γ'_i in (4.7) then imply that $\gamma'_i = 0$, for $i = 0, \dots, 7, 8, 10, 12, 14$ with $\gamma'_9 = -\gamma'_{13}$, $\gamma'_{11} = -\gamma'_{15}$. Thus,

$$(4.9) \quad \kappa^C(r_{vw}) = g(\gamma'_9, \gamma'_{11}) = \gamma'_9(xz - xyz) + \gamma'_{11}(x^3 z - x^3 yz).$$

For $r \neq r_{tu}, r_{vw}$, we have that $x \cdot \kappa^C(r) = \pm i \kappa^C(r)$ implies that $\kappa^C(r) = 0$.

We compute $\kappa^L(r)$ under condition (a) of Theorem 3.1. Fix $r \in I$ and let

$$\kappa^L(r) = t \otimes f_t + u \otimes f_u + v \otimes f_v + w \otimes f_w \in V \otimes H,$$

for some $f_t, f_u, f_v, f_w \in H$. Since y is central in H and $y \cdot r = r$ for each relation r , we have that

$$\begin{aligned} \kappa^L(r) &= y \cdot \kappa^L(r) = y \cdot t \otimes y f_t S(y) + y \cdot u \otimes y f_u S(y) + y \cdot v \otimes y f_v S(y) + y \cdot w \otimes y f_w S(y) \\ &= -t \otimes f_t - u \otimes f_u - v \otimes f_v - w \otimes f_w = -\kappa^L(r). \end{aligned}$$

Thus, $\kappa^L(r) = 0$.

To finish, we apply to κ conditions (b)-(d) of Theorem 3.1. Since $\kappa^L(r) = 0$ for all $r \in I$, only condition (c) is pertinent. Namely, we only need to impose $\kappa^C \otimes \text{id} = \text{id} \otimes \kappa^C$ as maps on $(I \otimes V) \cap (V \otimes I)$. This intersection is a 4-dimensional k -vector space with basis:

$$\begin{aligned} s_{tuv} &:= tuv - tvu - utv + uvt + vtu - vut, \\ s_{tuw} &:= tuw + twu - utw - uwt + wtu - wut, \\ s_{tvw} &:= tvw - twv - vtw + vwt + wtv - wvt, \\ s_{uvw} &:= uvw - uvw - vuw - vwu - wuv + wvu. \end{aligned}$$

Since $\kappa^C(r_{tv}) = \kappa^C(r_{uv}) = 0$, we get that

$$(4.10) \quad \begin{aligned} (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tuv}) &= \kappa^C(r_{uv})t - \kappa^C(r_{tv})u + \kappa^C(r_{tu})v - t\kappa^C(r_{uv}) + u\kappa^C(r_{tv}) - v\kappa^C(r_{tu}) \\ &= \kappa^C(r_{tu})v - v\kappa^C(r_{tu}). \end{aligned}$$

Identify $b \in V$ with $b\#1 \in A$, and $h \in H$ with $1\#h \in A$. Recall that in A , we have $(1\#h)(b\#1) = \sum(h_1 \cdot b)\#h_2$. Now by using (4.8) and by setting (4.10) equal to 0, we get that

$$(4.11) \quad \kappa^C(r_{tu}) = g(\gamma_0, \gamma_2) = \gamma_0 + \gamma_2 x^2.$$

Moreover,

$$\begin{aligned} (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tuvw}) &= -\kappa^C(r_{uv})t + \kappa^C(r_{tv})u + \kappa^C(r_{tu})w - t\kappa^C(r_{uv}) + u\kappa^C(r_{tv}) - w\kappa^C(r_{tu}) \\ &= \kappa^C(r_{tu})w - w\kappa^C(r_{tu}) = 0 \end{aligned}$$

imposes no new restrictions on $\kappa^C(r_{tu})$, nor do $(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) = 0$, $(\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{uvw}) = 0$. Therefore, $\kappa^C(r_{tu})$ is given by (4.11).

To compute $\kappa^C(r_{vw})$, consider the calculation below:

$$(4.12) \quad \begin{aligned} (\kappa^C \otimes \text{id} - \text{id} \otimes \kappa^C)(s_{tvw}) &= \kappa^C(r_{vw})t - \kappa^C(r_{tw})v + \kappa^C(r_{tv})w - t\kappa^C(r_{vw}) + v\kappa^C(r_{tw}) - w\kappa^C(r_{tv}) \\ &= \kappa^C(r_{vw})t - t\kappa^C(r_{vw}). \end{aligned}$$

Now by using (4.9) and by setting (4.12) equal to 0, we get that $\kappa^C(r_{vw}) = 0$.

Therefore, the filtered algebra $\mathcal{D}_{B,\kappa}$ is a PBW deformation of $B\#H_{a:1}$ if and only if the deformation parameter $\kappa = \kappa^C$ of $\mathcal{D}_{B,\kappa}$ is defined by (4.11) for the relation r_{tu} , and $\kappa^C(r) = 0$ for $r \neq r_{tu}$. Hence, we have a two parameter family of PBW deformations of $B\#H_{a:1}$.

4.2. Non-semisimple Hopf actions. Here, we consider non-semisimple Hopf actions to illustrate Theorem 3.1. We begin with an example of a Taft algebra action.

Example 4.13. Let $H = T(n)$, the n^2 -dimensional non-semisimple Taft algebra. We take $n \geq 3$ here and we consider the (slightly different) case $n = 2$ (the Sweedler algebra) in Example 4.16 below. Let $B = k[u, v]$. The Hopf algebra $T(n)$ is generated by g, x and the relations are $g^n = 1$, $x^n = 0$, and $xg = \zeta gx$, for $\zeta \in k^\times$ a primitive n -th root of unity. The rest of the structure of $T(n)$ and left $T(n)$ -action on B are given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= g \otimes x + x \otimes 1, & \epsilon(g) &= 1, & \epsilon(x) &= 0, & S(g) &= g^{-1} = g^{n-1}, & S(x) &= -g^{n-1}x, \\ & & & & g \cdot u &= u, & g \cdot v &= \zeta^{-1}v, & x \cdot u &= 0, & x \cdot v &= u. \end{aligned}$$

Let $r := uv - vu$ and note that $g \cdot r = \zeta^{-1}r$ and $x \cdot r = 0$. Hence, the ideal of relations $I = \langle r \rangle$ of B is H -stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. Now, we compute κ^C . Let $\kappa^C(r) = \sum_{i,j=0}^{n-1} \gamma_{ij} g^i x^j$. Since $h \cdot (\kappa^C(r)) = \sum h_1 (\kappa^C(r)) S(h_2)$, for $h \in H$, we have that $g \cdot (\kappa^C(r)) = \zeta^{-1} \kappa^C(r)$ implies that all terms equal zero except when $j = 1$; hence

$$\kappa^C(r) = \gamma_0 x + \gamma_1 g x + \cdots + \gamma_{n-1} g^{n-1} x,$$

for $\gamma_i \in k$. Also, $x \cdot (\kappa^C(r)) = 0$ implies that all terms equal zero except when $i = n - 1$, so

$$(4.14) \quad \kappa^C(r) = \gamma g^{n-1} x$$

for $\gamma \in k$.

On the other hand, let $\kappa^L(r) = u \otimes f + v \otimes f' \in V \otimes H$, for $f = \sum_{i,j=0}^{n-1} \lambda_{ij} g^i x^j$ and $f' = \sum_{i,j=0}^{n-1} \lambda'_{ij} g^i x^j$. Note that $h \cdot (\kappa^L(r)) = \sum h_1 \cdot u \otimes h_2 f S(h_3) + \sum h_1 \cdot v \otimes h_2 f' S(h_3)$ (see Section 1.1). Since $g \cdot (\kappa^L(r)) = \zeta^{-1} \kappa^L(r)$, all terms equal zero except possibly those in the first sum for which $j = 1$ and those in the second sum for which $j = 0$. Therefore $\kappa^L(r) = u \otimes f + v \otimes f'$ for

$$f = \lambda_0 x + \lambda_1 g x + \cdots + \lambda_{n-1} g^{n-1} x \quad \text{and} \quad f' = \lambda'_0 + \lambda'_1 g + \cdots + \lambda'_{n-1} g^{n-1}$$

with $\lambda_i, \lambda'_i \in k$. Applying x , we obtain

$$\begin{aligned} 0 = x \cdot \kappa^L(r) &= (g \cdot u) \otimes (gfS(x) + xfS(1)) + (x \cdot u) \otimes f + (g \cdot v) \otimes (gf'S(x) + xf'S(1)) + (x \cdot v) \otimes f' \\ &= u \otimes (-gfg^{n-1}x + xf + f') + \zeta^{-1}v \otimes (-gfg^{n-1}x + xf'). \end{aligned}$$

It follows that

$$-gfg^{n-1}x + xf + f' = 0 \quad \text{and} \quad -gfg^{n-1}x + xf' = 0.$$

Since f' is in the group algebra $kG(T(n)) \cong k\mathbb{Z}_n$ and $g^n = 1$, the second equation implies that $xf' = f'x$, and so $f' = \lambda_0$ is constant. The first equation further implies that $f' = 0$ and that all terms of f are equal to zero except when $i = n - 1$. Thus,

$$(4.15) \quad \kappa^L(r) = u \otimes \lambda g^{n-1} x,$$

for $\lambda \in k$.

In summary,

$$\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# T(n)}{(uv - vu - \kappa(uv - vu))}$$

is a PBW deformation of $k[u, v] \# T(n)$ if and only if the deformation map κ equals $\kappa^C + \kappa^L$ as given in (4.14) and (4.15). So, we have a two parameter family of PBW deformations of $k[u, v] \# T(n)$.

Example 4.16. Let H be $H_{Sw} = T(2)$, the 4-dimensional non-semisimple Sweedler algebra, which is a Taft algebra with $n = 2$. Let $B = k[u, v]$. Retaining the notation from Example 4.13, the Hopf algebra H_{Sw} is generated by g, x and acts on B by $g \cdot u = u, g \cdot v = -v, x \cdot u = 0, x \cdot v = u$. Similar to Example 4.13, let $r := uv - vu$ and note that $g \cdot r = -r$ and $x \cdot r = 0$. So, $I = \langle r \rangle$ is H -stable.

Let $\kappa^C(r) = \gamma_0 + \gamma_1 g + \gamma_2 x + \gamma_3 gx$. We have that $g \cdot (\kappa^C(r)) = -\kappa^C(r)$ implies that $\gamma_0 = \gamma_1 = 0$. Moreover, $x \cdot (\kappa^C(r)) = 0$ does not yield new restrictions on $\kappa^C(r)$. Thus, for $\gamma, \gamma' \in k$, we get that $\kappa^C(r) = \gamma x + \gamma' gx$. In the same fashion as Example 4.13, we also get that $\kappa^L(r) = u \otimes (\lambda x + \lambda' gx)$, for $\lambda, \lambda' \in k$.

In summary,

$$\mathcal{D}_{B, \kappa} = \frac{k\langle u, v \rangle \# H_{Sw}}{(uv - vu - \kappa(uv - vu))}$$

is a PBW deformation of $k[u, v] \# H_{Sw}$ if and only if the deformation map κ equals $\kappa^C + \kappa^L$, where

$$\kappa^C(uv - vu) = \gamma x + \gamma' gx \quad \text{and} \quad \kappa^L(uv - vu) = u \otimes (\lambda x + \lambda' gx)$$

for $\gamma, \gamma', \lambda, \lambda' \in k$. Thus, we have a four parameter family of PBW deformations of $k[u, v] \# H_{Sw}$.

Remark 4.17. The invariant ring resulting from the action of H_{Sw} on $k[u, v]$ is isomorphic to the polynomial ring $k[u, v^2]$, that is to say, $k[u, v]^{H_{Sw}}$ is regular. Recall that the Shephard-Todd-Chevalley Theorem states that when given a finite group (G -) action on a commutative polynomial ring R that is linear and faithful, R^G is regular if and only if G is a reflection group. Our results would then suggest that H_{Sw} is a ‘‘reflection Hopf algebra.’’ Ram and Shepler showed that there are no non-trivial PBW deformations of $k[v_1, \dots, v_n] \# kG$ for many complex reflection groups G ; such deformations are referred to as *graded Hecke algebras* [29]. Now by broadening their setting to Hopf actions on (possibly noncommutative) regular algebras, we consider new objects in representation theory: Hopf analogues of graded Hecke algebras. Non-trivial examples of these objects exist as we showed in the example above. Further examples and a general explanation of this phenomenon are worthy of further investigation.

Now we consider the well-known Hopf action of $U_q(\mathfrak{sl}_2)$ on $k\langle u, v \rangle / (uv - qvu)$, where $q \in k^\times$ with $q^2 \neq 1$. A PBW deformation of $(k\langle u, v \rangle / (uv - qvu)) \# U_q(\mathfrak{sl}_2)$ was studied by Gan and Khare in [10]; we recover their result below. Such algebras are known as *quantized symplectic oscillator algebras of rank 1*.

Example 4.18. Fix $q \in k^\times$, with $q^2 \neq 1$. Let H be the Hopf algebra $U_q(\mathfrak{sl}_2)$, and $B = k\langle u, v \rangle / (uv - qvu)$. As in [3, I.6.2], we take $U_q(\mathfrak{sl}_2)$ to be generated by E, F, K, K^{-1} with relations:

$$EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad KK^{-1} = K^{-1}K = 1.$$

So, $U_q(\mathfrak{sl}_2)$ has a k -vector space basis $\{E^i F^j K^m\}_{i, j \in \mathbb{N}; m \in \mathbb{Z}}$. The rest of the structure of $U_q(\mathfrak{sl}_2)$ and left $U_q(\mathfrak{sl}_2)$ -action on B is given by:

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K \otimes E, & \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, & \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \epsilon(E) &= 0, & \epsilon(F) &= 0, & \epsilon(K) &= 1, & \epsilon(K^{-1}) &= 1, \\ S(E) &= -K^{-1}E, & S(F) &= -FK, & S(K) &= K^{-1}, & S(K^{-1}) &= K, \\ E \cdot u &= 0, & F \cdot u &= v, & K \cdot u &= qu, & K^{-1} \cdot u &= q^{-1}u, \\ E \cdot v &= u, & F \cdot v &= 0, & K \cdot v &= q^{-1}v, & K^{-1} \cdot v &= qv. \end{aligned}$$

Let $r := uv - qvu$ and note that $E \cdot r = F \cdot r = 0$ and $K \cdot r = K^{-1} \cdot r = r$. Hence, the ideal of relations $I = \langle r \rangle$ of B is H -stable.

Since $\dim_k V = 2$, only condition (a) of Theorem 3.1 applies. Let us compute $\kappa^C(r)$. Since $K \cdot \kappa^C(r) = \kappa^C(K \cdot r) = \kappa^C(r)$, we have that $K\kappa^C(r)S(K) = \kappa^C(r)$ (see Section 1.1). Hence, $K\kappa^C(r) = \kappa^C(r)K$. Moreover,

$$0 = \kappa^C(E \cdot r) = E \cdot \kappa^C(r) = E\kappa^C(r)S(1) + K\kappa^C(r)S(E),$$

so $E\kappa^C(r) = \kappa^C(r)E$. Likewise, $F \cdot \kappa^C(r) = 0$ implies that $F\kappa^C(r) = \kappa^C(r)F$. So, $\kappa^C(r)$ is in the center of $U_q(\mathfrak{sl}_2)$. For q a non-root of unity, the center of $U_q(\mathfrak{sl}_2)$ is generated by the *quantum Casimir element* [15, Theorem VI.4.8],

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2},$$

whereas for q a root of unity, the elements E^e, F^e, K^e also belong to the center of $U_q(\mathfrak{sl}_2)$, where $e = \text{ord}(q^2)$.

To compute $\kappa^L(r)$, let $\kappa^L(r) = u \otimes \sum \gamma_{ijm} E^i F^j K^m + v \otimes \sum \gamma'_{ijm} E^i F^j K^m$ for $\gamma_{ijm}, \gamma'_{ijm} \in k$. Then,

$$\begin{aligned} \kappa^L(r) &= \kappa^L(K \cdot r) = \sum K \cdot u \otimes \gamma_{ijm} K(E^i F^j K^m) K^{-1} + \sum K \cdot v \otimes \gamma'_{ijm} K(E^i F^j K^m) K^{-1} \\ &= \sum qu \otimes \gamma_{ijm} q^{2(i-j)} E^i F^j K^m + \sum q^{-1}v \otimes \gamma'_{ijm} q^{2(i-j)} E^i F^j K^m \\ &= \sum q^{2(i-j)+1} u \otimes \gamma_{ijm} E^i F^j K^m + \sum q^{2(i-j)-1} v \otimes \gamma'_{ijm} E^i F^j K^m. \end{aligned}$$

Thus given $m \in \mathbb{Z}/n\mathbb{Z}$, define the subspace $V_m \subset U_q(\mathfrak{sl}_2)$ to be the k -span of all monomials $E^i F^j K^\ell$ such that $j - i \equiv m \pmod n$. Then $\kappa^L(uv - qvu) \in u \otimes V_{2-1} + v \otimes V_{-2-1}$ if q is a primitive root of unity of odd order, and $\kappa^L(uv - qvu) = 0$ otherwise.

Therefore,

$$\mathcal{D}_{B,\kappa} = \frac{k\langle u, v \rangle \# U_q(\mathfrak{sl}_2)}{(uv - qvu - \kappa(uv - qvu))}$$

is a PBW deformation of $(k\langle u, v \rangle / (uv - qvu)) \# U_q(\mathfrak{sl}_2)$ if and only if $\kappa = \kappa^C + \kappa^L$ where $\kappa^C(uv - qvu)$ is in the center of $U_q(\mathfrak{sl}_2)$ and $\kappa^L(uv - qvu)$ is given as above.

More generally, there is a standard $U_q(\mathfrak{sl}_n)$ -action on a q -polynomial ring B in n variables.

Question 4.19. Are there nontrivial PBW deformations of the resulting smash product algebra $B \# U_q(\mathfrak{sl}_n)$?

These would be *quantized symplectic oscillator algebras of rank $n - 1$* , and merit further investigation.

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