RESOLUTIONS FOR TWISTED TENSOR PRODUCTS

A.V. SHEPLER AND S. WITHERSPOON

ABSTRACT. We build resolutions for general twisted tensor products of algebras. These bimodule and module resolutions unify many constructions in the literature and are suitable for computing Hochschild (co)homology and more generally Ext and Tor for (bi)modules. We analyze in detail the case of Ore extensions, consequently obtaining Chevalley-Eilenberg resolutions for universal enveloping algebras of Lie algebras (defining the cohomology of Lie groups and Lie algebras). Other examples include semidirect products, crossed products, Weyl algebras, Sridharan enveloping algebras, and Koszul pairs.

1. INTRODUCTION

Motivated by questions in noncommutative geometry, Čap, Schichl, and Vanžura [5] introduced a very general *twisted tensor product* of algebras to replace the (commutative) tensor product. Their examples included noncommutative 2-tori and crossed products of C*-algebras with groups. Many other algebras of interest arise as twisted tensor product algebras: crossed products with Hopf algebras, algebras with triangular decomposition (e.g., universal enveloping algebras of Lie algebras and quantum groups), braided tensor products defined by R-matrices, and other biproduct constructions. In fact, twisted tensor product algebras are rather copious: If an algebra is isomorphic to $A \otimes B$ as a vector space for two of its subalgebras A and B under the canonical inclusion maps, then it must be isomorphic to a twisted tensor product $A \otimes_\tau B$ for some twisting map $\tau : B \otimes A \to A \otimes B$ (see [5]).

Modules over a twisted tensor product algebra arise from tensoring together modules for the individual algebras: If M and N are modules over algebras A and B, respectively, compatible with a twisting map τ, then $M \otimes N$ adopts the structure of a module over $A \otimes_\tau B$. We describe in this note a general method to twist together resolutions of A-modules and B-modules in order to construct resolutions for the corresponding modules over the twisted tensor product $A \otimes_\tau B$. A similar method works for bimodules. In particular, we twist together resolutions of algebras over a field to obtain a resolution for a twisted tensor product algebra as a bimodule over itself.

We are motivated by a desire to understand deformations of twisted tensor products and to describe the homology theory in terms of the homology of the original factor algebras. For example, under some finiteness assumptions, the Hochschild cohomology of a tensor product of algebras is the tensor product of their Hochschild cohomology rings. A similar statement is true of the cohomology of augmented algebras. Both results hold because the

tensor product of projective resolutions for the factor algebras is a projective resolution for the tensor product of the algebras.

In some particular settings, similar homological constructions have appeared for modified versions of the tensor product of algebras. We mention just a few examples. Gopalakrishnan and Sridharan [7] constructed resolutions for modules of Ore extensions. Bergh and Oppermann [1] twisted resolutions when the twisting arises from a bicharacter on grading groups. Jara Martinez, López Peña, and Ştefan [12] worked with Koszul pairs. Guccione and Guccione [8, 9] built resolutions for twisted tensor products, in particular crossed products with Hopf algebras, out of bar and Koszul resolutions of the factor algebras. We adapted this last construction in [16] to handle more general resolutions for the case of skew group algebras in order to understand deformations. Walton and the second author generalized these resolutions to smash products with Hopf algebras in [18].

In this paper, we unify many of these previous constructions and provide methods useful in new settings for finding resolutions of modules over twisted tensor product algebras: We show very generally that projective resolutions for bimodules of two factor algebras can be twisted together to construct a projective resolution for the resulting bimodule for the twisted tensor product given some compatibility conditions. This twisting of resolutions provides an efficient means for computing and handling Hochschild (co)homology in particular. A similar construction applies to projective (left) module resolutions used, for example, to compute (co)homology of augmented algebras.

We verify that many known resolutions may be viewed as twisted resolutions in this way, including some of those mentioned above. We give details in the case of Ore extensions. In particular, the bimodule Koszul resolution of a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a twisted resolution when $\mathfrak{g}$ is a finite dimensional supersolvable Lie algebra. More general Lie algebras can be handled via triangular decomposition. Our method also leads to standard resolutions for Weyl algebras and some Sridharan enveloping algebras. For an Ore extension, we adapt results of Gopalakrishnan and Sridharan [7] to construct twisted product resolutions of modules. We thus regard the Chevalley-Eilenberg complex of $\mathcal{U}(\mathfrak{g})$ as a twisted product resolution. This defines Lie algebra and Lie group cohomology in terms of an iterative twisting of resolutions.

In Section 2, we give definitions and some preliminary results. Bimodule twisted tensor product complexes are constructed in Section 3 and we show they give projective resolutions in Theorem 3.10. Section 4 gives applications to some types of Ore extensions. We construct twisted tensor product complexes for resolving modules in Section 5, and we show these complexes are projective resolutions in Theorem 5.12. Applications to Ore extensions appear in Section 6.

We fix a field k of arbitrary characteristic throughout. All tensor products are over k unless otherwise indicated, i.e., $\otimes = \otimes_k$, and all algebras are k-algebras. Modules are left modules unless otherwise described.

2. Twisted tensor product algebras and compatible resolutions

In this section, we recall twisted tensor product algebras from [5] and define a compatibility condition necessary for twisting resolutions together. Examples include skew group algebras and crossed products with Hopf algebras [13], twisted tensor products given by

bicharacters of grading groups [1], braided products arising from R-matrices [11], two-cocycle twists of Hopf algebras [15], and more.

Let A and B be associative algebras over k with multiplication maps $m_A : A \otimes A \to A$ and $m_B : B \otimes B \to B$ and multiplicative identities 1_A and 1_B, respectively. We write 1 for the identity map on any set.

Twisted tensor products. A *twisting map*

$$\tau : B \otimes A \to A \otimes B$$

is a bijective k-linear map for which $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$ and $b \in B$, and

$$(2.1) \qquad \tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1)$$

as maps $B \otimes B \otimes A \otimes A \to A \otimes B$. The *twisted tensor product algebra* $A \otimes_\tau B$ is the vector space $A \otimes B$ together with multiplication m_τ given by such a twisting map τ. By [5, Proposition/Definition 2.3], the algebra $A \otimes_\tau B$ is associative.

Note that the left-right distinction in a twisted tensor product algebra is artificial since $A \otimes_\tau B \cong B \otimes_{\tau^{-1}} A$. Indeed, one might identify $A \otimes_\tau B$ with the algebra generated by A and B (so that A and B are subalgebras) with relations given by Equation (2.1).

If A and B are $\mathbb{N}$-graded algebras, we take the standard $\mathbb{N}$-grading on $A \otimes B$ and $B \otimes A$ and say a twisting map τ is *strongly graded* if it takes $B_j \otimes A_i$ to $A_i \otimes B_j$ for all i, j following Conner and Goetz [4]. (Note that [12] leave off the adjective *strongly*.) In this case, the twisted tensor product algebra $A \otimes_\tau B$ is $\mathbb{N}$-graded.

Example 2.2. The Weyl algebra $\mathcal{W} = k\langle x, y \rangle / (xy - yx - 1)$ is isomorphic to the twisted tensor product $A \otimes_\tau B$ of $A = k[x]$ and $B = k[y]$ with twisting map $\tau : B \otimes A \to A \otimes B$ defined by $\tau(y \otimes x) = x \otimes y - 1 \otimes 1$. Likewise, the Weyl algebra $\mathcal{W}_n$ on $2n$ indeterminates,

$$\mathcal{W}_n = k\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle / (x_i x_j - x_j x_i, \ y_i y_j - y_j y_i, \ x_i y_j - y_j x_i - \delta_{i,j} : 1 \leq i, j \leq n),$$

is isomorphic to a twisted tensor product. These are examples of (iterated) Ore extensions, which we consider in detail in Section 4.

Example 2.3. A skew group algebra $S \rtimes G$ for a finite group G acting on an algebra S by automorphisms is isomorphic to the twisted tensor product $kG \otimes_\tau S$ of the group algebra kG and of S. The twisting map τ is defined by $\tau(s \otimes g) = g \otimes g^{-1}(s)$ for $s \in S$ and $g \in G$. We consider the special case that S is a Koszul algebra at the end of Section 3.

Bimodules over twisted tensor products. We fix a twisting map $\tau : B \otimes A \to A \otimes B$ for k-algebras A and B.

Definition 2.4. An A-bimodule M is *compatible with* τ if there is a bijective k-linear map $\tau_{B,M} : B \otimes M \to M \otimes B$ commuting with the bimodule structure of M and multiplication in B, i.e., as maps on $B \otimes B \otimes M$ and on $B \otimes A \otimes M \otimes A$, respectively,

$$(2.5) \qquad \tau_{B,M}(m_B \otimes 1) = (1 \otimes m_B)(\tau_{B,M} \otimes 1)(1 \otimes \tau_{B,M}) \qquad \text{and}$$

$$(2.6) \qquad \tau_{B,M}(1 \otimes \rho_{A,M}) = (\rho_{A,M} \otimes 1)(1 \otimes 1 \otimes \tau)(1 \otimes \tau_{B,M} \otimes 1)(\tau \otimes 1 \otimes 1),$$

where $\rho_{A,M} : A \otimes M \otimes A \to M$ is the bimodule structure map. If A is graded and M is a graded A-bimodule, we say that M is *compatible with a strongly graded twisting map* τ if there is a map $\tau_{B,M}$ as above that takes $B_i \otimes M_j$ to $M_j \otimes B_i$ for all i, j.
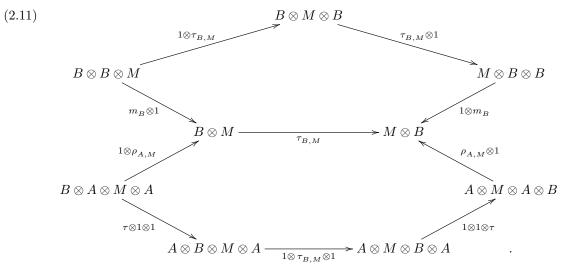
Remark 2.7. Note that the above definition applies to B-bimodules as well as A-bimodules by reversing the role of A and B. Indeed, we apply the definition to the algebra B, the twisted tensor product $B \otimes_{\tau^{-1}} A$, and the twisting map τ^{-1} to obtain conditions for a B-bimodule N to be compatible with τ^{-1}. We may rewrite these conditions using the convenient notation $\tau_{N,A} = (\tau_{A,N}^{-1})^{-1}$. We obtain an equivalent right version of the above definition: A given B-bimodule N is *compatible* with τ^{-1} when there is some bijective k-linear map $\tau_{N,A} : N \otimes A \to A \otimes N$ satisfying

$$(2.8) \qquad \tau_{N,A}(1 \otimes m_A) = (m_A \otimes 1)(1 \otimes \tau_{N,A})(\tau_{N,A} \otimes 1) \qquad \text{and}$$

$$(2.9) \qquad \tau_{N,A}(\rho_{B,N} \otimes 1) = (1 \otimes \rho_{B,N})(\tau \otimes 1 \otimes 1)(1 \otimes \tau_{N,A} \otimes 1)(1 \otimes 1 \otimes \tau),$$

as maps on $N \otimes A \otimes A$ and on $B \otimes N \otimes B \otimes A$, respectively, where $\rho_{B,N} : B \otimes N \otimes B \to N$ is the bimodule structure map.

In light of the last remark, we will say a bimodule is *compatible with* τ when it is either an A-bimodule compatible with τ or a B-bimodule compatible with τ^{-1}, since one often identifies $A \otimes_\tau B$ and the isomorphic algebra $B \otimes_{\tau^{-1}} A$ in practice.

Remark 2.10. An A-bimodule M is compatible with the twisting map τ exactly when there is a bijective k-linear map $\tau_{B,M} : B \otimes M \to M \otimes B$ making the following diagram commute:

(2.11)

A similar diagram expresses compatibility of a B-bimodule N with τ.

Example 2.12. Let $M = A$, an A-bimodule via multiplication. Then A is compatible with τ via $\tau_{B,A} = \tau$. Similarly $N = B$ is compatible with τ.

Bimodule structure. When M and N are compatible with τ, the tensor product $M \otimes N$ is naturally an $A \otimes_\tau B$-bimodule via the following composition of maps:

(2.13)

$$A \otimes_\tau B \otimes M \otimes N \otimes A \otimes_\tau B \xrightarrow{1 \otimes \tau_{B,M} \otimes \tau_{N,A} \otimes 1} A \otimes M \otimes B \otimes A \otimes N \otimes B$$

$$\xrightarrow{1 \otimes 1 \otimes \tau \otimes 1 \otimes 1} A \otimes M \otimes A \otimes B \otimes N \otimes B \xrightarrow{\rho_{A,M} \otimes \rho_{B,N}} M \otimes N \,.$$

Bimodule resolutions. For any k-algebra A, let $A^e = A \otimes A^{op}$ be its enveloping algebra, with A^{op} the opposite algebra to A. We view an A-bimodule M as a left A^e-module. In Lemma 3.1 below, we will construct a projective resolution of an $(A \otimes_\tau B)^e$-module $M \otimes N$ from individual resolutions of M and N using some compatibility conditions. Let $P_\bullet(M)$ be an A^e-projective resolution of M and let $P_\bullet(N)$ be a B^e-projective resolution of N:

(2.14) $\cdots \to P_2(M) \to P_1(M) \to P_0(M) \to M \to 0,$

(2.15) $\cdots \to P_2(N) \to P_1(N) \to P_0(N) \to N \to 0.$

Bar and reduced bar resolutions. For example, M could be A itself and $P_\bullet(A)$ could be the *bar resolution* $\mathrm{Bar}_\bullet(A)$ given by $\mathrm{Bar}_n(A) = A^{\otimes(n+2)}$ with differential

$$a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \;\mapsto\; \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $n \geq 0$ and $a_0, a_1, \ldots, a_{n+1} \in A$. We also use the *reduced bar resolution* $\overline{\mathrm{Bar}}_\bullet(A)$ with $\overline{\mathrm{Bar}}_n(A) = A \otimes \bar{A}^{\otimes n} \otimes A$ for $\bar{A} = A/k1_A$ and differential given by the same formula.

Compatibility conditions. We now define what it means for resolutions to be compatible with the twisting map τ. We tensor arbitrary resolutions (2.15) and (2.14) with A and B on the right and left to obtain complexes

$$P_\bullet(N) \otimes A, \; A \otimes P_\bullet(N), \; P_\bullet(M) \otimes B, \; \text{ and } B \otimes P_\bullet(M) \,.$$

Viewing these simply as exact sequences of vector spaces, we note that any k-linear maps $\tau_{N,A} : N \otimes A \to A \otimes N$ and $\tau_{B,M} : B \otimes M \to M \otimes B$ can be lifted to k-linear chain maps

(2.16) $\tau_{P_\bullet(N),A} : P_\bullet(N) \otimes A \to A \otimes P_\bullet(N) \text{ and } \tau_{B,P_\bullet(M)} : B \otimes P_\bullet(M) \to P_\bullet(M) \otimes B \,.$

For simplicity in the sequel, we will write $\tau_{i,A} = \tau_{P_i(N),A}$ and $\tau_{B,i} = \tau_{B,P_i(M)}$, for each i, when no confusion will arise. We will use such maps to glue the two resolutions together provided they satisfy the following compatibility conditions. These conditions just state that the chain maps commute with multiplication and with bimodule structure maps. There are many settings in which compatible chain maps do exist, as we will see.

Definition 2.17. Let M be an A-bimodule that is compatible with τ. A projective A-bimodule resolution $P_\bullet(M)$ is *compatible with the twisting map* τ if each $P_i(M)$ is compatible with τ via a map

$$\tau_{B,i} : B \otimes P_i(M) \longrightarrow P_i(M) \otimes B$$

with $\tau_{B,\bullet}$ a chain map lifting $\tau_{B,M}$. If A is graded, M is a graded A-bimodule, and $P_\bullet(M)$ is a graded projective A-bimodule resolution, we say that $P_\bullet(M)$ is *compatible with a strongly graded twisting map* τ if there are maps $\tau_{B,i}$ as above taking $B_j \otimes (P_i(M))_l$ to $(P_i(M))_l \otimes B_j$ for all j, l.

Remark 2.18. The above definition applies to B-bimodule resolutions as well as A-bimodule resolutions by reversing the role of A and B in the definition, again as $A \otimes_\tau B = B \otimes_{\tau^{-1}} A$. For a B-bimodule N that is compatible with τ, the definition implies that a projective B-bimodule resolution $P_\bullet(N)$ of N is *compatible with the twisting map* τ when each $P_i(N)$ is compatible with τ via a map $\tau_{i,A} : P_i(N) \otimes A \to A \otimes P_i(N)$, with $\tau_{\bullet,A}$ a chain map lifting $\tau_{N,A}$. Thus we say a resolution is compatible with τ if it is either an A-bimodule resolution or a B-bimodule resolution compatible with τ.

We give some small examples later: Example 2.21 (Weyl algebra) and Example 3.13 (skew group algebra). First a remark on embedding resolutions and some general results.

Remark 2.19. Note that compatibility is preserved under embedding of resolutions so long as the extensions of the twisting map τ preserve the embedding. Specifically, assume

$$\phi_\bullet : Q_\bullet(A) \hookrightarrow P_\bullet(A)$$

is an embedding of resolutions of the algebra A, and $P_\bullet(A)$ is compatible with a twisting map $\tau : B \otimes A \to A \otimes B$ via chain maps

$$\tau_{B,i} : B \otimes P_i(A) \to P_i(A) \otimes B.$$

If the maps $\tau_{B,i}$ preserve the embedding in the obvious sense that each $\tau_{B,i}$ restricts to a surjective map $B \otimes \mathrm{Im}(\phi) \twoheadrightarrow \mathrm{Im}(\phi) \otimes B$, then $Q_\bullet(A)$ is compatible with τ via these restrictions.

Compatibility of bar and Koszul resolutions. If A and B are both Koszul algebras and τ is a strongly graded twisting map, then the algebra $A \otimes_\tau B$ is known to be Koszul (see [14, Example 4.7.3], [12, Corollary 4.1.9], or [19, Proposition 1.8]). Conner and Goetz [4] examine the situation when τ is not strongly graded. We show next that both bar and Koszul resolutions are compatible with twisting maps. We always assume our Koszul algebras are connected graded algebras, so that they are quotients of tensor algebras on generating vector spaces in degree 1. Note that the roles of A and B may be exchanged in the next proposition.

Proposition 2.20. *Let τ be a twisting map for some k-algebras A and B.*
- (i) *The bar resolution $Bar_\bullet(A)$ is compatible with τ.*
- (ii) *The reduced bar resolution $\overline{Bar}_\bullet(A)$ is compatible with τ.*
- (iii) *If A is a Koszul algebra, B is a graded algebra, and τ is strongly graded, then the Koszul resolution $Kos_\bullet(A)$ is compatible with τ.*

Proof. (i) The bar resolution of A may be twisted by repeated application of the map τ, e.g., define $\tau_{B,i} : B \otimes A^{\otimes(i+2)} \to A^{\otimes(i+2)} \otimes B$ by applying τ to the first two tensor factors on the left, then applying τ to next two tensor factors, and so on:

$$\tau_{B,i} = (1 \otimes \cdots \otimes 1 \otimes \tau)(1 \otimes \cdots \otimes 1 \otimes \tau \otimes 1) \cdots (1 \otimes \tau \otimes 1 \otimes \cdots \otimes 1)(\tau \otimes 1 \otimes \cdots \otimes 1).$$

Then $\mathrm{Bar}_\bullet(A)$ is compatible with τ via $\tau_{B,i}$, as may be verified directly by repeated use of equation (2.1).

(ii) Write the terms in the bar complex $\mathrm{Bar}_\bullet(A)$ as $P_i = A^{\otimes(i+2)}$ for each i, and define the terms in the reduced bar complex $\overline{\mathrm{Bar}}_\bullet(A)$ by $\bar{P}_i = A \otimes \bar{A}^{\otimes i} \otimes A$. For each i, let T_i be the kernel of the quotient map $\mathrm{Bar}_i(A) \to \overline{\mathrm{Bar}}_i(A)$. Then $T_\bullet$ is a subcomplex of $\mathrm{Bar}_\bullet(A)$ and $\overline{\mathrm{Bar}}_\bullet(A) \cong \mathrm{Bar}_\bullet(A)/T_\bullet$. By definition of twisting map τ, the multiplicative identity 1_A commutes with elements of B under τ, implying that $\tau_{B,i}$ of part (i) takes $B \otimes T_i$ onto $T_i \otimes B$ for each i. Let $\bar{\tau}_{B,i} : B \otimes \overline{\mathrm{Bar}}_i(A) \to \overline{\mathrm{Bar}}_i(A) \otimes B$ be the corresponding map on quotients. Then $\overline{\mathrm{Bar}}_\bullet(A)$ is compatible with τ via the maps $\bar{\tau}_{B,i}$.

(iii) The proof of [19, Proposition 1.8] shows that the embedding $\mathrm{Kos}_\bullet(A) \hookrightarrow \mathrm{Bar}_\bullet(A)$ of bimodule resolutions is preserved by the iterated twisting in part (i) above (see Remark 2.19). Thus $\mathrm{Kos}_\bullet(A)$ satisfies compatibility. □

We give an example next showing how Proposition 2.20 can be used for Koszul resolutions even when the twisting map τ is not strongly graded.

Example 2.21. As in Example 2.2, let $\mathcal{W}$ be the Weyl algebra on x, y with $A = k[x]$ and $B = k[y]$. Let $\mathrm{Kos}_\bullet(A)$ be the Koszul resolution of A as an A-bimodule,

$$0 \to A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{m} A \to 0 \,,$$

where $V = \mathrm{Span}_k\{x\} \subset A$, $d_1(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$, and m is multiplication. Let $\bar{\tau} : B \otimes V \to V \otimes B$ be the swap map $b \otimes v \mapsto v \otimes b$ for all b in B and v in V, and define

$$\bar{\tau}_{B,\bullet} : B \otimes \mathrm{Kos}_\bullet(A) \to \mathrm{Kos}_\bullet(A) \otimes B$$

by iterations of τ and $\bar{\tau}$:

$$\bar{\tau}_{B,0} : B \otimes A \otimes A \xrightarrow{\ \tau \otimes 1\ } A \otimes B \otimes A \xrightarrow{\ 1 \otimes \tau\ } A \otimes A \otimes B \,, \quad \text{and}$$

$$\bar{\tau}_{B,1} : B \otimes A \otimes V \otimes A \xrightarrow{\ \tau \otimes 1 \otimes 1\ } A \otimes B \otimes V \otimes A$$

$$\xrightarrow{\ 1 \otimes \bar{\tau} \otimes 1\ } A \otimes V \otimes B \otimes A \xrightarrow{\ 1 \otimes 1 \otimes \tau\ } A \otimes V \otimes A \otimes B \,.$$

Define $\bar{\tau}_{\bullet,A} : \mathrm{Kos}_\bullet(B) \otimes A \to A \otimes \mathrm{Kos}_\bullet(B)$ similarly for the Koszul resolution $\mathrm{Kos}_\bullet(B)$ of B. Note that τ is not strongly graded, so part (iii) of Proposition 2.20 does not apply even though both A and B are Koszul algebras. Instead, we appeal to part (ii) and Remark 2.19 after taking canonical embeddings $\mathrm{Kos}_\bullet(A) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(B)$. (For example, view $A \otimes V \otimes A$ as a subspace of $A \otimes \bar{A} \otimes A$; the terms in other degrees are either 0 or the same as in the bar resolution.) The maps $\bar{\tau}_{B,\bullet}$ and $\bar{\tau}_{\bullet,A}$ above are the restrictions to $B \otimes \mathrm{Kos}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B) \otimes A$ of the maps of the same name in the proof of Proposition 2.20(ii) (after identifying V with its image under the quotient map $A \to \bar{A}$). In this way, we see that the Koszul resolutions $\mathrm{Kos}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B)$ are compatible with the twisting map τ via $\bar{\tau}_{B,\bullet}$ and $\bar{\tau}_{\bullet,A}$. We extend these ideas in Theorem 4.2.
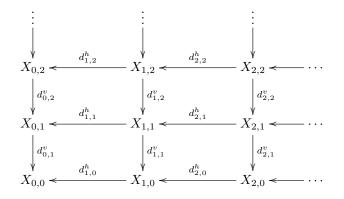
3. Twisted product resolutions for Bimodules

Again, we fix k-algebras A and B with a twisting map $\tau : B \otimes A \to A \otimes B$ and consider an A-bimodule M and B-bimodule N. We build a projective resolution of $M \otimes N$ as a bimodule over $A \otimes_\tau B$ from resolutions $P_\bullet(M)$ and $P_\bullet(N)$ under our compatibility assumptions. We give the construction in Lemma 3.1, prove exactness in Lemma 3.5, and show in Lemma 3.9 that the modules in the construction are indeed projective under an additional assumption.

Lemma 3.1. *Let M be an A-bimodule and let N be a B-bimodule, both compatible with a twisting map τ. Let $P_\bullet(M)$ and $P_\bullet(N)$ be projective A- and B-bimodule resolutions of M and N, respectively, that are compatible with τ. For each $i, j \geq 0$, let*

$$(3.2) \qquad X_{i,j} = P_i(M) \otimes P_j(N),$$

an $A \otimes_\tau B$-bimodule via diagram (2.13). Then $X_{\bullet,\bullet}$ is a bicomplex of $A \otimes_\tau B$-bimodules with horizontal and vertical differentials given by $d_{i,j}^h = d_i \otimes 1$ and $d_{i,j}^v = (-1)^i \otimes d_j$, where d_i and d_j denote the differentials of the appropriate resolutions:

Proof. The k-vector spaces $X_{i,j}$ form a tensor product bicomplex with differentials as stated. The bimodule action of $A \otimes_\tau B$ on $X_{i,j}$ commutes with the horizontal and vertical differentials since $\tau_{\bullet,B}$ and $\tau_{A,\bullet}$ are chain maps. Therefore this is an $A \otimes_\tau B$-bimodule bicomplex. $\qquad \square$

Definition 3.3. The *twisted product complex* $X_\bullet$ is the total complex of $X_{\bullet,\bullet}$, i.e., when augmented by $M \otimes N$, it is the complex

$$(3.4) \qquad \cdots \to X_2 \to X_1 \to X_0 \to M \otimes N \to 0$$

with $X_n = \oplus_{i+j=n} X_{i,j}$, and nth differential $\sum_{i+j=n} d_{i,j}$ where $d_{i,j} = d_i \otimes 1 + (-1)^i \otimes d_j$.

Lemma 3.5. *The twisted product complex (3.4) is exact.*

Proof. By the Künneth Theorem [20, Theorem 3.6.3], for each n there is an exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} \mathrm{H}_i\left(P_\bullet(M)\right) \otimes \mathrm{H}_j\left(P_\bullet(N)\right) \longrightarrow \mathrm{H}_n\left(P_\bullet(M) \otimes P_\bullet(N)\right)$$

$$\longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1^k\left(\mathrm{H}_i\left(P_\bullet(M)\right), \mathrm{H}_j\left(P_\bullet(N)\right)\right) \longrightarrow 0.$$

Now $P_\bullet(M)$ and $P_\bullet(N)$ are exact other than in degree 0, where they have homology M and N, respectively. Therefore

$$\mathrm{H}_i\left(P_\bullet(M)\right) = 0 \text{ for all } i > 0 \quad \text{and} \quad \mathrm{H}_j\left(P_\bullet(N)\right) = 0 \text{ for all } j > 0 \,.$$

The Tor term is 0 since k is a field. Thus for all $n > 0$, $\mathrm{H}_n\left(P_\bullet(M) \otimes P_\bullet(N)\right) = 0$, and

$$\mathrm{H}_0\left(P_\bullet(M) \otimes P_\bullet(N)\right) \cong \mathrm{H}_0\left(P_\bullet(M)\right) \otimes \mathrm{H}_0\left(P_\bullet(N)\right) \cong M \otimes N$$

as vector spaces. Thus the complex (3.4) is exact. $\qquad\qquad\square$

In practice, one often can show directly that each $X_{i,j}$ is projective as an $A \otimes_\tau B$-bimodule, for example, when working with bar resolutions and/or Koszul resolutions. For the general case, we need an extra compatibility assumption, which we explain next. As each $P_i(N)$ is a projective B-bimodule, it embeds into a free B^e-module $(B^e)^{\oplus J}$ for some indexing set J. In the following definition, we use the map $(\tau \otimes 1)(1 \otimes \tau) : B^e \otimes A \to A \otimes B^e$.

Definition 3.6. A chain map $\tau_{i,A} : P_i(N) \otimes A \to A \otimes P_i(N)$ is *compatible with a chosen embedding* $P_i(N) \hookrightarrow (B^e)^{\oplus J}$ (for some indexing set J) if the corresponding diagram is commutative:

Similarly, the map $\tau_{B,i}$ of (2.16) is *compatible with a chosen embedding* of $P_i(M)$ into a free A^e-module $(A^e)^{\oplus I}$ (for some indexing set I) if the corresponding diagram is commutative, i.e., if $\tau_{B,i}$ is the restriction of the map $((1 \otimes \tau)(\tau \otimes 1))^{\oplus I}$ to $B \otimes P_i(M)$.

Remark 3.7. In many settings, one sees directly that each $X_{i,j}$ is projective, in which case one need not consider this extra compatibility condition, as the next lemma is not needed. This is the case, for example, when twisting by a bicharacter on grading groups (see [1, Lemma 3.3]). In other settings, $\tau_{i,A}$ and $\tau_{B,i}$ are automatically compatible with chosen embeddings into free modules, for example if A and B are Koszul algebras and the embeddings are standard embeddings into bar resolutions (see [19, Proposition 1.8]).

Example 3.8. As in Examples 2.2 and 2.21, let $\mathcal{W} \cong A \otimes_\tau B$ be the Weyl algebra on x, y, $A = k[x]$, and $B = k[y]$. By construction, each map $\bar\tau_{i,A}$ is compatible with the canonical embedding $\mathrm{Kos}_i(A) \hookrightarrow \overline{\mathrm{Bar}}_i(A)$ and likewise $\bar\tau_{B,i}$ is compatible with $\mathrm{Kos}_i(B) \hookrightarrow \overline{\mathrm{Bar}}_i(B)$.

Lemma 3.9. *If $\tau_{B,i}$ and $\tau_{j,A}$ are compatible with chosen embeddings of $P_i(M)$ and $P_j(N)$ into free modules, then $X_{i,j} = P_i(M) \otimes P_j(N)$ is a projective $A \otimes_\tau B$-bimodule.*

Proof. First we verify the lemma in case $P_i(M) = A^e$, $P_j(N) = B^e$, and the chosen embeddings are the identity maps. In this case, $X_{i,j} = A^e \otimes B^e = A \otimes A^{op} \otimes B \otimes B^{op}$. One checks that the map

$$1 \otimes \tau \otimes 1 : \ A \otimes B \otimes (A \otimes B)^{op} \longrightarrow A \otimes A^{op} \otimes B \otimes B^{op}$$

is an isomorphism of $(A \otimes_\tau B)^e$-modules by equation (2.1) and the definition of the action given in the proof of Lemma 3.1. If $P_i(M)$ and $P_j(N)$ are arbitrary free modules, and the chosen embeddings are identity maps, we apply the above map to each summand $A^e \otimes B^e$ of $P_i(M) \otimes P_j(N)$ to see that $X_{i,j}$ is a free $(A \otimes_\tau B)^e$-module.

Now we consider the general case, including the possibility that at least one of $P_i(M)$, $P_j(N)$ is free but the corresponding chosen embedding into a (possibly different) free module is not the identity map. The first part of the proof together with the compatibility hypothesis implies that the embedding of k-vector spaces $P_i(M) \otimes P_j(N) \hookrightarrow (A^e)^{\oplus I} \otimes (B^e)^{\oplus J}$ given by the tensor product of the two embedding maps is a map of $(A \otimes_\tau B)^e$-modules. $\square$

We combine the lemmas to obtain the following theorem.

Theorem 3.10. *Let A and B be k-algebras, and let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Let M be an A-bimodule and N a B-bimodule with projective A- and B-bimodule resolutions $P_\bullet(M)$ and $P_\bullet(N)$, respectively. Assume that M, N, $P_\bullet(M)$, and $P_\bullet(N)$ are compatible with τ and the corresponding maps $\tau_{B,i}$ and $\tau_{j,A}$ are compatible with chosen embeddings of $P_i(M)$ and $P_i(N)$ into free modules. Then the twisted product complex with*

$$X_n = \oplus_{i+j=n} X_{i,j} \qquad for \qquad X_{i,j} = P_i(M) \otimes P_j(N)$$

gives a projective resolution of $M \otimes N$ as $A \otimes_\tau B$-bimodule:

$$\cdots \to X_2 \to X_1 \to X_0 \to M \otimes N \to 0.$$

Proof. The result follows from Lemmas 3.1, 3.5, and 3.9. $\square$

Remark 3.11. The theorem generally unifies known constructions of resolutions in several different contexts, for example, twisted tensor products given by bicharacters of grading groups [1], crossed products [9], skew group algebras (semidirect products) of Koszul algebras and finite groups [16], and smash products of Koszul algebras with Hopf algebras [18].

Theorem 3.10 combined with Proposition 2.20 and Remark 3.7 implies that a twisted product resolution of $A \otimes_\tau B$ as a bimodule always exists, since bar resolutions may always be twisted (and likewise Koszul resolutions, when one or both of the algebras is Koszul, see also [12, 14, 19]):

Corollary 3.12. *Let A and B be k-algebras with twisting map $\tau : B \otimes A \to B \otimes A$. The following are projective resolutions of $A \otimes_\tau B$ as a bimodule over itself.*

- *The twisted product complex of two bar resolutions.*
- *The twisted product complex of two Koszul resolutions when A and B are Koszul algebras and τ is strongly graded.*
- *The twisted product complex of one bar resolution and one Koszul resolution in case one of A or B is Koszul and the other is graded, for τ strongly graded.*

Moreover, bar resolutions may be replaced by reduced bar resolutions in the above statements.

Examples: Skew group algebras. We give some details for a class of examples introduced in Example 2.3. The resolutions in [16] for $S \rtimes G$, where G is a finite group acting by graded automorphisms on a Koszul algebra S, appear different from but are equivalent to (3.4) when $M = kG$ (the group algebra) and $N = S$. Note that $kG \otimes S$ is isomorphic to $S \rtimes G$ as an $(S \rtimes G)$-bimodule via the twisting map τ. In [16], the modules $X_{i,j}$ are given as

$$(S \rtimes G) \otimes C_i' \otimes D_j' \otimes (S \rtimes G)$$

where $P_i(kG) = kG \otimes C_i' \otimes kG$, $P_j(S) = S \otimes D_j' \otimes S$ are free $(kG)^e$- and S^e-modules determined by vector spaces C_i', D_j', respectively. We assume $P_i(kG)$ is G-graded and the grading is compatible with the kG-bimodule action. We assume $P_j(S)$ is a kG-module in such a way that the differentials are kG-module homomorphisms, and this action is compatible with that of S, so that $P_j(S)$ becomes an $S \rtimes G$-module. Compatibility with τ follows from these assumptions. There is an isomorphism of $S \rtimes G$-bimodules,

$$(kG \otimes C_i' \otimes kG) \otimes (S \otimes D_j' \otimes S) \xrightarrow{\sim} (S \rtimes G) \otimes C_i' \otimes D_j' \otimes (S \rtimes G),$$

similar to that used in the proof of [16, Theorem 4.3], given by

$$g \otimes x \otimes g' \otimes s \otimes y \otimes s' \mapsto g((hg')s) \otimes x \otimes (g'y) \otimes g's'$$

for all $g, g' \in G$, $s, s' \in S$, x in the h-component of C_i', and $y \in D_j'$.

Example 3.13. In particular, [16, Example 4.6] involves a resolution that is neither a Koszul resolution nor a bar resolution and yet satisfies compatibility. In that example, k is a field of positive characteristic p, $S = k[x, y]$, and $G = \langle g \rangle$ is a group of order p acting on S by $g \cdot x = x$, $g \cdot y = x + y$. The resolution $P_\bullet(S)$ is the Koszul resolution $\mathrm{Kos}_\bullet(S)$ of S,

$$0 \to S \otimes \textstyle\bigwedge^2 V \otimes S \to S \otimes \textstyle\bigwedge^1 V \otimes S \to S \otimes S \to S \to 0,$$

where $V = \mathrm{Span}_k\{x, y\}$. The resolution $P_\bullet(kG)$ is the bimodule resolution of kG,

$$(3.14) \qquad \cdots \xrightarrow{\eta \cdot} kG \otimes kG \xrightarrow{\gamma \cdot} kG \otimes kG \xrightarrow{\eta \cdot} kG \otimes kG \xrightarrow{\gamma \cdot} kG \otimes kG \xrightarrow{m} kG \longrightarrow 0,$$

where $\gamma = g \otimes 1 - 1 \otimes g$, $\eta = g^{p-1} \otimes 1 + g^{p-2} \otimes g + \cdots + 1 \otimes g^{p-1}$, and m is multiplication. Compatibility follows from Proposition 2.20(i) using Remark 2.19 after taking the standard embedding $\mathrm{Kos}_\bullet(S) \hookrightarrow \mathrm{Bar}_\bullet(S)$ and embedding (3.14) into $\mathrm{Bar}_\bullet(kG)$ (see, e.g., [3]).

4. Bimodule resolutions of Ore extensions

Many algebras of interest are Ore extensions of other algebras. We show how to twist bimodule resolutions for such extensions in this section.

Ore extensions as twisted tensor products. Let R be a k-algebra and fix a k-algebra automorphism σ of R. Let $\delta : R \to R$ be a left σ-derivation of R, that is,

$$(4.1) \qquad\qquad \delta(rs) = \delta(r)s + \sigma(r)\delta(s) \quad \text{for all } r, s \in R.$$

The *Ore extension* $R[x; \sigma, \delta]$ is the algebra with underlying vector space $R[x]$ and multiplication determined by that of R and of $k[x]$ and the additional Ore relation

$$xr = \sigma(r)x + \delta(r) \quad \text{for all } r \in R.$$

An Ore extension $R[x; \sigma, \delta]$ is thus isomorphic to a twisted tensor product $A \otimes_\tau B$ where $A = R$, $B = k[x]$, and the twisting map $\tau : B \otimes A \to A \otimes B$ satisfies

$$\tau(x \otimes r) = \sigma(r) \otimes x + \delta(r) \otimes 1 \quad \text{for all } r \in R.$$

Free resolutions for iterated Ore extensions. We will work with general Ore extensions in Section 6. Here for simplicity we restrict to the case that the automorphism on R is the identity, $\sigma = 1_R$, so the Ore relation sets commutators $xr - rx$ equal to elements in R. In this case, the Ore extension is also known as a *ring of formal differential operators*. We consider an *iterated Ore extension* $S = (\cdots(k[x_1][x_2; 1, \delta_2])\cdots)[x_t; 1, \delta_t]$, which we abbreviate as

$$S = k[x_1, \ldots, x_t; \delta_2, \ldots, \delta_t] = k\langle x_1, \ldots, x_t \rangle / (x_j x_i - x_i x_j - \delta_j(x_i) : 1 \le i < j \le t)$$

with $S \cong k[x_1, \ldots, x_t]$ as a k-vector space. We assume that S is a filtered algebra with $\deg(x_i) = 1$ for all i. Then each δ_j is a filtered map, i.e., $\delta_j(x_i) \in k \oplus k\text{-span}\{x_1, \ldots, x_{j-1}\}$ for $i < j$. This setting includes Weyl algebras and universal enveloping algebras of supersolvable Lie algebras.

Theorem 4.2. *Consider an iterated Ore extension $S = k[x_1, \ldots, x_t; \delta_2, \ldots, \delta_t]$ with identity automorphisms $\sigma_i = 1$ and filtered derivations δ_i. There is an iterated twisted product resolution $K_\bullet$ that is a free resolution of S as a bimodule over itself:*

$$K_n = S \otimes {\textstyle\bigwedge}^n V \otimes S$$

for $V = k\text{-span}\{x_1, \ldots, x_t\}$ with differentials given by (for $1 \le l_1 < \cdots < l_n \le t$)

$$
\begin{aligned}
d_n&(1 \otimes x_{l_1} \wedge \cdots \wedge x_{l_n} \otimes 1) \\
&= \sum_{1 \le i \le n} (-1)^{i+1} \big(x_{l_i} \otimes x_{l_1} \wedge \cdots \wedge \hat{x}_{l_i} \wedge \cdots \wedge x_{l_n} \otimes 1 - 1 \otimes x_{l_1} \wedge \cdots \wedge \hat{x}_{l_i} \wedge \cdots \wedge x_{l_n} \otimes x_{l_i} \big) \\
&\quad + \sum_{1 \le i < j \le n} (-1)^j \otimes x_{l_1} \wedge \cdots \wedge x_{l_{i-1}} \wedge \bar{\delta}_{l_j}(x_{l_i}) \wedge x_{l_{i+1}} \wedge \cdots \wedge \hat{x}_{l_j} \wedge \cdots \wedge x_{l_n} \otimes 1,
\end{aligned}
$$

where $\bar{\delta}_{l_j}(x_{l_i})$ is the image of $\delta_{l_j}(x_{l_i})$ under the projection $k \oplus V \twoheadrightarrow V$.

Proof. We induct on t. For each i, the Koszul resolution of $k[x_i]$ is embedded in the (reduced) bar resolution of $k[x_i]$ as

$$(4.3) \qquad 0 \to k[x_i] \otimes \text{Span}_k\{x_i\} \otimes k[x_i] \xrightarrow{d_1} k[x_i] \otimes k[x_i] \xrightarrow{m} k[x_i] \to 0,$$

where $d_1(1 \otimes x_i \otimes 1) = x_i \otimes 1 - 1 \otimes x_i$ and m is multiplication. For $t = i = 1$, the complex (4.3) is a resolution of S satisfying the statement of the theorem.

Now assume $t \ge 2$ and that the iterated Ore extension $A = k[x_1, \ldots, x_{t-1}; \delta_2, \ldots, \delta_{t-1}]$ has a free bimodule resolution $P_\bullet(A)$ as in the theorem. Let $B = k[x_t]$ and let $P_\bullet(B)$ be the Koszul resolution (4.3) for $i = t$. Then $S = A \otimes_\tau B$ where

$$\tau(x_t \otimes a) = a \otimes x_t + \delta_t(a) \otimes 1 \quad \text{for all } a \in A.$$

Embedding into the reduced bar resolution. We embed $P_\bullet(A)$ into the reduced bar resolution $\overline{\mathrm{Bar}}_\bullet(A)$ and then define twisting maps for $P_\bullet(A)$ via this embedding: Let $\phi_n : P_n(A) \to A^{\otimes(n+2)}$ be the standard symmetrization map defined by

$$\phi_n(1 \otimes x_{l_1} \wedge \cdots \wedge x_{l_n} \otimes 1) = \sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sgn}\,\sigma \otimes x_{l_{\sigma(1)}} \otimes \cdots \otimes x_{l_{\sigma(n)}} \otimes 1$$

for all $1 \leq l_1 < \cdots < l_n \leq t-1$. This is a chain map from $P_\bullet(A)$ to $\mathrm{Bar}_\bullet(A)$. Compose with the quotient map $\mathrm{Bar}_\bullet(A) \to \overline{\mathrm{Bar}}_\bullet(A)$ to obtain a chain map

$$\bar{\phi}_\bullet : P_\bullet(A) \longrightarrow \overline{\mathrm{Bar}}_\bullet(A)\,.$$

Note that the image of $P_\bullet(A)$ in the bar resolution $\mathrm{Bar}_\bullet(A)$, under $\phi_\bullet$, intersects the kernel of this quotient map trivially. Thus the induced map $\bar{\phi}_\bullet$ is injective.

Iterated twisting. The reduced bar resolution is compatibile with τ via the map

$$\bar{\tau}_{B,\bullet} : B \otimes \overline{\mathrm{Bar}}_\bullet(A) \longrightarrow \overline{\mathrm{Bar}}_\bullet(A) \otimes B$$

from the proof of Proposition 2.20(ii). We argue that $\bar{\tau}_{B,\bullet}$ restricts to a surjective map

$$\tilde{\tau}_{B,\bullet} : B \otimes P_\bullet(A) \longrightarrow P_\bullet(A) \otimes B$$

by verifying that it preserves the image of $\bar{\phi}_\bullet$, i.e., $\bar{\tau}_{B,n}$ takes $B \otimes \mathrm{Im}(\bar{\phi}_n)$ onto $\mathrm{Im}(\bar{\phi}_n) \otimes B$ for all n. We apply $\bar{\tau}_{B,n}$ to

$$x_t \otimes \bar{\phi}_n(a_0 \otimes y_1 \wedge \cdots \wedge y_n \otimes a_{n+1}) = \sum_{\pi \in \mathrm{Sym}_n} \mathrm{sgn}\,\pi\;(x_t \otimes a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1}),$$

for some a_0, a_{n+1} in A, in order to move x_t to the far right, obtaining

$$\Big(\sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi)\, a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1}\Big) \otimes x_t \in \mathrm{Im}(\bar{\phi}_n) \otimes B$$

plus additional terms that arise from the relation $\tau(x_t \otimes y_{\pi(i)}) = y_{\pi(i)} \otimes x_t + \delta_t(y_{\pi(i)}) \otimes 1$. (We use the same notation for elements of A and their images under the quotient map $A \to \bar{A}$ in cases where no confusion can arise.) Since $\tau(1 \otimes y_j) = y_j \otimes 1$ for all j, these additional terms sum to

$$\sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi)\, \delta_t(a_0) \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1} \otimes 1$$

$$+ \sum_{\pi \in \mathrm{Sym}_n} \sum_{1 \leq i \leq n} (\mathrm{sgn}\,\pi)\, a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes \bar{\delta}_t(y_{\pi(i)}) \otimes y_{\pi(i+1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1} \otimes 1$$

$$+ \sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi)\, a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes \delta_t(a_{n+1}) \otimes 1$$

$$= \bar{\phi}_n\big(\delta_t(a_0) \otimes y_1 \wedge \cdots \wedge y_n \otimes a_{n+1}\big) \otimes 1 \;+\; \bar{\phi}_n\big(a_0 \otimes y_1 \wedge \cdots \wedge y_n \otimes \delta_t(a_{n+1})\big) \otimes 1$$

$$+ \sum_{1 \leq i \leq n} \bar{\phi}_n\big(a_0 \otimes y_1 \wedge \cdots \wedge \bar{\delta}_t(y_i) \wedge y_{i+1} \wedge y_n \otimes a_{n+1}\big) \otimes 1 \;\in\; \mathrm{Im}(\bar{\phi}_n) \otimes B\,.$$

We may replace x_t by x_t^m in the above computation using induction after noting that $\tau(x_t^m \otimes x_i) = (1 \otimes m_B)\tau(x_t \otimes \big(\tau(x_t^{m-1} \otimes x_i)\big)$ for $i < t$. The above arguments can be

modified to apply to $\bar{\tau}_{B,i}^{-1}$ as well. Thus the chain map $\bar{\tau}_{B,\bullet}$ preserves the image of $\bar{\phi}_\bullet$ and restricts to a surjective chain map $\tilde{\tau}_{B,\bullet} : B \otimes P_\bullet(A) \longrightarrow P_\bullet(A) \otimes B$ as claimed.

Compatibility on one side. The complex $P_\bullet(A)$ inherits compatibility with τ from the compatibility of the reduced bar complex $\overline{\mathrm{Bar}}_\bullet(A)$ with τ. Indeed, since $\overline{\mathrm{Bar}}_\bullet(A)$ is compatible with τ via a map $\bar{\tau}_{B,\bullet}$ which preserves the embedding $\bar{\phi}_\bullet : P_\bullet(A) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(A)$, the complex $P_\bullet(A)$ is compatible with τ via the restriction $\tilde{\tau}_{B,\bullet}$ of $\bar{\tau}_{B,\bullet}$ to $B \otimes P_\bullet(A)$. (See Proposition 2.20(ii) and its proof and Remark 2.19.)

Compatibility on the other side. Define a chain map $\tau_{\bullet,A} : P_\bullet(B) \otimes A \to A \otimes P_\bullet(B)$ by setting $\tau_{0,A} = (\tau \otimes 1)(1 \otimes \tau)$ and

$$\tau_{1,A}((1 \otimes x_t \otimes 1) \otimes x_i) = x_i \otimes (1 \otimes x_t \otimes 1)$$

and then extending (uniquely) to $P_1(B) \otimes A$ by requiring that compatibility conditions (2.8) and (2.9) hold. A calculation shows that $\tau_{\bullet,A}$ is a chain map and that $P_\bullet(B)$ is compatible with τ. By their definitions, $\tau_{0,A}$ and $\tau_{1,A}$ are compatible with the embeddings of $P_0(B)$ and $P_1(B)$ into corresponding terms of the (reduced) bar resolution.

Twisted product resolution. By construction, the twisted product resolution $K_\bullet$ arising from $P_\bullet(A)$ and $P_\bullet(B)$ in degree n is isomorphic to $S \otimes \bigwedge^n V \otimes S$ as an S-bimodule via the isomorphisms

$$A \otimes {\textstyle\bigwedge}^i \mathrm{Span}_k\{x_1, \ldots, x_{t-1}\} \otimes A \otimes B \otimes {\textstyle\bigwedge}^j \mathrm{Span}_k\{x_t\} \otimes B$$
$$\xrightarrow{\sim} A \otimes B \otimes {\textstyle\bigwedge}^i \mathrm{Span}_k\{x_1, \ldots, x_{t-1}\} \otimes {\textstyle\bigwedge}^j \mathrm{Span}_k\{x_t\} \otimes A \otimes B \,,$$

for $j = 0, 1$, given by applying τ^{-1} (properly interpreted for each factor) to the innermost tensor factors A and B. We check the differentials: On $X_{n,0}$, the differential is just that arising from the factor $P_n(A)$. Now consider on $X_{n-1,1}$, again writing $x_{l_i} = y_i$ for some indices $1 \leq l_1 < \cdots < l_n \leq t-1$:

$$d_n(1 \otimes y_1 \wedge \cdots \wedge y_{n-1} \otimes 1 \otimes 1 \otimes x_t \otimes 1)$$
$$= \Big(\sum_{1 \leq i \leq n-1} (-1)^{i+1}\big(y_i \otimes y_1 \wedge \cdots \hat{y}_i \wedge \cdots \wedge y_{n-1} \otimes 1 - 1 \otimes y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_{n-1} \otimes y_i\big)$$
$$+ \sum_{1 \leq i < j \leq n-1} (-1)^j \otimes y_1 \wedge \cdots \wedge \bar{\delta}_j(y_i) \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_{n-1} \otimes 1 \Big) \otimes (1 \otimes x_t \otimes 1)$$
$$+ (-1)^{n-1}(1 \otimes y_1 \wedge \cdots \wedge y_{n-1} \otimes 1) \otimes (x_t \otimes 1 - 1 \otimes x_t) \,,$$

which may be rewritten, under the above isomorphism, as

$$\sum_{1\leq i\leq n-1} (-1)^{i+1} y_i \otimes y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_{n-1} \otimes x_t \otimes 1$$

$$- \sum_{1\leq i\leq n-1} (-1)^{i+1} \otimes y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_{n-1} \otimes x_t \otimes y_i$$

$$+ \sum_{1\leq i<j\leq n-1} (-1)^j \otimes y_1 \wedge \cdots \wedge \bar{\delta}_j(y_i) \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_{n-1} \otimes x_t \otimes 1$$

$$+ (-1)^{n-1} x_t \otimes y_1 \wedge \cdots \wedge y_{n-1} \otimes 1 + (-1)^n \otimes y_1 \wedge \cdots \wedge y_{n-1} \otimes x_t$$

$$+ (-1)^n \sum_{1\leq i\leq n-1} 1 \otimes y_1 \wedge \cdots \wedge \bar{\delta}_t(y_i) \wedge \cdots \wedge y_{n-1} \otimes 1\,.$$

Once one sets $y_n = x_t$, identifies $y_1 \wedge \cdots \wedge y_{n-1} \otimes x_t$ with $y_1 \wedge \cdots \wedge y_{n-1} \wedge x_t$, and makes other similar identifications, this agrees with the differential in the statement. $\square$

Examples. The theorem applies in particular to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite dimensional solvable Lie algebra $\mathfrak{g}$. Here, we assume the underlying field k is algebraically closed, else $\mathfrak{g}$ should be supersolvable; see [6, 1.3.14] and [2, Section 3]. The theorem gives a bimodule Koszul resolution of $\mathcal{U}(\mathfrak{g})$. Semisimple Lie algebras can then be handled via triangular decomposition. Other examples include Weyl algebras and Sridharan enveloping algebras [17].

5. TWISTED PRODUCT RESOLUTIONS FOR (LEFT) MODULES

We now consider a twisted product resolution of left modules instead of bimodules. We give the one-sided version of bimodule constructions in Sections 2 and 3. Again, we fix k-algebras A and B with a twisting map $\tau : B \otimes A \to A \otimes B$. In the constructions below, we consider compatible A-modules, but note that we as easily could have started with compatible B-modules instead of A-modules using the inverse twisting map τ^{-1} instead of τ in order to lift (left) modules of A and B to (left) modules of $A \otimes B = B \otimes_{\tau^{-1}} A$.

Let M be an A-module with module structure map $\rho_{A,M} : A \otimes M \to M$ and recall the multiplication map $m_B : B \otimes B \to B$.

Definition 5.1. The A-module M is *compatible with the twisting map* τ if there is a bijective k-linear map $\tau_{B,M} : B \otimes M \to M \otimes B$ such that

(5.2) $$\tau_{B,M}(m_B \otimes 1) = (1 \otimes m_B)(\tau_{B,M} \otimes 1)(1 \otimes \tau_{B,M}) \quad \text{and}$$
(5.3) $$\tau_{B,M}(1 \otimes \rho_{A,M}) = (\rho_{A,M} \otimes 1)(1 \otimes \tau_{B,M})(\tau \otimes 1)$$

as maps on $B \otimes B \otimes M$ and on $B \otimes A \otimes M$, respectively.

Note that this definition is equivalent to the commutativity of a diagram similar to (2.11), where $\rho_{A,M}$ is replaced by a one-sided module structure map.

Let N be a B-module with module structure map $\rho_{B,N} : B \otimes N \to N$. In case M is compatible with τ, the tensor product $M \otimes N$ may be given the structure of an $A \otimes_\tau B$-module via the following composition of maps:

$$(5.4) \qquad A \otimes_\tau B \otimes M \otimes N \xrightarrow{1 \otimes \tau_{B,M} \otimes 1} A \otimes M \otimes B \otimes N \xrightarrow{\rho_{A,M} \otimes \rho_{B,N}} M \otimes N .$$

Let $P_\bullet(M)$ be an A-projective resolution of M and $P_\bullet(N)$ a B-projective resolution of N:

$$\cdots \to P_2(M) \to P_1(M) \to P_0(M) \to k \to 0 ,$$
$$\cdots \to P_2(N) \to P_1(N) \to P_0(N) \to k \to 0 .$$

Definition 5.5. Let M be an A-module that is compatible with τ. The projective module resolution $P_\bullet(M)$ of the A-module M is *compatible with the twisting map* τ if each $P_i(M)$ is compatible with τ via maps $\tau_{B,i}$ for which $\tau_{B,\bullet} : B \otimes P_\bullet(M) \to P_\bullet(M) \otimes B$ is a k-linear chain map lifting $\tau_{B,M} : B \otimes M \to M \otimes B$.

Under the assumption of compatibility, we make the following definition.

Definition 5.6. Let M be an A-module compatible with τ and $P_\bullet(M)$ a projective resolution of M that is compatible with τ. Let N be a B-module. The *twisted product complex* $Y_\bullet$ is the total complex of the bicomplex $Y_{\bullet,\bullet}$ defined by

$$(5.7) \qquad\qquad\qquad\qquad Y_{i,j} = P_i(M) \otimes P_j(N) ,$$

with $A \otimes_\tau B$-module structure given by the maps $\tau_{B,\bullet}$ as in equation (5.4) and with vertical and horizontal differentials given by $d_{i,j}^h = d_i \otimes 1$ and $d_{i,j}^v = (-1)^i \otimes d_j$. In other words, $Y_n = \oplus_{i+j=n} Y_{i,j}$ with $d_n = \sum_{i+j=n} d_{i,j}$ where $d_{i,j} = d_{i,j}^h + d_{i,j}^v$.

Lemma 5.8. *Assume M and $P_\bullet(M)$ are compatible with τ. Then the twisted product complex $Y_\bullet$ is a complex of $A \otimes_\tau B$-modules.*
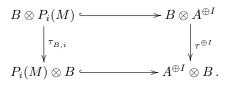
Proof. Each space $Y_{i,j}$ is given the structure of an $A \otimes_\tau B$-module via diagram (5.4). The differentials are module homomorphisms since $\tau_{B,\bullet}$ is a chain map. $\qquad\square$

Lemma 5.9. *The twisted product complex $\cdots \to Y_2 \to Y_1 \to Y_0 \to M \otimes N \to 0$ is exact.*

Proof. As in the proof of Lemma 3.5, apply the Künneth Theorem to obtain $\mathrm{H}_n(Y_\bullet) = 0$ for all $n > 0$ and $\mathrm{H}_0(Y_\bullet) \cong M \otimes N$. $\qquad\square$

We wish to prove in general that the modules $Y_{i,j}$ are projective, so we make an additional assumption in the next lemma. Since $P_\bullet(M)$ is a projective resolution of M as an A-module, each $P_i(M)$ embeds in a free A-module $A^{\oplus I}$.

Definition 5.10. For each $i \geq 0$, the map $\tau_{B,i}$ is *compatible with a chosen embedding* $P_i(M) \hookrightarrow A^{\oplus I}$ (for some indexing set I) if the corresponding diagram is commutative:

$$
\begin{array}{ccc}
B \otimes P_i(M) & \hookrightarrow & B \otimes A^{\oplus I} \\
\downarrow{\scriptstyle \tau_{B,i}} & & \downarrow{\scriptstyle \tau^{\oplus I}} \\
P_i(M) \otimes B & \hookrightarrow & A^{\oplus I} \otimes B .
\end{array}
$$

In many settings, one proves directly that the modules $Y_{i,j}$ are projective—e.g. the Ore extensions in the next section—and so one does not need this additional compatibility assumption, nor the next lemma.

Lemma 5.11. *For $i \geq 0$, if $\tau_{B,i}$ is compatible with a chosen embedding of $P_i(M)$ into a free A-module, then $Y_{i,j} = P_i(M) \otimes P_j(N)$ is a projective $A \otimes_\tau B$-module.*

Proof. By the hypothesis, it suffices to prove the lemma in case $P_i(A) = A$ and $P_j(B) = B$. In that case, $A \otimes B$ is the right regular module $A \otimes_\tau B$ by definition, and so is free. □

Combining Lemmas 5.8, 5.9, and 5.11, we obtain the following theorem.

Theorem 5.12. *Let A and B be k-algebras with twisting map $\tau : B \otimes A \to A \otimes B$. Let $P_\bullet(M)$ and $P_\bullet(N)$ be projective A- and B-module resolutions of M and N, respectively. Assume M and $P_\bullet(M)$ are compatible with τ and that the corresponding maps $\tau_{B,i}$ are compatible with chosen embeddings of $P_i(M)$ into free A-modules. Then the twisted product complex with*

$$Y_n = \oplus_{i+j=n} Y_{i,j} \qquad for \qquad Y_{i,j} = P_i(M) \otimes P_j(N)$$

gives a projective resolution of $M \otimes N$ as a module over the twisted tensor product $A \otimes_\tau B$:

$$\cdots \to Y_2 \to Y_1 \to Y_0 \to M \otimes N \to 0 \,.$$

Examples. Resolutions that may be constructed in this way include the Koszul resolution of k for a twisted tensor product of two Koszul algebras (see the proof of [19, Proposition 1.8]) and a resolution for a twisted tensor product of algebras whose twisting map is given by a bicharacter on grading groups (see [1]). We give another class of examples in the next section.

6. Resolutions for Ore extensions

In Section 4, we considered resolutions of an Ore extension algebra as a bimodule over itself. Here, we consider (left) modules over an Ore extension and show how to construct projective resolutions of these modules by regarding the Ore extension as a twisted tensor product. Gopalakrishnan and Sridharan [7] studied Ore extensions $R[x; \sigma, \delta]$ in case σ is the identity automorphism. They showed that if M is a (left) module over $R[x; 1, \delta]$, then an R-projective resolution of M lifts to an $R[x; 1, \delta]$-projective resolution. Here we allow arbitrary automorphisms σ of R and give conditions under which an R-projective resolution of an $R[x; \sigma, \delta]$-module M lifts to an $R[x; \sigma, \delta]$-projective resolution.

Again, let R be a k-algebra and σ a k-algebra automorphism of R. Let δ be a left σ-derivation of R (see (4.1)) and consider the Ore extension $R[x; \sigma, \delta]$. Let $A = R$, $B = k[x]$, and $\tau : B \otimes A \to A \otimes B$ be the twisting map determined by $\tau(x \otimes r) = \sigma(r) \otimes x + \delta(r) \otimes 1$ for all $r \in R$, as in Section 4, so that $R[x; \sigma, \delta]$ is the twisted tensor product $A \otimes_\tau B$.

Modules over Ore extensions. Consider an $R[x; \sigma, \delta]$-module M. Assume that on restriction to R, there is an isomorphism of R-modules, $\phi : M \xrightarrow{\sim} M^\sigma$, where M^σ is the vector space M with R-module action given by $r \cdot_\sigma m = \sigma(r) \cdot m$ for all $r \in R$ and $m \in M$. Then M is compatible with τ: We define $\tau_{B,M} := B \otimes M \to M \otimes B$ by setting

$$
\begin{aligned}
\tau_{B,M}(1 \otimes m) &= m \otimes 1 \,, \\
\tau_{B,M}(x \otimes m) &= \phi(m) \otimes x + xm \otimes 1 \qquad \text{for all} \quad m \in M
\end{aligned}
$$

and extending by applying compatibility condition (5.2). That is, since the algebra $B = k[x]$ is free on the generator x, for each element m of M, we may define $\tau_{B,M}(x^n \otimes m)$ by applying (5.2) to $x \otimes x^{n-1} \otimes m$. We check that (5.3) holds for elements of the form $x \otimes r \otimes m$, where $r \in R$ and $m \in M$. Then a careful induction on the power of x shows that (5.3) holds for all elements of the form $x^n \otimes r \otimes m$.

For example, if $R[x; \sigma, \delta]$ is an augmented algebra with augmentation $\varepsilon : R[x; \sigma, \delta] \to k$ for which $\varepsilon\sigma = \varepsilon$, then $\varepsilon\delta = 0$ and the field k as a module over $R[x; \sigma, \delta]$ via ε has the property that $k \cong k^\sigma$, and so k is compatible with τ.

Projective resolutions. Let $P_\bullet(M)$ be a projective resolution of M as an R-module:

$$
\cdots \xrightarrow{d_2} P_1(M) \xrightarrow{d_1} P_0(M) \xrightarrow{\mu} M \to 0.
$$

For each i, set $P_i^\sigma(M) = (P_i(M))^\sigma$. Then

$$
\cdots \xrightarrow{d_2} P_1^\sigma(M) \xrightarrow{d_2} P_0^\sigma(M) \xrightarrow{\phi^{-1}\mu} M \to 0
$$

is also a projective resolution of M as an R-module. By the Comparison Theorem, there is an R-module chain map from $P_\bullet(M)$ to $P_\bullet^\sigma(M)$ lifting the identity map $M \to M$, which we view as a k-linear chain map

$$
\text{(6.1)} \qquad\qquad\qquad \sigma_\bullet : P_\bullet(M) \to P_\bullet(M)
$$

with $\sigma_i(rz) = \sigma(r)\sigma_i(z)$ for all $i \geq 0$, $r \in R$, and $z \in P_i(M)$. We will assume for Theorem 6.6 below that each σ_i is bijective. Let $P_\bullet(B)$ be the Koszul resolution of k for $B = k[x]$,

$$
\text{(6.2)} \qquad\qquad\qquad 0 \to k[x] \xrightarrow{x\cdot} k[x] \xrightarrow{\epsilon} k \to 0 \,,
$$

where $\epsilon(x) = 0$. The following two lemmas are proven as in [7] (where Gopalakrishnan and Sridharan proved the special case $\sigma = 1$). We include details for completeness.

Lemma 6.3. *Let P be a projective R-module. There is an $R[x; \sigma, \delta]$-module structure on P that extends the action of R.*

Proof. First consider the case that $P = R$, the left regular module. Let x act on R by $x \cdot r = \delta(r)$ for all $r \in R$. One checks that the action of xr in $R[x; \sigma, \delta]$ agrees with that of $\sigma(r)x + \delta(r)$ on P, for all $r \in R$. Next, if P is a free module, it is a direct sum of copies of R, and x acts on each copy in this way. Finally, in general, P is a direct summand of a free R-module F. Let $\iota : P \to F$ and $\pi : F \to P$ be R-module homomorphisms for which $\pi\iota$ is the identity map. Define $x \cdot p = \pi(x \cdot \iota(p))$ for all $p \in P$, where the action of x on $\iota(p)$ is as given previously for a free module. Again one checks that the actions of xr and of $\sigma(r)x + \delta(r)$ agree, and so P is an $R[x; \sigma, \delta]$-module as claimed. $\qquad \square$

Compatibility requirements. We will use the next lemma to show that the resolution $P_\bullet(M)$ of M as an R-module is compatible with the twisting map τ (see Lemma 6.5). Let $f : M \to M$ be the function given by the action of x on the $R[x; \sigma, \delta]$-module M.

Lemma 6.4. *There is a k-linear chain map $\delta_\bullet : P_\bullet(M) \to P_\bullet(M)$ lifting $f : M \to M$ such that for each $i \geq 0$, $\delta_i(rz) = \sigma(r)\delta_i(z) + \delta(r)z$ for all $r \in R$ and $z \in P_i(M)$.*

Proof. If $i = 0$, let δ_0' be the action of x on $P_0(M)$ given by Lemma 6.3. Then
$$\delta_0'(rz) - \sigma(r)\delta_0'(z) = \delta(r)z$$
for $r \in R$, $z \in P_0(M)$. One checks that $\mu\delta_0' - f\mu : P_0(M) \to M^\sigma$ is an R-module homomorphism. As $P_0(M)$ is a projective R-module, there is an R-module homomorphism $\delta_0'' : P_0(M) \to P_0^\sigma(M)$ such that $\mu\delta_0' - f\mu = \mu\delta_0''$. Let $\delta_0 = \delta_0' - \delta_0''$. One may check this satisfies the equation in the lemma.

Now fix $i > 0$ and assume there are k-linear maps $\delta_j : P_j(M) \to P_j(M)$ such that $\delta_j(rz) = \sigma(r)\delta_j(z) + \delta(r)z$ and $d_j\delta_j = \delta_{j-1}d_j$ for all j, $0 \leq j < i$, and $r \in R$, $z \in P_j(M)$. Let $\delta_i' : P_i(M) \to P_i(M)$ be the action of x on $P_i(M)$ given in Lemma 6.3, so that $\delta_i'(rz) = \sigma(r)\delta_i'(z) + \delta(r)z$ for all $r \in R$, $z \in P_i(M)$. Consider the map
$$d_i\delta_i' - \delta_{i-1}d_i : P_i(M) \longrightarrow P_{i-1}^\sigma(M) \,.$$
A calculation shows that it is an R-module homomorphism. Since δ_{i-1} is a chain map,
$$d_{i-1}(d_i\delta_i' - \delta_{i-1}d_i) = 0 \,,$$
and so the image of $d_i\delta_i' - \delta_{i-1}d_i$ lies in $\text{Ker}\,(d_{i-1}) = \text{Im}(d_i)$. Since $P_i(M)$ is projective as an R-module, there is an R-homomorphism $\delta_i'' : P_i(M) \to P_i^\sigma(M)$ such that $d_i\delta_i' - \delta_{i-1}d_i = d_i\delta_i''$. Let $\delta_i = \delta_i' - \delta_i''$, so that $d_i\delta_i = \delta_{i-1}d_i$ by construction. One checks that for all $r \in R$ and $z \in P_i(M)$,
$$\delta_i(rz) = \delta_i'(rz) - \delta_i''(rz) = \sigma(r)\delta_i'(z) + \delta(r)z - \sigma(r)\delta_i''(z) = \sigma(r)\delta_i(z) + \delta(r)z \,.$$
$\square$

Lemma 6.5. *The resolution $P_\bullet(M)$ is compatible with the twisting map τ.*

Proof. Define $\tau_{B,i} : B \otimes P_i(M) \to P_i(M) \otimes B$ by
$$\begin{aligned}
\tau_{B,i}(1 \otimes z) &= z \otimes 1 \,, \\
\tau_{B,i}(x \otimes z) &= \sigma_i(z) \otimes x + \delta_i(z) \otimes 1 \qquad \text{for all} \quad z \in P_i(M) \,,
\end{aligned}$$
where $\sigma_\bullet$ is the chain map of (6.1), $\delta_\bullet$ is the chain map of Lemma 6.4, and we extend $\tau_{B,i}$ to $B \otimes P_i(M)$ as before by requiring that compatibility conditions (5.2) and (5.3) hold. We check condition (5.3) in one case as an example:
$$\tau_{B,i}(x \otimes rz) = \sigma_i(rz) \otimes x + \delta_i(rz) \otimes 1 = \sigma(r)\sigma_i(z) \otimes x + \sigma(r)\delta_i(z) \otimes 1 + \delta(r)z \otimes 1 \,,$$
for all $r \in R$, and $z \in P_i(M)$, while on the other hand,
$$\begin{aligned}
(\rho_{A,i} &\otimes 1)(1 \otimes \tau_{B,i})(\tau \otimes 1)(x \otimes r \otimes z) \\
&= (\rho_{A,i} \otimes 1)(1 \otimes \tau_{B,i})(\sigma(r) \otimes x \otimes z + \delta(r) \otimes 1 \otimes z) \\
&= (\rho_{A,i} \otimes 1)(\sigma(r) \otimes \sigma_i(z) \otimes x + \sigma(r) \otimes \delta_i(z) \otimes 1 + \delta(r) \otimes z \otimes 1) \\
&= \sigma(r)\sigma_i(z) \otimes x + \sigma(r)\delta_i(z) \otimes 1 + \delta(r)z \otimes 1 \,;
\end{aligned}$$

Condition (5.3) holds for all $x^n \otimes rz$ by induction on n. $\square$

Twisting resolutions for an Ore extension. We now construct a projective resolution of M as an $R[x; \sigma, \delta]$-module from a projective resolution of M as an R-module. We take the twisted product of two resolutions: the R-projective resolution of M and the Koszul resolution (6.2) of k as a module over $B = k[x]$.

Theorem 6.6. *Let $R[x; \sigma, \delta]$ be an Ore extension. Let M be an $R[x; \sigma, \delta]$-module for which $M^\sigma \cong M$ as R-modules. Consider a projective resolution $P_\bullet(M)$ of M as an R-module and suppose that each map $\sigma_i : P_i(M) \to P_i(M)$ of (6.1) is bijective. For each $i \geq 0$, set*

$$Y_{i,0} = Y_{i,1} = P_i(M) \otimes k[x] \quad and \quad Y_{i,j} = 0 \quad for \ all \ j > 1$$

as in Lemma 5.8. Then $Y_\bullet$ is a projective resolution of M as an $R[x; \sigma, \delta]$-module.

Proof. By Lemma 6.5, $P_\bullet(M)$ is compatible with τ, and so by Lemmas 5.8 and 5.9, the complex $\cdots \to Y_1 \to Y_0 \to M \to 0$ is an exact complex of $R[x; \sigma, \delta]$-modules. We verify directly that each $Y_{i,j}$ is a projective module: For each $i \geq 0$ and $j = 0, 1$,

$$(6.7) \hspace{4cm} Y_{i,j} \cong R[x; \sigma, \delta] \otimes_R P_i(M)$$

via the $R[x; \sigma, \delta]$-homomorphism given by

$$R[x; \sigma, \delta] \otimes_R P_i(M) \longrightarrow Y_{i,j}, \qquad x \otimes z \mapsto \sigma_i(z) \otimes x + \delta_i(z) \otimes 1,$$

with inverse map given by

$$z \otimes x \mapsto x \otimes \sigma_i^{-1}(z) - 1 \otimes \delta_i(\sigma_i^{-1}(z)).$$

Then $R[x; \sigma, \delta] \otimes_R P_i(M)$ is projective since it is a tensor-induced module and $R[x; \sigma, \delta]$ is flat over R. $\square$

Remark 6.8. When σ is the identity, the complex $Y_\bullet$ is precisely that of Gopalakrishnan and Sridharan [7, Theorem 1], under the isomorphism (6.7) above. As a specific class of examples, we obtain in this way, via iterated Ore extension, the Chevalley-Eilenberg resolution of the $\mathcal{U}(\mathfrak{g})$-module k for a finite dimensional supersolvable Lie algebra $\mathfrak{g}$.

REFERENCES

[1] P. A. Bergh and S. Oppermann, "Cohomology of twisted tensor products," J. Algebra 320 (2008), 3327–3338.
[2] K.A. Brown, S. O'Hagan, J.J. Zhang, and G. Zhuang, "Connected Hopf algebras and iterated Ore extensions", J. Pure Appl. Algebra 219 (2015), no. 6, 2405–2433.
[3] The Buenos Aires Cyclic Homology Group, "Cyclic homology of algebras with one generator," K-Theory 5 (1991), 51–69.
[4] A. Conner and P. Goetz, "The Koszul Property for graded twisted tensor products," submitted. See https://arxiv.org/pdf/1708.02514.pdf.
[5] A. Čap, H. Schichl, and J. Vanžura, "On twisted tensor products of algebras," Comm. Algebra 23 (1995), no. 12, 4701–4735.

[6] J. Dixmier, *Enveloping Algebras.* (English Summary) Revised reprint of the 1977 translation. Graduate Studies in Mathematics, 11. American Mathematical Society, Providence, RI, 1996. xx+379 pp. ISBN: 0-8218-0560-6.

[7] N. S. Gopalakrishnan and R. Sridharan, "Homological dimension of Ore-extensions," Pacific J. Math. 19 (1966), no. 1, 67–75.

[8] J. A. Guccione and J. J. Guccione, "Hochschild homology of twisted tensor products," K-Theory 18 (1999), 363–400.

[9] J. A. Guccione and J. J. Guccione, "Hochschild (co)homology of Hopf crossed products," K-Theory 25 (2002), no. 2, 139–169.

[10] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer-Verlag, New York, 1971.

[11] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montréal, 1988.

[12] P. Jara Martinez, J. López Piña, and D. Ştefan, "Koszul pairs and applications," arxiv:1011.4243.

[13] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, 82, Regional Conference Series in Mathematics, AMS, Providence, RI, 1993.

[14] A. Polishchuk and L. Positselski, *Quadratic Algebras*, University Lecture Series 37, Amer. Math. Soc., 2005.

[15] D. E. Radford and H. J. Schneider, "Biproducts and two-cocycle twists of Hopf algebras," *Modules and Comodules*, 331–355, Trends Math., Birkhäuser Verlag, Basel, 2008.

[16] A.V. Shepler and S. Witherspoon, "A Poincaré-Birkhoff-Witt Theorem for quadratic algebras with group actions," Trans. Amer. Math. Soc. 366 (2014), no. 12, 6483–6506.

[17] R. Sridharan, "Filtered algebras and representations of Lie algebras," Trans. Amer. Math. Soc. 100 (1961), 530–550.

[18] C. Walton and S. Witherspoon, "Poincaré-Birkhoff-Witt deformations of smash product algebras from Hopf actions on Koszul algebras," Algebra and Number Theory 8 (2014), no. 7, 1701–1731.

[19] C. Walton and S. Witherspoon, "PBW deformations of braided products," J. Algebra 504 (2018), 536–567.

[20] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, volume 38, Cambridge University Press, Cambridge, 1994.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TEXAS 76203, USA
E-mail address: ashepler@unt.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843, USA
E-mail address: sjw@math.tamu.edu