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RESTRICTED TWO-PARAMETER QUANTUM GROUPS

Georgia Benkart
Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706, USA
benkart@math.wisc.edu

Sarah Witherspoon
Department of Mathematics and Computer Science
Amherst College
Amherst, Massachusetts 01002, USA
sjw@cs.amherst.edu

Abstract. We construct a family of finite-dimensional Hopf algebras from two-parameter quantum groups and show that these Hopf algebras are pointed and are Drinfel'd doubles. Using their left and right integrals, we determine necessary and sufficient conditions for them to possess ribbon elements.

1 Introduction

Hopf algebras first appeared in Hopf's pioneering work on cohomology and Lie groups. Their study as algebraic entities began in earnest in the mid 1960s, and it continues today because major open problems in the theory remain, and because new applications of Hopf algebras to diverse topics such as knot theory, Lie theory, and combinatorics continue to be found. A longstanding problem in the area is the classification of the finite-dimensional Hopf algebras. Kaplansky's ten conjectures [24] have been the impetus for much research on the subject, and there have been a number of significant advances in the last ten years. Many classification results have been proven under constraints on the dimension or under the assumption that the Hopf algebra is semisimple, pointed, or triangular. In [45], Zhu gave a proof of Kaplansky's 8th conjecture – a Hopf algebra of dimension p (a prime) over an algebraically closed field of characteristic 0 must be the group algebra of a cyclic group of order p (see also [23]). The semisimple Hopf algebras of dimension p^2, $2p$, p^3, pq (p and q distinct primes) and 16 have been classified in [30], [31], [32], [16], [25] (see also [34]), and it is known that there are only finitely many types of semisimple and cosemisimple Hopf algebras of a given dimension (see [42]). Triangular, semisimple and cosemisimple Hopf algebras over an algebraically closed field of any characteristic must be a twist of a group algebra, as shown in [16], [17].

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Since the complete classification of the finite-dimensional Hopf algebras still seems very far off, it is quite useful to have various means of constructing examples of finite-dimensional Hopf algebras. Iterated Ore extensions provide all the pointed Hopf algebras of dimension p^n with commutative coradical of dimension p^{n-1} ([7], [8]) (see also [1], [4], [5], [14] for further work on pointed Hopf algebras and extensions). Pointed Hopf algebras also play an essential role in [3], [6], [19], where it is proven that there can exist infinitely many nonisomorphic Hopf algebras of a given prime power dimension, thereby answering Kaplansky's 10th conjecture in the negative.

The starting point for our investigations is the family of two-parameter special linear quantum groups $U_{r,s}(\mathfrak{sl}_n)$ introduced by Takeuchi [43] and studied in [12], [13]. We assume here that r is a primitive dth root of unity and s is a primitive d'th root of unity, and let ℓ denote the least common multiple of d and d'. We construct a finite-dimensional pointed Hopf algebra $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ of dimension $\ell^{(n+2)(n-1)}$ as a quotient of $U_{r,s}(\mathfrak{sl}_n)$ by a Hopf ideal I_n, which is generated by certain central elements. The restricted quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ are analogues of the restricted enveloping algebras of Lie algebras of prime characteristic and the restricted one-parameter quantum groups $\mathfrak{u}_q(\mathfrak{g})$ of Lusztig [28], [29] for q a root of unity (see also [20]), which play a pivotal part in the representation theory of algebraic groups and Lie algebras of characteristic $p > 0$.

Our results on the skew-primitive elements of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ in Section 3 enable us to determine when two such restricted two-parameter quantum groups are isomorphic. In Section 4, we prove that $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a Drinfel'd double of a certain Hopf subalgebra $\mathfrak{b}$. Previously, we have shown in [12] that the infinite-dimensional algebra $U_{r,s}(\mathfrak{sl}_n)$ is always a Drinfel'd double for any choice of r and s, and in [13], have used the corresponding R-matrix to investigate the representation theory of $U_{r,s}(\mathfrak{sl}_n)$. In Section 5, we determine the left and right integrals of $\mathfrak{b}$ and use these results in conjunction with some from [26] to give necessary and sufficient conditions for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ to have a ribbon element. In the one-parameter setting such a criterion has been found by Gelaki and Westreich [21].

One of the most studied finite-dimensional Hopf algebras is the Taft algebra A_ℓ, which has generators a, x satisfying the relations

$$a^\ell = 1, \qquad x^\ell = 0, \qquad ax = \theta xa,$$

where θ is a primitive ℓth root of unity. Its coalgebra structure is determined by

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = x \otimes a + 1 \otimes x.$$

A nonsemisimple Hopf algebra of dimension p^2 (p a prime) is isomorphic to the Taft algebra A_p (see [2], [35]). Extending an earlier result of Hennings [22], Kauffman and Radford [26, Prop. 7] have shown that the Drinfel'd double $D(A_\ell)$ of the Taft algebra A_ℓ has a ribbon element if and only if ℓ is odd. In this case, $D(A_\ell^{\mathrm{coop}}) \cong \mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_2)$, where q^2 is a primitive ℓth root of unity, and thus the results of our paper give natural generalizations of well-known, useful facts. Indeed, the ribbon element of $D(A_\ell)$ (ℓ odd) provides an important invariant of 3-manifolds (see [22]). It has played a major role in the investigations of Kauffman and Radford on quantum algebras and link invariants. Although much of the motivation for our investigations came from the desire to produce invariants of framed links embedded in 3-dimensional space from ribbon Hopf algebras, we do not address that issue here.

2 Central elements and restricted quantum groups

First we recall the definition of the two-parameter quantum groups from [12], and some basics about their structure. Let $\epsilon_1, \ldots, \epsilon_n$ denote an orthonormal basis of a Euclidean space $E = \mathbb{R}^n$ with an inner product $\langle\ ,\ \rangle$. Set $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \ldots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\}$. Then Φ is a finite root system of type A_{n-1} with Π a base of simple roots.

Throughout we fix nonzero elements r, s in a field $\mathbb{K}$ with $r \ne s$.

Let $U_{r,s}(\mathfrak{sl}_n)$ be the unital associative algebra over $\mathbb{K}$ generated by elements $e_j,\ f_j, (1 \le j < n)$, and $\omega_i^{\pm 1},\ (\omega_i')^{\pm 1}\quad (1 \le i < n)$, which satisfy the following relations.

(R1) The $\omega_i^{\pm 1},\ (\omega_j')^{\pm 1}$ all commute with one another and $\omega_i\omega_i^{-1} = \omega_j'(\omega_j')^{-1} = 1$,

(R2) $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j\rangle} s^{\langle \epsilon_{i+1}, \alpha_j\rangle} e_j \omega_i$ and $\omega_i f_j = r^{-\langle \epsilon_i, \alpha_j\rangle} s^{-\langle \epsilon_{i+1}, \alpha_j\rangle} f_j \omega_i$,

(R3) $\omega_i' e_j = r^{\langle \epsilon_{i+1}, \alpha_j\rangle} s^{\langle \epsilon_i, \alpha_j\rangle} e_j \omega_i'$ and $\omega_i' f_j = r^{-\langle \epsilon_{i+1}, \alpha_j\rangle} s^{-\langle \epsilon_i, \alpha_j\rangle} f_j \omega_i'$,

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i')$.

(R5) $[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if} \quad |i-j| > 1$,

(R6) $e_i^2 e_{i+1} - (r+s)e_i e_{i+1} e_i + rs e_{i+1} e_i^2 = 0$,
$e_i e_{i+1}^2 - (r+s)e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i = 0$,

(R7) $f_i^2 f_{i+1} - (r^{-1}+s^{-1}) f_i f_{i+1} f_i + r^{-1}s^{-1} f_{i+1} f_i^2 = 0$,
$f_i f_{i+1}^2 - (r^{-1}+s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1}s^{-1} f_{i+1}^2 f_i = 0$.

The subalgebra generated by the two elements e_i, e_{i+1} (or f_i, f_{i+1}) for $i = 1, \ldots, n-2$ is isomorphic to the down-up algebra $A(r+s, -rs, 0)$ of [9], [11]. Down-up algebras generalize the algebra generated by the down and up operators on a partially ordered set [41], [18]. Such operators encode essential enumerative and structural information about the partially ordered set.

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega_i' - \omega_i^{-1},\ \ 1 \le i < n$, is isomorphic to the one-parameter quantum special linear group $U_q(\mathfrak{sl}_n)$.

The algebra $U = U_{r,s}(\mathfrak{sl}_n)$ is a Hopf algebra, where the $\omega_i^{\pm 1}, (\omega_i')^{\pm 1}$ are group-like elements, and the remaining Hopf structure is given by

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i',$$
$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \qquad S(e_i) = -\omega_i^{-1} e_i, \qquad S(f_i) = -f_i(\omega_i')^{-1}.$$

Moreover, U has a triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$, where U^0 is the subalgebra generated by the elements $\omega_i^{\pm 1}, (\omega_i')^{\pm 1}$, and U^+ (resp. U^-) is the subalgebra generated by the elements e_i (resp. f_i). Applying Gröbner-Shirshov basis techniques, Benkart, Kang, and Lee [10] have constructed a monomial PBW-type basis for U^+ as follows:

Suppose

$$\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1}\mathcal{E}_{i-1,j} e_i \quad (i > j). \tag{2.1}$$

Then the defining relations for U^+ in (R6) can be reformulated as saying

$$\begin{aligned} e_{i+1}\mathcal{E}_{i+1,i} &= s^{-1}\mathcal{E}_{i+1,i} e_{i+1} \\ \mathcal{E}_{i+1,i} e_i &= s^{-1} e_i \mathcal{E}_{i+1,i}. \end{aligned} \tag{2.2}$$

Theorem 2.3 [10]

(1) *The set*

$$\{\mathcal{E}_{i_1,j_1}\mathcal{E}_{i_2,j_2}\cdots\mathcal{E}_{i_p,j_p} \mid (i_1,j_1)\le(i_2,j_2)\le\cdots\le(i_p,j_p) \text{ lexicographically}\}$$

is a linear basis of the algebra U^+.

(2) *The set*

$$\{e_{i_1,j_1}e_{i_2,j_2}\cdots e_{i_p,j_p} \mid (i_1,j_1)\le(i_2,j_2)\le\cdots\le(i_p,j_p) \text{ lexicographically}\}$$

is a linear basis of the algebra U^+, *where* $e_{i,j}=e_ie_{i-1}\cdots e_j$ *for* $i\ge j$.

We consider the Hopf subalgebra B (the so-called Borel subalgebra) of U generated by the elements ω_i, e_i, $(1\le i<n)$, and the Hopf subalgebra B' generated by the elements ω_i', f_i, $(1\le i<n)$. From the defining relations, it is easy to check that $(B')^{\mathrm{coop}}$ (that is, the algebra B' but with the opposite coproduct) is isomorphic to the Borel subalgebra $B_{s^{-1},r^{-1}}$ of $U_{s^{-1},r^{-1}}(\mathfrak{sl}_n)$ generated by its ω_i and e_i elements. Explicitly, the isomorphism $\psi:(B')^{\mathrm{coop}}\to B_{s^{-1},r^{-1}}$ is given by $f_i\mapsto e_i$, $\omega_i'\mapsto\omega_i$. Since the computations to prove Theorem 2.3 just involve the relations in (R6) (or equivalently, those in (2.2)), we see that a monomial basis for U^- can be constructed as follows. Let

$$\mathcal{F}_{j,j}=f_j \quad\text{and}\quad \mathcal{F}_{i,j}=f_i\mathcal{F}_{i-1,j}-s\mathcal{F}_{i-1,j}f_i \quad (i>j). \tag{2.4}$$

Corollary 2.5

(1) *The set*

$$\{\mathcal{F}_{i_1,j_1}\mathcal{F}_{i_2,j_2}\cdots\mathcal{F}_{i_p,j_p} \mid (i_1,j_1)\le(i_2,j_2)\le\cdots\le(i_p,j_p) \text{ lexicographically}\}$$

is a linear basis of the algebra U^-.

(2) *The set*

$$\{f_{i_1,j_1}f_{i_2,j_2}\cdots f_{i_p,j_p} \mid (i_1,j_1)\le(i_2,j_2)\le\cdots\le(i_p,j_p) \text{ lexicographically}\}$$

is a linear basis of the algebra U^-, *where* $f_{i,j}=f_if_{i-1}\cdots f_j$ *for* $i\ge j$.

Henceforth we restrict the parameters r *and* s *to be roots of unity:* r *is a primitive* d*th root of unity,* s *is a primitive* d'*th root of unity, and* ℓ *is the least common multiple of* d *and* d'. *We will assume further that* $\mathbb{K}$ *contains a primitive* ℓ*th root of unity.*

Our first goal in this section is to establish the following theorem:

Theorem 2.6 *The elements* $\mathcal{E}_{k,l}^\ell$, $\mathcal{F}_{k,l}^\ell$ $(1\le l\le k<n)$ *and* $\omega_k^\ell-1$, $(\omega_k')^\ell-1$ $(1\le k<n)$ *are central in* $U_{r,s}(\mathfrak{sl}_n)$.

Towards this end, we will need the r,s-integers, factorials, and binomial coefficients defined for positive integers c and d by

$$[c]:=\frac{r^c-s^c}{r-s},\qquad [c]!:=[c]\,[c-1]\cdots[2]\,[1],\qquad \begin{bmatrix} c\\ d\end{bmatrix}:=\frac{[c]!}{[d]!\,[c-d]!}. \tag{2.7}$$

By convention $[0]=0$ and $[0]!=1$. Also we will need the following commutation relations determined in [10] along with some of their consequences.

Proposition 2.8 [10, Thm. 3.4] *Assume $(k,l) > (i,j)$ in the lexicographic order. Then the following relations hold*:

$$\begin{array}{lll} (1) & \mathcal{E}_{k,l}\mathcal{E}_{i,j} - r^{-1}\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,j} = 0 & \text{if } \ l = i+1, \\ (2) & \mathcal{E}_{k,l}\mathcal{E}_{i,j} - \mathcal{E}_{i,j}\mathcal{E}_{k,l} = 0 & \text{if } \ k > i \geq j > l \text{ or } \ l > i+1, \\ (3) & \mathcal{E}_{k,l}\mathcal{E}_{i,j} - s^{-1}\mathcal{E}_{i,j}\mathcal{E}_{k,l} = 0 & \text{if } \ k = i \geq l > j \text{ or } \ k > i \geq l = j. \end{array}$$

Corollary 2.9 *For $k \geq l$ and any positive integer a,*

$$\mathcal{E}_{k,l}^{a} e_{l-1} = r^{-a} e_{l-1} \mathcal{E}_{k,l}^{a} + \frac{r^{-a} - s^{-a}}{r^{-1} - s^{-1}} \mathcal{E}_{k,l-1} \mathcal{E}_{k,l}^{a-1} \tag{2.10}$$

$$e_{k+1} \mathcal{E}_{k,l}^{a} = r^{-a} \mathcal{E}_{k,l}^{a} e_{k+1} + \frac{r^{-a} - s^{-a}}{r^{-1} - s^{-1}} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k+1,l} \tag{2.11}$$

Proof An inductive argument (with $i = j = l-1$ in (1) of Proposition 2.8 for the $a = 1$ case) proves (2.10). Similarly, (2.11) can be done by induction with the $a = 1$ case coming from (1) of Proposition 2.8 by switching the roles of the pairs (i,j) and (k,l), and then setting $i = j = k+1$. □

Our proof of Theorem 2.6 now proceeds in a series of steps.

Step 1. *$\mathcal{E}_{k,l}^{\ell}$ commutes with e_j for all $k \geq l$ and $j = 1, \ldots, n-1$.*

Proof When j satisfies either $k > j > l$ or $l - 1 > j$, the assertion follows directly from (2) of Proposition 2.8. Switching the roles of (k,l) and (i,j) in (2) and then letting $i = j$ shows that $\mathcal{E}_{k,l}$ commutes with e_j whenever $j > k+1$, so the result is clear in that case. Setting $i = j = l$ in (3) of Proposition 2.8 gives $\mathcal{E}_{k,l} e_l = s^{-1} e_l \mathcal{E}_{k,l}$ for $k > l$, and using that relation ℓ times gives the desired conclusion. Similarly, the case $j = k > l$ may be done by interchanging the pairs (i,j) and (k,l) in (3) and then taking $i = j = k$. (When $k = l$, the fact that $\mathcal{E}_{k,l}^{\ell}$ commutes with e_k is obvious.) What remains after these considerations are the cases $j = l-1$ and $j = k+1$. But these are simple consequences of (2.10) and (2.11) with $a = \ell$. □

Step 2. *$\mathcal{E}_{k,l}^{\ell}$ commutes with ω_j and ω_j' for all $k \geq l$ and $j = 1, \ldots, n-1$.*

Proof We have

$$\omega_j \mathcal{E}_{k,l} = r^{\langle \epsilon_j, \alpha_k + \cdots + \alpha_l \rangle} s^{\langle \epsilon_{j+1}, \alpha_k + \cdots + \alpha_l \rangle} \mathcal{E}_{k,l} \omega_j,$$

from which it is apparent that $\omega_j \mathcal{E}_{k,l}^{\ell} = \mathcal{E}_{k,l}^{\ell} \omega_j$. (Note that when $k = l$, the coefficient is simply $r^{\langle \epsilon_j, \alpha_k \rangle} s^{\langle \epsilon_{j+1}, \alpha_k \rangle}$.) The argument for ω_j' is equally easy. □

Step 3. *Assume $k > l$, and let a be any positive integer. Then*

(i) $\mathcal{E}_{k,l}^{a} f_k = f_k \mathcal{E}_{k,l}^{a} + r^{-a+1} [a] \dfrac{r^{-1} - s^{-1}}{r - s} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k-1,l} \omega_k'$, *and*

(ii) $\mathcal{E}_{k,l}^{a} f_l = f_l \mathcal{E}_{k,l}^{a} + s^{-a+1} [a] \dfrac{1 - r^{-1}s}{r - s} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k,l+1} \omega_l$.

In particular, $\mathcal{E}_{k,l}^{\ell}$ commutes with both f_k and f_l.

Proof (i) First we set $a = 1$ and calculate:

$$\begin{aligned}
\mathcal{E}_{k,l} f_k &= e_k \mathcal{E}_{k-1,l} f_k - r^{-1} \mathcal{E}_{k-1,l} e_k f_k \\
&= \left(f_k e_k + \frac{\omega_k - \omega_k'}{r-s} \right) \mathcal{E}_{k-1,l} - r^{-1} \mathcal{E}_{k-1,l} \left(f_k e_k + \frac{\omega_k - \omega_k'}{r-s} \right) \\
&= f_k \mathcal{E}_{k,l} + \frac{\omega_k - \omega_k'}{r-s} \mathcal{E}_{k-1,l} - r^{-1} \mathcal{E}_{k-1,l} \frac{\omega_k - \omega_k'}{r-s} \\
&= f_k \mathcal{E}_{k,l} + \frac{r^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k - \frac{s^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k' - \frac{r^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k + \frac{r^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k' \\
&= f_k \mathcal{E}_{k,l} + \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k'.
\end{aligned}$$

Now assuming (i) holds for $a - 1$, we have the computation for a:

$$\begin{aligned}
\mathcal{E}_{k,l}^a f_k &= \mathcal{E}_{k,l} \left(f_k \mathcal{E}_{k,l}^{a-1} + r^{-a+2} [a-1] \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k,l}^{a-2} \mathcal{E}_{k-1,l} \omega_k' \right) \\
&= \left(f_k \mathcal{E}_{k,l} + \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k-1,l} \omega_k' \right) \mathcal{E}_{k,l}^{a-1} + r^{-a+2} [a-1] \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k-1,l} \omega_k' \\
&= f_k \mathcal{E}_{k,l}^a + (r^{-a+2} [a-1] + r^{-a+1} s^{a-1}) \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k-1,l} \omega_k' \\
&= f_k \mathcal{E}_{k,l}^a + r^{-a+1} [a] \frac{r^{-1} - s^{-1}}{r-s} \mathcal{E}_{k,l}^{a-1} \mathcal{E}_{k-1,l} \omega_k'.
\end{aligned}$$

(ii) We suppose initially that $a = 1$ and $k = l + 1$ and show:

$$\begin{aligned}
\mathcal{E}_{l+1,l} f_l &= (e_{l+1} e_l - r^{-1} e_l e_{l+1}) f_l \\
&= e_{l+1} \left(f_l e_l + \frac{\omega_l - \omega_l'}{r-s} \right) - r^{-1} \left(f_l e_l + \frac{\omega_l - \omega_l'}{r-s} \right) e_{l+1} \\
&= f_l \mathcal{E}_{l+1,l} + e_{l+1} \frac{\omega_l - \omega_l'}{r-s} - \frac{r^{-1} s}{r-s} e_{l+1} \omega_l + \frac{1}{r-s} e_{l+1} \omega_l' \\
&= f_l \mathcal{E}_{l+1,l} + \frac{1 - r^{-1} s}{r-s} e_{l+1} \omega_l.
\end{aligned}$$

In the calculations for the inductive step below, we will apply the following helpful identity

$$[x, [y, z]_{r^{-1}}] = [[x, y], z]_{r^{-1}} + [y, [x, z]]_{r^{-1}} \tag{2.12}$$

where $[x, y]_{r^{-1}} = xy - r^{-1} yx$.

Now assume (ii) is true for $a = 1$ and $k - 1$, and use (2.12) to compute:

$$\begin{aligned}[f_l, \mathcal{E}_{k,l}] &= [f_l, [e_k, \mathcal{E}_{k-1,l}]_{r^{-1}}] \\ &= [[f_l, e_k], \mathcal{E}_{k-1,l}]_{r^{-1}} + [e_k, [f_l, \mathcal{E}_{k-1,l}]]_{r^{-1}} \\ &= -[e_k, \frac{1-r^{-1}s}{r-s}\mathcal{E}_{k-1,l+1}\omega_l]_{r^{-1}} \\ &= r^{-1}\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k-1,l+1}\omega_l e_k - \frac{1-r^{-1}s}{r-s}e_k\mathcal{E}_{k-1,l+1}\omega_l \\ &= -\frac{1-r^{-1}s}{r-s}[e_k, \mathcal{E}_{k-1,l+1}]_{r^{-1}}\omega_l \\ &= -\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l+1}\omega_l.\end{aligned}$$

Finally, assume (ii) is true for arbitrary $k > l$ and $a-1$. Then

$$\begin{aligned}\mathcal{E}_{k,l}^a f_l &= \mathcal{E}_{k,l}\left(f_l\mathcal{E}_{k,l}^{a-1} + s^{-a+2}[a-1]\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l}^{a-2}\mathcal{E}_{k,l+1}\omega_l\right) \\ &= \left(f_l\mathcal{E}_{k,l} + \frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l+1}\omega_l\right)\mathcal{E}_{k,l}^{a-1} + s^{-a+2}[a-1]\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l}^{a-1}\mathcal{E}_{k,l+1}\omega_l \\ &= f_l\mathcal{E}_{k,l}^a + (s^{-a+2}[a-1] + r^{a-1}s^{-a+1})\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l}^{a-1}\mathcal{E}_{k,l+1}\omega_l \\ &= f_l\mathcal{E}_{k,l}^a + s^{-a+1}[a]\frac{1-r^{-1}s}{r-s}\mathcal{E}_{k,l}^{a-1}\mathcal{E}_{k,l+1}\omega_l.\end{aligned}$$

□

Step 4. *$\mathcal{E}_{k,l}^{\ell}$ commutes with f_j for all $k \geq l$ and $j = 1, \ldots, n-1$.*

Proof If $j > k$ or $j < l$, this statement is immediate from the defining relations; while if $j = k$ or $j = l$, it is the content of Step 3. We may suppose $k > j > l$, so that by (2.12),

$$[f_j, \mathcal{E}_{k,l}] = \frac{1}{r-s}[e_k, \cdots[[e_{j+1}, [(\omega_j' - \omega_j), \mathcal{E}_{j-1,l}]_{r^{-1}}]_{r^{-1}}\cdots]_{r^{-1}}, \tag{2.13}$$

while

$$\begin{aligned}&[e_{j+1}, [(\omega_j' - \omega_j), \mathcal{E}_{j-1,l}]_{r^{-1}}]_{r^{-1}} \\ &\quad= e_{j+1}(\omega_j' - \omega_j)\mathcal{E}_{j-1,l} - r^{-1}e_{j+1}\mathcal{E}_{j-1,l}(\omega_j' - \omega_j) \\ &\qquad -r^{-1}(\omega_j' - \omega_j)\mathcal{E}_{j-1,l}e_{j+1} + r^{-2}\mathcal{E}_{j-1,l}(\omega_j' - \omega_j)e_{j+1} \\ &\quad= e_{j+1}\Big(\omega_j' - \omega_j - r^{-1}s\omega_j' + \omega_j \\ &\qquad -\omega_j' + r^{-1}s\omega_j + r^{-1}s\omega_j' - r^{-1}s\omega_j\Big)\mathcal{E}_{j-1,l} \\ &\quad= 0.\end{aligned}$$

Thus $[f_j, \mathcal{E}_{k,l}] = 0$ in this case, and hence Step 4 is known to hold for $k > l$.

Finally, the following formula from [13, Lemma 2.3] may be proved by induction:

$$e_k^a f_k = f_k e_k^a + [a]e_k^{a-1}\frac{s^{1-a}\omega_k - r^{1-a}\omega_k'}{r-s},$$

and so e_k^{ℓ} commutes with f_k. As e_k commutes with f_j for $j \neq k$, we have finished the proof of Step 4. □

Step 5. *Conclusion of the proof of Theorem 2.6.*

Proof We know from Steps 1-4 that the elements $\mathcal{E}_{k,l}^{\ell}$ are central in $U_{r,s}(\mathfrak{sl}_n)$ for all $k \geq l$. Reversing the roles of e_k and f_k (and interchanging r, s with s^{-1}, r^{-1}), we see that $\mathcal{F}_{k,l}^{\ell}$ is also central. It is easy to check directly that $\omega_k^{\ell} - 1$ and $(\omega_k')^{\ell} - 1$ are central also, so the proof is complete. □

Remark 2.14 For a one-parameter quantum group $U_q(\mathfrak{g})$ with q a primitive ℓth root of unity, DeConcini and Kac [15] have shown that the ℓth power of each of the generators is central and have used that fact in an essential way to develop the representation theory of those algebras.

Definition 2.15 Let I_n denote the ideal of $U_{r,s}(\mathfrak{sl}_n)$ generated by all $\mathcal{E}_{k,l}^{\ell}$, $\mathcal{F}_{k,l}^{\ell}$, $\omega_k^{\ell} - 1$, $(\omega_k')^{\ell} - 1$ $(1 \leq \ell \leq k < n)$. The *restricted two-parameter quantum group* is the quotient

$$\mathfrak{u}_{r,s}(\mathfrak{sl}_n) := U_{r,s}(\mathfrak{sl}_n)/I_n.$$

By Theorem 2.3 and Corollary 2.5, $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is an algebra of dimension $\ell^{(n+2)(n-1)}$ with linear basis all

$$\mathcal{E}_{i_1,j_1}^{a_1} \cdots \mathcal{E}_{i_p,j_p}^{a_p} \omega_1^{b_1} \cdots \omega_{n-1}^{b_{n-1}} (\omega_1')^{b_1'} \cdots (\omega_{n-1}')^{b_{n-1}'} \mathcal{F}_{i_1',j_1'}^{a_1'} \cdots \mathcal{F}_{i_p',j_p'}^{a_p'} \tag{2.16}$$

where $(i_1, j_1) < \cdots < (i_p, j_p)$ and $(i_1', j_1') < \cdots < (i_p', j_p')$ lexicographically, and all powers range between 0 and $\ell - 1$.

The remainder of the section is devoted to proving

Theorem 2.17 *The ideal I_n is a Hopf ideal, so that $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a finite-dimensional Hopf algebra.*

First note that the generators of I_n are contained in the kernel of the counit ε, and so I_n is as well. Because the coproduct Δ is an algebra homomorphism, and the antipode S is an algebra antihomomorphism, it suffices to show that $\Delta(x) \in I_n \otimes U + U \otimes I_n$ (where $U = U_{r,s}(\mathfrak{sl}_n)$) and $S(x) \in I_n$ for each of the generators x of I_n. To accomplish this, we rely on the following computations:

$$\begin{aligned} \Delta(\omega_i^{\ell} - 1) &= \omega_i^{\ell} \otimes \omega_i^{\ell} - 1 \otimes 1 \\ &= \omega_i^{\ell} \otimes \omega_i^{\ell} - 1 \otimes \omega_i^{\ell} + 1 \otimes \omega_i^{\ell} - 1 \otimes 1 \\ &= (\omega_i^{\ell} - 1) \otimes \omega_i^{\ell} + 1 \otimes (\omega_i^{\ell} - 1) \in I_n \otimes U + U \otimes I_n, \end{aligned}$$

and $S(\omega_i^{\ell} - 1) = \omega_i^{-\ell} - 1 = -\omega_i^{-\ell}(\omega_i^{\ell} - 1) \in I_n$. The argument for $(\omega_i')^{\ell} - 1$ is similar. In determining $\Delta(\mathcal{E}_{k,l}^{\ell})$ and $\Delta(\mathcal{F}_{k,l}^{\ell})$, we adopt the following notational conventions:

$$\omega_{k,l} := \omega_k \omega_{k-1} \cdots \omega_l \quad \text{and} \quad \zeta := 1 - r^{-1}s. \tag{2.18}$$

Lemma 2.19

$$\Delta(\mathcal{E}_{k,l}) = \mathcal{E}_{k,l} \otimes 1 + \omega_{k,l} \otimes \mathcal{E}_{k,l} + \zeta \sum_{j=l}^{k-1} \mathcal{E}_{k,j+1} \omega_{j,l} \otimes \mathcal{E}_{j,l}.$$

Proof The statement is true when $k = l$, as this just amounts to the coproduct of e_k in that case. Assume it is true for $k-1$. Then as $\mathcal{E}_{k,l} = e_k\mathcal{E}_{k-1,l} - r^{-1}\mathcal{E}_{k-1,l}e_l$,

$$\begin{aligned}
\Delta(\mathcal{E}_{k,l}) &= \Delta(e_k)\Delta(\mathcal{E}_{k-1,l}) - r^{-1}\Delta(\mathcal{E}_{k-1,l})\Delta(e_k) \\
&= (e_k \otimes 1 + \omega_k \otimes e_k)\left(\mathcal{E}_{k-1,l} \otimes 1 + \omega_{k-1,l} \otimes \mathcal{E}_{k-1,l} + \zeta \sum_{j=l}^{k-2} \mathcal{E}_{k-1,j+1}\omega_{j,l} \otimes \mathcal{E}_{j,l}\right) \\
&\quad -r^{-1}\left(\mathcal{E}_{k-1,l} \otimes 1 + \omega_{k-1,l} \otimes \mathcal{E}_{k-1,l} + \zeta \sum_{j=l}^{k-2} \mathcal{E}_{k-1,j+1}\omega_{j,l} \otimes \mathcal{E}_{j,l}\right)(e_k \otimes 1 + \omega_k \otimes e_k) \\
&= \mathcal{E}_{k,l} \otimes 1 + \omega_{k,l} \otimes \mathcal{E}_{k,l} + \zeta e_k\omega_{k-1,l} \otimes \mathcal{E}_{k-1,k} + \zeta \sum_{j=l}^{k-2} \mathcal{E}_{k,j+1}\omega_{j,l} \otimes \mathcal{E}_{j,l},
\end{aligned}$$

and combining all but the first two terms into one sum, we obtain the desired result. □

Next we generalize Lemma 2.19 to an expression for $\Delta(\mathcal{E}_{k,l}^{a})$ in Lemma 2.22 below. The formula involves certain exponents p_m of s that are defined recursively on ordered m-tuples of nonnegative integers, as follows:

$$\begin{aligned}
p_m(0,\ldots,0) &:= 0 \\
p_m(c_1+1, c_2, \ldots, c_m) &:= p_m(c_1, c_2, \ldots, c_i) - c_2 - c_3 - \cdots - c_m \\
p_m(c_1, c_2, \ldots, c_{m-1}, c_m+1) &:= p_m(c_1, c_2, \ldots, c_m) - c_1 - c_2 - \cdots - c_{m-1} \\
p_m(c_1, c_2, \ldots, c_j+1, \ldots, c_m) &:= p_m(c_1, c_2, \ldots, c_m) - c_1 - c_2 - \cdots - c_{j-1} \\
&\qquad + c_j - c_{j+1} - \cdots - c_m \quad (1 < j < m).
\end{aligned}$$

An inductive argument on $c_1 + \cdots + c_m$ shows that p_m is well-defined.

The following r,s-multinomial formula may be checked directly.

Lemma 2.20 *Let $c_1, \ldots, c_m$ be positive integers. Then*

$$\begin{aligned}
&\begin{bmatrix} c_1 + \cdots + c_m \\ c_m \end{bmatrix}\begin{bmatrix} c_1 + \cdots + c_{m-1} \\ c_{m-1} \end{bmatrix}\cdots\begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} = \\
&\qquad \sum_{j=1}^{m} r^{C_j}s^{D_j}\begin{bmatrix} c_1 + \cdots + c_m - 1 \\ c_m \end{bmatrix}\cdots\begin{bmatrix} c_1 + \cdots + c_j - 1 \\ c_j - 1 \end{bmatrix}\cdots\begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}.
\end{aligned}$$

where $C_j = c_{j+1} + c_{j+2} + \cdots + c_m$, $D_j = c_1 + \cdots + c_{j-1}$, and $C_m = 0 = D_1$.

Remark 2.21 In the above sum, the binomial coefficients have 1 subtracted from their top number to the one with $c_1 + \cdots + c_j$ on top and not thereafter, as in the jth summand we have replaced c_j by $c_j - 1$.

Lemma 2.22 *Let $1 \leq l \leq k < n$, and $m = k - l + 2$. Then*

$$\Delta(\mathcal{E}_{k,l}^a) = \sum s^{p_m(c_1,\ldots,c_m)}\zeta^{a-c_1-c_m} \begin{bmatrix} c_1+\cdots+c_m \\ c_m \end{bmatrix} \begin{bmatrix} c_1+\cdots+c_{m-1} \\ c_{m-1} \end{bmatrix} \cdots \begin{bmatrix} c_1+c_2 \\ c_2 \end{bmatrix} \times \mathcal{E}_{k,l}^{c_1}\mathcal{E}_{k,l+1}^{c_2}\cdots\mathcal{E}_{k,k}^{c_{m-1}}\omega_{l,l}^{c_2}\omega_{l+1,l}^{c_3}\cdots\omega_{k,l}^{c_m} \otimes \mathcal{E}_{l,l}^{c_2}\mathcal{E}_{l+1,l}^{c_3}\cdots\mathcal{E}_{k,l}^{c_m},$$

the sum taken over all m-tuples $(c_1,\ldots,c_m)$ of nonnegative integers with $c_1+\cdots+c_m = a$.

Proof Observe first that if $a = 1$, the above expression coincides with that in Lemma 2.19. To simplify notation, we will assume without loss of generality that $l = 1$ (and hence $m = k+1$). Now suppose that the above formula holds for $a-1$, that is

$$\Delta(\mathcal{E}_{k,1}^{a-1}) = \sum s^{p_{k+1}(c_1,\ldots,c_{k+1})}\zeta^{a-1-c_1-c_{k+1}} \begin{bmatrix} c_1+\cdots+c_{k+1} \\ c_{k+1} \end{bmatrix} \begin{bmatrix} c_1+\cdots+c_k \\ c_k \end{bmatrix} \cdots \begin{bmatrix} c_1+c_2 \\ c_2 \end{bmatrix} \times \mathcal{E}_{k,1}^{c_1}\mathcal{E}_{k,2}^{c_2}\cdots\mathcal{E}_{k,k}^{c_k}\omega_{1,1}^{c_2}\omega_{2,1}^{c_3}\cdots\omega_{k,1}^{c_{k+1}} \otimes \mathcal{E}_{1,1}^{c_2}\mathcal{E}_{2,1}^{c_3}\cdots\mathcal{E}_{k,1}^{c_{k+1}},$$

where the sum is over all $(k+1)$-tuples $(c_1,\ldots,c_{k+1})$ with $c_1+\cdots+c_{k+1} = a-1$.

Then $\Delta(\mathcal{E}_{k,1}^a) = \Delta(\mathcal{E}_{k,1}^{a-1})\Delta(\mathcal{E}_{k,1})$, which by Lemma 2.19 is equal to the above sum times

$$\mathcal{E}_{k,1} \otimes 1 + \zeta \sum_{j=1}^{k-1} \mathcal{E}_{k,j+1}\omega_{j,1} \otimes \mathcal{E}_{j,1} + \omega_{k,1} \otimes \mathcal{E}_{k,1}.$$

Expanding and re-summing over all $(d_1,\ldots,d_{k+1})$ with $d_1+\cdots+d_{k+1} = a$, we find that the coefficient of $\mathcal{E}_{k,1}^{d_1}\mathcal{E}_{k,2}^{d_2}\cdots\mathcal{E}_{k,k}^{d_k}\omega_{1,1}^{d_2}\cdots\omega_{k,1}^{d_{k+1}} \otimes \mathcal{E}_{1,1}^{d_2}\cdots\mathcal{E}_{k,1}^{d_{k+1}}$ is

$$s^{p_{k+1}(d_1,\ldots,d_{k+1})}\zeta^{a-d_1-d_{k+1}} \times \left\{ \sum_{j=1}^{k+1} r^{C'_j}s^{D'_j} \begin{bmatrix} d_1+\cdots+d_{k+1}-1 \\ d_{k+1} \end{bmatrix} \cdots \begin{bmatrix} d_1+\cdots+d_j-1 \\ d_j-1 \end{bmatrix} \cdots \begin{bmatrix} d_1+d_2 \\ d_2 \end{bmatrix} \right\}$$

where $C'_j = d_{j+1}+d_{j+2}+\cdots+d_{k+1}$, $D'_j = d_1+\cdots+d_{j-1}$, and $C'_{k+1} = 0 = D'_1$. By Lemma 2.20, the desired formula results. □

As a special case of the lemma, we have the following formula for $e_k = \mathcal{E}_{k,k}$:

$$\Delta(e_k^a) = \sum_{j=0}^{a} s^{j(j-a)} \begin{bmatrix} a \\ j \end{bmatrix} e_k^j\omega_k^{a-j} \otimes e_k^{a-j}. \tag{2.23}$$

Completion of the Proof of Theorem 2.17. We now have the tools to finish the proof that I_n is a Hopf ideal. As $[\ell] = 0$, it follows from Lemma 2.22 that the only nonzero terms of $\Delta(\mathcal{E}_{k,l}^\ell)$ are those having $c_p = \ell$ for some p. As a result,

$$\Delta(\mathcal{E}_{k,l}^\ell) = \mathcal{E}_{k,l}^\ell \otimes 1 + \omega_{k,l}^\ell \otimes \mathcal{E}_{k,l}^\ell + s^{\ell(\ell-1)/2}\zeta^\ell \sum_{p=l}^{k-1} \mathcal{E}_{k,p+1}^\ell\omega_{p,l}^\ell \otimes \mathcal{E}_{p,l}^\ell, \tag{2.24}$$

which is clearly in $I_n \otimes U + U \otimes I_n$ (for $U = U_{r,s}(\mathfrak{sl}_n)$). Applying the antipode property to (2.24), we obtain

$$0 = \varepsilon(\mathcal{E}_{k,l}^{\ell}) = \mathcal{E}_{k,l}^{\ell} + \omega_{k,l}^{\ell} S(\mathcal{E}_{k,l}^{\ell}) + s^{\ell(\ell-1)/2}\zeta^{\ell} \sum_{p=l}^{k-1} \mathcal{E}_{k,p+1}^{\ell} \omega_{p,l}^{\ell} S(\mathcal{E}_{p,l}^{\ell}).$$

Therefore

$$S(\mathcal{E}_{k,l}^{\ell}) = -\omega_{k,l}^{-\ell} \left(\mathcal{E}_{k,l}^{\ell} + s^{\ell(\ell-1)/2}\zeta^{\ell} \sum_{p=l}^{k-1} \mathcal{E}_{k,p+1}^{\ell} \omega_{p,l}^{\ell} S(\mathcal{E}_{p,l}^{\ell}) \right). \tag{2.25}$$

If $k = l$, we have $S(\mathcal{E}_{k,l}^{\ell}) = S(e_k^{\ell}) = (-\omega_k^{-1} e_k)^{\ell}$, which is a scalar multiple of $\omega_k^{-\ell} e_k^{\ell} \in I_n$. Using (2.25) and induction on $k - l$, we see that $S(\mathcal{E}_{k,l}^{\ell}) \in I_n$ for all $k > l$ as well. A similar argument applies to $\mathcal{F}_{k,l}^{\ell}$, using the isomorphism $(B'_{r,s})^{\mathrm{coop}} \cong B_{s^{-1},r^{-1}}$ described in the text following Theorem 2.3. By applying this isomorphism to the formula in Lemma 2.22 (replace (r, s) by (s^{-1}, r^{-1}), $\omega_{k,l}$ by $\omega'_{k,l} := \omega'_k \omega'_{k-1} \cdots \omega'_l$, and $\mathcal{E}_{k,l}$ by $\mathcal{F}_{k,l}$), we find

$$\Delta(\mathcal{F}_{k,l}^{a}) = \sum r^{-p_m(c_1,\ldots,c_m)} \zeta^{a-c_1-c_m} \left[\begin{array}{c} c_1 + \cdots + c_m \\ c_m \end{array} \right]_{s^{-1},r^{-1}} \cdots \left[\begin{array}{c} c_1 + c_2 \\ c_2 \end{array} \right]_{s^{-1},r^{-1}} \cdot \tag{2.26}$$

$$\mathcal{F}_{l,l}^{c_2} \mathcal{F}_{l+1,l}^{c_3} \cdots \mathcal{F}_{k,l}^{c_m} \otimes \mathcal{F}_{k,l}^{c_1} \mathcal{F}_{k,l+1}^{c_2} \cdots \mathcal{F}_{k,k}^{c_{m-1}} (\omega'_{l,l})^{c_2} (\omega'_{l+1,l})^{c_3} \cdots (\omega'_{k,l})^{c_m}.$$

Consequently, the ideal I_n is a Hopf ideal, and $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a finite-dimensional Hopf algebra. □

Example 2.27 Letting $n = 2$, $r = 1$, and $s = -1$, we obtain a 16-dimensional Hopf algebra $\mathfrak{u}_{1,-1}(\mathfrak{sl}_2)$. This Hopf algebra has as a left and right integral $(1+\omega_1+\omega'_1+\omega_1\omega'_1)e_1 f_1$. As ε applied to this integral yields 0, $\mathfrak{u}_{1,-1}(\mathfrak{sl}_2)$ is not semisimple (see the remarks preceding Proposition 5.5 below).

Remark 2.28 If $r = q$ and $s = q^{-1}$, the quotient of $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ by the ideal generated by all $\omega'_i - \omega_i^{-1}$ $(1 \le i < n)$ is isomorphic to a subalgebra of $\mathfrak{u}_q(\mathfrak{sl}_n)$. (However, the subalgebra depends on the particular definition of the restricted one-parameter quantum group $\mathfrak{u}_q(\mathfrak{sl}_n)$, and there several different versions; see for example [20], [28], [29], [40], [44].)

3 Isomorphisms among restricted two-parameter quantum groups

For ease of notation we write $\mathfrak{u}_{r,s} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$. If $n = 2$, there is an isomorphism of Hopf algebras $\varphi : \mathfrak{u}_{r,s} \to \mathfrak{u}_{r',s'}$ whenever $rs^{-1} = r'(s')^{-1}$ and $\ell = \ell'$ (where ℓ' is the least common multiple of the orders of r' and s' as roots of unity). This isomorphism is given by $\varphi(e_1) = e_1$, $\varphi(f_1) = r^{-1}r' f_1$, $\varphi(\omega_1) = \omega_1$, $\varphi(\omega'_1) = \omega'_1$. When $n \ge 3$, there is an isomorphism of Hopf algebras $\varphi : \mathfrak{u}_{r,s} \to \mathfrak{u}_{s^{-1},r^{-1}}$ defined by $\varphi(e_i) = e_{n-i}$, $\varphi(f_i) = r^{-1}s^{-1} f_{n-i}$, $\varphi(\omega_i) = \omega_{n-i}$, and $\varphi(\omega'_i) = \omega'_{n-i}$. This turns out to be the only nontrivial isomorphism between two such restricted two-parameter quantum groups if $n \ge 3$ (under an additional mild hypothesis), as we will show in Theorem 3.8. This result depends on a description of the skew-primitive elements in $\mathfrak{u}_{r,s}$, given in Lemma 3.3 below and the text following it.

Now let G denote the group generated by ω_i, ω'_i $(1 \le i < n)$ in the restricted quantum group $\mathfrak{u} := \mathfrak{u}_{r,s}$. Define linear subspaces $\mathfrak{a}_k$ of $\mathfrak{u}$ by

$$\mathfrak{a}_0 = \mathbb{K}G, \qquad \mathfrak{a}_1 = \mathbb{K}G + \sum_{i=1}^{n-1}(\mathbb{K}e_iG + \mathbb{K}f_iG), \quad \text{and} \tag{3.1}$$
$$\mathfrak{a}_k = (\mathfrak{a}_1)^k \quad \text{for} \quad k \geq 1.$$

Note that $1 \in \mathfrak{a}_0$, $\Delta(\mathfrak{a}_0) \subseteq \mathfrak{a}_0 \otimes \mathfrak{a}_0$, $\mathfrak{a}_1$ generates $\mathfrak{u}$ as an algebra, and $\Delta(\mathfrak{a}_1) \subseteq \mathfrak{a}_1 \otimes \mathfrak{a}_0 + \mathfrak{a}_0 \otimes \mathfrak{a}_1$. By [33, Lemma 5.5.1], $\{\mathfrak{a}_k\}$ is a coalgebra filtration of $\mathfrak{u}$ and $\mathfrak{u}_0 \subseteq \mathfrak{a}_0$, where the *coradical* $\mathfrak{u}_0$ of $\mathfrak{u}$ is the sum of all the simple subcoalgebras of $\mathfrak{u}$. Clearly $\mathfrak{a}_0 \subseteq \mathfrak{u}_0$ as well, and so $\mathfrak{u}_0 = \mathbb{K}G$. This implies that $\mathfrak{u}$ is *pointed*, that is every simple subcoalgebra of $\mathfrak{u}$ is one-dimensional.

Let $\mathfrak{b}$ be the Hopf subalgebra of $\mathfrak{u} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by $e_i, \omega_i^{\pm 1}$ $(1 \leq i < n)$, and $\mathfrak{b}'$ the Hopf subalgebra generated by $f_i, (\omega_i')^{\pm 1}$ $(1 \leq i < n)$. The same type of argument shows that $\mathfrak{b}$ and $\mathfrak{b}'$ are pointed as well. Thus, we have

Proposition 3.2 *The restricted two-parameter quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a pointed Hopf algebra, as are its subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ generated by the elements ω_i, e_i and ω_i', f_i $(1 \leq i < n)$ respectively.*

It also follows from [33, Lemma 5.5.1] that $\mathfrak{a}_k \subseteq \mathfrak{u}_k$ for all k, where $\{\mathfrak{u}_k\}$ is the *coradical filtration* of $\mathfrak{u}$ defined inductively by $\mathfrak{u}_k = \Delta^{-1}(\mathfrak{u} \otimes \mathfrak{u}_{k-1} + \mathfrak{u}_0 \otimes \mathfrak{u})$. In particular, $\mathfrak{a}_1 \subseteq \mathfrak{u}_1$. By [33, Theorem 5.4.1], as $\mathfrak{u}$ is pointed, $\mathfrak{u}_1$ is spanned by the set of group-like elements G together with all of the skew-primitive elements of $\mathfrak{u}$. We claim that under the additional hypothesis of the lemma below, $\mathfrak{u}_1 = \mathfrak{a}_1$. That is, each skew-primitive element of $\mathfrak{u}$ is a linear combination of elements of G, of e_iG, and of f_iG $(1 \leq i < n)$.

Let m be the smallest positive integer such that $r^m = s^m$. Note that $[m] = 0$ while $[1], [2], \ldots, [m-1]$ are all nonzero. It follows from this observation and (2.23) that e_i^m, f_i^m $(1 \leq k < n)$ are skew-primitive if $m < \ell$. However, if we assume that $m = \ell$, then the following lemma gives a precise description of $\mathfrak{u}_1$.

Lemma 3.3 *Assume rs^{-1} is a primitive ℓth root of unity. Then*

$$\mathfrak{u}_1 = \mathbb{K}G + \sum_{i=1}^{n-1}(\mathbb{K}e_iG + \mathbb{K}f_iG).$$

Proof By the observations stated before the lemma, it suffices to show that each skew-primitive element X of $\mathfrak{u} = \mathfrak{u}_{r,s}$ is in $\mathbb{K}G + \sum_{i=1}^{n-1}(\mathbb{K}e_iG + \mathbb{K}f_iG)$. By (2.16), any element X is a linear combination of distinct elements of the form

$$\mathcal{E}_{k_1,l_1}^{b_1}\mathcal{E}_{k_2,l_2}^{b_2}\cdots\mathcal{E}_{k_p,l_p}^{b_p}\sigma\mathcal{F}_{k_1',l_1'}^{b_1'}\mathcal{F}_{k_2',l_2'}^{b_2'}\cdots\mathcal{F}_{k_p',l_p'}^{b_p'},$$

where $\sigma \in G$.

Applying Lemma 2.22 and (2.26), we find that $\Delta(X)$ contains a term of the form

$$\mathcal{E}_{k_1,l_1}^{b_1}\mathcal{E}_{k_2,l_2}^{b_2}\cdots\mathcal{E}_{k_p,l_p}^{b_p}\sigma\otimes\sigma\mathcal{F}_{k_1',l_1'}^{b_1'}\mathcal{F}_{k_2',l_2'}^{b_2'}\cdots\mathcal{F}_{k_p',l_p'}^{b_p'},$$

which by its nature cannot arise from any other term of X, and so it has nonzero coefficient in $\Delta(X)$. As X is skew-primitive, we are forced to take $b_1 = b_2 = \cdots = b_p = 0$ or $b_1' = b_2' = \cdots = b_p' = 0$. Therefore, each term of X involves only factors of the form $\mathcal{E}_{k,l}^{b}$ or $\mathcal{F}_{k,l}^{b'}$, but not both.

We will consider those terms involving factors of $\mathcal{E}_{k,l}^{b}$. Those involving $\mathcal{F}_{k,l}^{b'}$ may be dealt with similarly.

Now we assume that X is a linear combination of elements of the form

$$\mathcal{E}_{k_1,l_1}^{b_1}\mathcal{E}_{k_2,l_2}^{b_2}\cdots\mathcal{E}_{k_p,l_p}^{b_p}\sigma, \tag{3.4}$$

with $\sigma \in G$, $(k_1,l_1) < (k_2,l_2) < \cdots < (k_p,l_p)$ in the lexicographic ordering, and $b_1, b_2, \ldots, b_p > 0$. We may use Lemma 2.22 to compute the coproduct of each monomial (3.4), and we find that one of the terms in the coproduct is

$$\mathcal{E}_{k_1,l_1}^{b_1}\mathcal{E}_{k_2,l_2}^{b_2}\cdots\mathcal{E}_{k_{p-1},l_{p-1}}^{b_{p-1}}\omega_{k_p,l_p}^{b_p}\sigma \otimes \mathcal{E}_{k_p,l_p}^{b_p}\sigma. \tag{3.5}$$

Due to the ordering of the factors and the formula in Lemma 2.22, this term appears exactly once in the coproduct of (3.4), with a nonzero coefficient of 1. Suppose another monomial in X has coproduct containing a term of the form (3.5). Clearly such a monomial has σ as a factor. Examination of the second factor, and then the first factor in the formula in Lemma 2.22 shows that $\mathcal{E}_{k_p,l_p}^{b_p}$ must also appear in such a monomial. This further forces the monomial to be precisely a scalar multiple of (3.4) itself. As (3.5) is then a nonzero term in the coproduct of the skew-primitive element X, and since $b_1, b_2, \ldots, b_p > 0$, it must be that $p = 1$. Therefore X is a linear combination of elements of the form $\mathcal{E}_{k,l}^{b}\sigma$. If we assume $k > l$ and $b \geq 1$, then by Lemma 2.22, $\Delta(\mathcal{E}_{k,l}^{b}\sigma)$ contains the nonzero term $s^{b(b-1)/2}\zeta^{b}\mathcal{E}_{k,l+1}^{b}\omega_l^{b} \otimes e_l^{b}$ (which cannot arise from the coproduct of any other term $\mathcal{E}_{k',l'}^{b'}\sigma'$ of X). This contradicts the assumption that X is skew-primitive, so it must be that $k = l$ for all terms $\mathcal{E}_{k,l}^{b}\sigma$ in X. As we have assumed that rs^{-1} is a *primitive* ℓth root of unity, examination of the formula (2.23) for $\Delta(e_k^b)$ shows that such an element is skew-primitive if and only if $b = 1$. □

We may use Lemma 3.3 to determine explicitly certain subspaces of skew-primitive elements needed to prove the isomorphism result. If g and h are group-like elements in a Hopf algebra H, let $P_{g,h}(H)$ denote the set of skew-primitive elements of H given by

$$P_{g,h}(H) := \{x \in H \mid \Delta(x) = x \otimes g + h \otimes x\}.$$

We wish to compute $P_{1,\sigma}(\mathfrak{u}_{r,s})$ and $P_{\sigma,1}(\mathfrak{u}_{r,s})$ for $\sigma \in G$. Assuming rs^{-1} is a primitive ℓth root of unity, we have by [33, Theorem 5.4.1] and Lemma 3.3:

$$\mathbb{K}G + \sum_{g,h\in G} P_{g,h}(\mathfrak{u}_{r,s}) = \mathbb{K}G + \sum_{i=1}^{n-1}(\mathbb{K}e_iG + \mathbb{K}f_iG).$$

Therefore, an element x of $P_{1,\sigma}(\mathfrak{u}_{r,s})$ (where $\sigma \in G$) may be written as a linear combination

$$x = \sum_{g\in G}\vartheta_g g + \sum_{g\in G}\sum_{i=1}^{n-1}\alpha_{i,g}e_i g + \sum_{g\in G}\sum_{i=1}^{n-1}\beta_{i,g}f_i g,$$

where ϑ_g, $\alpha_{i,g}$, and $\beta_{i,g}$ are scalars. Comparing $\Delta(x) = x \otimes 1 + \sigma \otimes x$ with the coproduct of the right side, which is

$$\sum_{g\in G}\vartheta_g g \otimes g + \sum_{g\in G}\sum_{i=1}^{n-1}\alpha_{i,g}(e_i g \otimes g + \omega_i g \otimes e_i g) + \sum_{g\in G}\sum_{i=1}^{n-1}\beta_{i,g}(g \otimes f_i g + f_i g \otimes \omega_i' g),$$

we find that $\beta_{i,g} = 0$ for all i, g and $\alpha_{i,g} = 0$ for all $g \neq 1$. A further comparison of the group-like components yields $\vartheta_\sigma = -\vartheta_1$ and $\vartheta_g = 0$ for all $g \notin \{1, \sigma\}$. Finally, comparing $\Delta(x)$ with

$\Delta(\vartheta_1(1-\sigma)+\sum_{i=1}^{n-1}\alpha_{i,1}e_i)$ yields $\alpha_{i,1}=0$ for all i when $\sigma\notin\{\omega_1,\omega_2,\ldots,\omega_{n-1}\}$. In the special case that $\sigma=\omega_i$ for some i, we have $\alpha_{j,1}=0$ for all $j\neq i$, and so $x=\vartheta_1(1-\omega_i)+\alpha_{i,1}e_i$. Thus

$$\begin{aligned} P_{1,\omega_i}(\mathfrak{u}_{r,s}) &= \mathbb{K}(1-\omega_i)+\mathbb{K}e_i \quad (1\le i<n) \\ P_{1,\sigma}(\mathfrak{u}_{r,s}) &= \mathbb{K}(1-\sigma) \text{ if } \sigma\neq\omega_i \text{ for any } i. \end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned} P_{\omega_i',1}(\mathfrak{u}_{r,s}) &= \mathbb{K}(1-\omega_i')+\mathbb{K}f_i \quad (1\le i<n) \\ P_{\sigma,1}(\mathfrak{u}_{r,s}) &= \mathbb{K}(1-\sigma) \text{ if } \sigma\neq\omega_i \text{ for any } i. \end{aligned} \tag{3.7}$$

Theorem 3.8 *Assume that $n\geq 3$. Then $\mathfrak{u}_{r,s}\cong\mathfrak{u}_{r',s'}$ as Hopf algebras if and only if $(r',s')=(r,s)$ or $(r',s')=(s^{-1},r^{-1})$.*

Proof Suppose $\varphi:\mathfrak{u}_{r,s}\to\mathfrak{u}_{r',s'}$ is a Hopf algebra isomorphism. We will write $\check{e}_i$, $\check{f}_i$, $\check{\omega}_i$, and $\check{\omega}_i'$ to distinguish the generators of $\mathfrak{u}_{r',s'}$. Note that φ takes group-like elements to group-like elements. Because $\Delta(\varphi(e_i))=\varphi(\Delta(e_i))=\varphi(e_i)\otimes 1+\varphi(\omega_i)\otimes\varphi(e_i)$, we have $\varphi(e_i)\in P_{1,\varphi(\omega_i)}(\mathfrak{u}_{r',s'})$. As φ is an isomorphism, $\varphi(e_i)$ cannot be an element of $\mathbb{K}\check{G}$, and so by (3.6), we have $\varphi(\omega_i)=\check{\omega}_{j_i}$ for some j_i, and $\varphi(e_i)=\alpha(1-\check{\omega}_{j_i})+\beta\check{e}_{j_i}$, for some $\alpha,\beta\in\mathbb{K}$.

Now applying φ to relation (R2) with $j=i$ yields

$$\begin{aligned} \varphi(\omega_i)\varphi(e_i) &= rs^{-1}\varphi(e_i)\varphi(\omega_i) \\ \check{\omega}_{j_i}(\alpha(1-\check{\omega}_{j_i})+\beta\check{e}_{j_i}) &= rs^{-1}(\alpha(1-\check{\omega}_{j_i})+\beta\check{e}_{j_i})\check{\omega}_{j_i} \\ \alpha(1-\check{\omega}_{j_i})\check{\omega}_{j_i}+\beta\check{\omega}_{j_i}\check{e}_{j_i} &= rs^{-1}\alpha(1-\check{\omega}_{j_i})\check{\omega}_{j_i}+rs^{-1}\beta\check{e}_{j_i}\check{\omega}_{j_i}. \end{aligned}$$

As $r\neq s$, this forces α to be 0, so that $\varphi(e_i)=\beta\check{e}_{j_i}$, for some $\beta\neq 0$. Moreover, because $\check{\omega}_{j_i}\check{e}_{j_i}=r'(s')^{-1}\check{e}_{j_i}\check{\omega}_{j_i}$, it must be that $r'(s')^{-1}=rs^{-1}$.

Since $r\neq s$, either $s\neq 1$ or $r\neq 1$. Assuming $s\neq 1$, we apply φ again to relation (R2) but with $i=1, j=2$ (recall $n\geq 3$) to obtain

$$\begin{aligned} \varphi(\omega_1)\varphi(e_2) &= s\varphi(e_2)\varphi(\omega_1) \\ \check{\omega}_{j_1}\check{e}_{j_2} &= s\check{e}_{j_2}\check{\omega}_{j_1}. \end{aligned}$$

On the other hand, relation (R2) for the algebra $\mathfrak{u}_{r',s'}$ gives

$$\check{\omega}_{j_1}\check{e}_{j_2}=(r')^{\langle\epsilon_{j_1},\alpha_{j_2}\rangle}(s')^{\langle\epsilon_{j_1+1},\alpha_{j_2}\rangle}\check{e}_{j_2}\check{\omega}_{j_1}.$$

As $s\neq 1$, this implies $|j_1-j_2|=1$. If $j_2=j_1+1$, this further forces $s'=s$, and consequently $r'=r$ as well. If $j_2=j_1-1$, we find $(r')^{-1}=s$, and so $(s')^{-1}=r$. Under these conditions, an explicit isomorphism $\varphi:\mathfrak{u}_{r,s}\to\mathfrak{u}_{r',s'}$ was provided at the beginning of this section. A similar argument applies when $s=1$ and $r\neq 1$. □

Example 3.9 Let $\mathbb{K}=\mathbb{C}$, $n=3$, $\ell=3$, and $\theta=\exp(2\pi i/3)$. Then by Theorem 3.8, $\mathfrak{u}_{1,\theta}$, $\mathfrak{u}_{1,\theta^2}$, $\mathfrak{u}_{\theta,\theta^2}$, and $\mathfrak{u}_{\theta^2,\theta}$ are four distinct Hopf algebras of dimension 3^{10} (the last two are related to restricted one-parameter quantum groups).

4 $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a Drinfel'd double

Assume as before that $\mathfrak{b}$ is the Hopf subalgebra of $\mathfrak{u} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by $e_i, \omega_i^{\pm 1}$ $(1 \le i < n)$, and $\mathfrak{b}'$ is the Hopf subalgebra generated by $f_i, (\omega_i')^{\pm 1}$ $(1 \le i < n)$. We show next that $\mathfrak{u}$ is isomorphic to the Drinfel'd double $D(\mathfrak{b})$ of $\mathfrak{b}$ under a few assumptions. This extends the known result that a restricted one-parameter quantum group is a Hopf algebra quotient of a Drinfel'd double (see [20, 40]) under some mild conditions. First we need a lemma.

Let θ be a primitive ℓth root of unity in $\mathbb{K}$, and write $r = \theta^y, s = \theta^z$. Note that in case $r = q, s = q^{-1}$, the first hypothesis in the following lemma states that n and ℓ are relatively prime, and the second that ℓ is odd.

Lemma 4.1 *Assume that $(y^{n-1} - y^{n-2}z + - \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1$ and rs^{-1} is a primitive ℓth root of unity. There is an isomorphism of Hopf algebras $(\mathfrak{b}')^{coop} \cong \mathfrak{b}^*$.*

Proof We define elements γ_i, η_i of $\mathfrak{b}^*$ $(1 \le i < n)$ as follows: The γ_i are algebra homomorphisms with

$$\gamma_i(\omega_j) = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} = r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle} \quad \text{and} \quad \gamma_i(e_j) = 0, \tag{4.2}$$

and so they are group-like elements in $\mathfrak{b}^*$. (The second expression for $\gamma_i(\omega_j)$ comes from the identity $\langle \epsilon_j, \alpha_i \rangle = -\langle \epsilon_{i+1}, \alpha_j \rangle$, which can be found in [12, (2.3)].) Let

$$\eta_i = \sum_{g \in G(\mathfrak{b})} (e_i g)^*, \tag{4.3}$$

where $G(\mathfrak{b})$ is the group generated by ω_i $(1 \le i < n)$, and the asterisk denotes the dual basis element relative to the PBW-basis of $\mathfrak{b}$ (see (2.16)). The isomorphism $\phi : (\mathfrak{b}')^{\mathrm{coop}} \to \mathfrak{b}^*$ is then defined by

$$\phi(\omega_i') = \gamma_i \quad \text{and} \quad \phi(f_i) = \eta_i.$$

First we will check that ϕ is a Hopf algebra homomorphism, and then we will show that it is a bijection.

Clearly the γ_i are invertible elements in $\mathfrak{b}^*$ that commute with one another, and $\gamma_i^\ell = 1$. We also observe that $\eta_i^\ell = 0$, as it is 0 on any basis element of $\mathfrak{b}$. We calculate $\gamma_i \eta_j \gamma_i^{-1}$: It is nonzero only on basis elements of the form $e_j \omega_1^{k_1} \cdots \omega_{n-1}^{k_{n-1}}$, and on such an element it takes the value

$$\begin{aligned}
&(\gamma_i \otimes \eta_j \otimes \gamma_i^{-1})((e_j \otimes 1 \otimes 1 + \omega_j \otimes e_j \otimes 1 + \omega_j \otimes \omega_j \otimes e_j)(\omega_1^{k_1} \cdots \omega_{n-1}^{k_{n-1}})^{\otimes 3}) \\
&\quad = \gamma_i(\omega_j \omega_1^{k_1} \cdots \omega_{n-1}^{k_{n-1}}) \eta_j(e_j \omega_1^{k_1} \cdots \omega_{n-1}^{k_{n-1}}) \gamma_i^{-1}(\omega_1^{k_1} \cdots \omega_{n-1}^{k_{n-1}}) \\
&\quad = \gamma_i(\omega_j) = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle}.
\end{aligned}$$

Therefore we have

$$\gamma_i \eta_j \gamma_i^{-1} = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} \eta_j,$$

which corresponds to relation (R3) for $\mathfrak{b}'$. Next we check relation (R7):

$$\begin{aligned}
&(\eta_i^2 \eta_{i+1})(e_i^2 e_{i+1}) \\
&\quad = (\eta_i \otimes \eta_i \otimes \eta_{i+1}) \Big((e_i \otimes 1 \otimes 1 + \omega_i \otimes e_i \otimes 1 + \omega_i \otimes \omega_i \otimes e_i)^2 \cdot \\
&\qquad\qquad (e_{i+1} \otimes 1 \otimes 1 + \omega_{i+1} \otimes e_{i+1} \otimes 1 + \omega_{i+1} \otimes \omega_{i+1} \otimes e_{i+1}) \Big) \\
&\quad = (\eta_i \otimes \eta_i \otimes \eta_{i+1})(e_i \omega_i \omega_{i+1} \otimes e_i \omega_{i+1} \otimes e_{i+1} + \omega_i e_i \omega_{i+1} \otimes e_i \omega_{i+1} \otimes e_{i+1}) \\
&\quad = (\eta_i \otimes \eta_i \otimes \eta_{i+1}) \Big((1 + rs^{-1}) e_i \omega_i \omega_{i+1} \otimes e_i \omega_{i+1} \otimes e_{i+1} \Big) = 1 + rs^{-1},
\end{aligned}$$

and similarly, by Lemma 2.19, $(\eta_i^2\eta_{i+1})(e_i\mathcal{E}_{i+1,i}) = 0$. Thus, for any $k_1, \ldots, k_{n-1}$, we have $\eta_i^2\eta_{i+1}(e_i^2 e_{i+1}\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) = 1 + rs^{-1}$ and $\eta_i^2\eta_{i+1}(e_i\mathcal{E}_{i+1,i}\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) = 0$. On all other basis elements, $\eta_i^2\eta_{i+1}$ is 0, and so

$$\eta_i^2\eta_{i+1} = \sum_{g\in G}(1 + rs^{-1})(e_i^2 e_{i+1} g)^*.$$

Similarly, we calculate

$$\begin{aligned}
&(\eta_i\eta_{i+1}\eta_i)(e_i^2 e_{i+1}) \\
&\quad = (\eta_i \otimes \eta_{i+1} \otimes \eta_i)(e_i\omega_i\omega_{i+1} \otimes \omega_i e_{i+1} \otimes e_i + \omega_i e_i \omega_{i+1} \otimes \omega_i e_{i+1} \otimes e_i) = r + s, \\
&(\eta_i\eta_{i+1}\eta_i)(e_i\mathcal{E}_{i+1,i}) = 1 - r^{-1}s.
\end{aligned}$$

So we have

$$\eta_i\eta_{i+1}\eta_i = \sum_{g\in G}\Big((r+s)(e_i^2 e_{i+1} g)^* + (1 - r^{-1}s)(e_i\mathcal{E}_{i+1,i} g)^*\Big).$$

Finally, we compute

$$\begin{aligned}
(\eta_{i+1}\eta_i^2)(e_i^2 e_{i+1}) &= (\eta_{i+1} \otimes \eta_i \otimes \eta_i)(\omega_i^2 e_{i+1} \otimes e_i\omega_i \otimes e_i + \omega_i^2 e_{i+1} \otimes \omega_i e_i \otimes e_i) \\
&= s^2 + rs \\
(\eta_{i+1}\eta_i^2)(e_i\mathcal{E}_{i+1,i}) &= r - r^{-1}s^2
\end{aligned}$$

which implies

$$\eta_{i+1}\eta_i^2 = \sum_{g\in G}\Big((s^2 + rs)(e_i^2 e_{i+1} g)^* + (r - r^{-1}s^2)(e_i\mathcal{E}_{i+1,i} g)^*\Big).$$

We use the above results to establish the relation

$$\begin{aligned}
&\eta_i^2\eta_{i+1} - (r^{-1} + s^{-1})\eta_i\eta_{i+1}\eta_i + r^{-1}s^{-1}\eta_{i+1}\eta_i^2 \\
&\quad = \sum_{g\in G}\Big((1 + rs^{-1} - (r^{-1} + s^{-1})(r+s) + r^{-1}s^{-1}(s^2 + rs))(e_i^2 e_{i+1} g)^* \\
&\qquad\qquad + (-(1 - r^{-1}s)(r^{-1} + s^{-1}) + r^{-1}s^{-1}(r - r^{-1}s^2))(e_i\mathcal{E}_{i+1,i} g)^*\Big) \\
&\quad = 0.
\end{aligned}$$

Similarly, it may be verified that $\eta_i\eta_{i+1}^2 - (r^{-1} + s^{-1})\eta_{i+1}\eta_i\eta_{i+1} + r^{-1}s^{-1}\eta_{i+1}^2\eta_i = 0$. Therefore ϕ is an algebra homomorphism.

Now we will check that ϕ preserves coproducts. We have already seen that γ_i is a group-like element in $\mathfrak{b}^*$. We calculate

$$\begin{aligned}
\Delta(\eta_i)(e_i\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}} \otimes \omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) &= \eta_i(e_i\omega_j^{j_1+k_1}\cdots\omega_{n-1}^{j_{n-1}+k_{n-1}}) = 1, \quad \text{and} \\
\Delta(\eta_i)(\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}} \otimes e_i\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) &= \eta_i(\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}} e_i\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) \\
&= s^{j_{i-1}}(rs^{-1})^{j_i} r^{-j_{i+1}}.
\end{aligned}$$

These are the only basis elements of $\mathfrak{b}\otimes\mathfrak{b}$ on which $\Delta(\eta_i)$ is nonzero. Correspondingly, we have

$$\begin{aligned}
(\eta_i \otimes 1 + \gamma_i \otimes \eta_i)(e_i\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}} \otimes \omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) &= 1, \quad \text{and} \\
(\eta_i \otimes 1 + \gamma_i \otimes \eta_i)(\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}} \otimes e_i\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) &= s^{j_{i-1}}(rs^{-1})^{j_i} r^{-j_{i+1}}.
\end{aligned}$$

Therefore $\Delta(\eta_i) = \eta_i \otimes 1 + \gamma_i \otimes \eta_i$. This shows that ϕ is a Hopf algebra homomorphism.

Finally, we prove that ϕ is bijective. As $\mathfrak{b}^*$ and $(\mathfrak{b}')^{\mathrm{coop}}$ have the same dimension, it suffices to show that ϕ is injective. By [33, Theorem 5.3.1], we need only show that $\phi|_{(\mathfrak{b}')_1^{\mathrm{coop}}}$ is injective.

The proof of Lemma 3.3 yields $(\mathfrak{b}')_1^{\rm coop} = \mathbb{K}G(\mathfrak{b}') + \sum_{i=1}^{n-1} \mathbb{K}f_i G(\mathfrak{b}')$, where $G(\mathfrak{b}')$ is the group generated by ω_i' $(1 \le i < n)$. First we claim that

$$\mathrm{Span}_{\mathbb{K}}\{\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}}\} = \mathrm{Span}_{\mathbb{K}}\{(\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}})^*\}, \tag{4.4}$$

where in each set, $k_1,\ldots,k_{n-1}$ range from 0 to $\ell-1$. This is equivalent to the statement that the $\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}}$ span the space of characters over $\mathbb{K}$ of the finite group $\mathbb{Z}/\ell\mathbb{Z}\times\cdots\times\mathbb{Z}/\ell\mathbb{Z}$ generated by $\omega_1,\ldots,\omega_{n-1}$. We have assumed that $\mathbb{K}$ contains a primitive ℓth root of unity. Therefore the irreducible characters of this group are the functions $\chi_{i_1,\ldots,i_{n-1}}$ given by

$$\chi_{i_1,\ldots,i_{n-1}}(\omega_1^{k_1}\cdots\omega_{n-1}^{k_{n-1}}) = \theta^{i_1k_1+\cdots+i_{n-1}k_{n-1}},$$

where θ is a primitive ℓth root of unity in $\mathbb{K}$. Note that

$$\begin{aligned}
\gamma_1 &= \chi_{y-z,-y,0,\ldots,0}\\
\gamma_2 &= \chi_{z,y-z,-y,0,\ldots,0}\\
\gamma_3 &= \chi_{0,z,y-z,-y,0,\ldots,0}\\
&\;\vdots\\
\gamma_{n-1} &= \chi_{0,\ldots,0,z,y-z}
\end{aligned}$$

We must show that, given $i_1,\ldots,i_{n-1}$, there are $k_1,\ldots,k_{n-1}$ such that

$$\chi_{i_1,\ldots,i_{n-1}} = \gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}},$$

which is equivalent to the existence of a solution to the matrix equation

$$\begin{pmatrix}
y-z & z & 0 & 0 & \cdots & & & 0\\
-y & y-z & z & 0 & \cdots & & & 0\\
0 & -y & y-z & z & 0 & \cdots & & 0\\
\vdots & & & & & & & \vdots\\
0 & & \cdots & & 0 & -y & y-z & z\\
0 & & \cdots & & 0 & 0 & -y & y-z
\end{pmatrix}
\begin{pmatrix} k_1\\ \\ \vdots\\ \\ k_{n-1}\end{pmatrix}
=
\begin{pmatrix} i_1\\ \\ \vdots\\ \\ i_{n-1}\end{pmatrix}$$

in $\mathbb{Z}/\ell\mathbb{Z}$ (as these are powers of θ). The determinant of the coefficient matrix is $y^{n-1}-y^{n-2}z+-\cdots(-1)^{n-1}z^{n-1}$, which is invertible in $\mathbb{Z}/\ell\mathbb{Z}$ by the hypothesis in the lemma. Therefore (4.4) holds. In particular, this implies that the matrix

$$\Big((\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}})(\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}})\Big)_{\overline{k}\times\overline{j}} \tag{4.5}$$

is invertible, and that ϕ is a bijection on group-like elements.

Next we will show for each i $(1 \le i < n)$ that the following matrix is invertible:

$$\Big((\eta_i\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}})(e_i\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}})\Big)_{\overline{k}\times\overline{j}}. \tag{4.6}$$

This will complete the proof that ϕ is injective on $(\mathfrak{b}')_1^{\rm coop}$, as desired. We will show that the matrix is block upper-triangular. Each matrix entry is

$$\begin{aligned}
&(\eta_i\otimes\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}})\big(\Delta(e_i)\Delta(\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}})\big)\\
&= (\eta_i\otimes\gamma_1^{k_1}\cdots\gamma_{n-1}^{k_{n-1}})(e_i\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}}\otimes\omega_1^{j_1}\cdots\omega_{n-1}^{j_{n-1}}).
\end{aligned}$$

Thus, (4.6) is precisely the invertible matrix (4.5). □

Remark 4.7 We note that the hypothesis that rs^{-1} is a *primitive* ℓth root of unity (required for Lemma 3.3) is not necessary in the above lemma. A more complicated argument involving the $\mathcal{E}_{i,j}$ (which we omit here to simplify the calculations) may be made in general.

The *Drinfel'd double* $D(\mathfrak{b})$ of the finite-dimensional Hopf algebra $\mathfrak{b}$ is a Hopf algebra whose underlying coalgebra is $\mathfrak{b}\otimes(\mathfrak{b}^*)^{\text{coop}}$ (that is, the vector space $\mathfrak{b}\otimes(\mathfrak{b}^*)^{\text{coop}}$ with the tensor product coalgebra structure). As an algebra, $D(\mathfrak{b})$ contains the subalgebras $\mathfrak{b}\otimes 1 \cong \mathfrak{b}$ and $1\otimes\mathfrak{b}^* \cong \mathfrak{b}^*$, and if $\alpha\in\mathfrak{b}$ and $\beta\in(\mathfrak{b}^*)^{\text{coop}}$, then $(\alpha\otimes 1)(1\otimes\beta)=\alpha\otimes\beta$ and

$$(1\otimes\beta)(\alpha\otimes 1)=\sum \beta_{(1)}(S^{-1}\alpha_{(1)})\beta_{(3)}(\alpha_{(3)})\alpha_{(2)}\otimes\beta_{(2)},$$

where S^{-1} is the composition inverse of the antipode S for $\mathfrak{b}$.

Theorem 4.8 *Assume that* $(y^{n-1}-y^{n-2}z+-\cdots+(-1)^{n-1}z^{n-1},\ell)=1$, *and let* $\mathfrak{b}$ *be the subalgebra of the restricted quantum group* $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ *generated by the elements* $\omega_i, e_i,\ 1\le i<n$. *There is an isomorphism of Hopf algebras* $D(\mathfrak{b})\cong\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$.

Proof We will denote the image $e_i\otimes 1$ of e_i in $D(\mathfrak{b})$ by $\check{e}_i$, and similarly for ω_i,η_i and γ_i. Define $\psi: D(\mathfrak{b})\to\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ on the generators by

$$\psi(\check{e}_i)=e_i,\quad \psi(\check{\eta}_i)=(s-r)f_i,$$
$$\psi(\check{\omega}_i^{\pm1})=\omega_i^{\pm1},\quad \psi(\check{\gamma}_i^{\pm1})=(\omega_i')^{\pm1}.$$

By Lemma 4.1 and its proof (see also (2.16)), ψ is bijective and restricts to a Hopf algebra isomorphism on $\mathfrak{b}$ (respectively, $(\mathfrak{b}^*)^{\text{coop}}$). It remains to check the mixed relations in (R1), (R2), and (R3), and the relation (R4).

For (R4), we use

$$(\Delta^{\text{op}})^2(\eta_i)=1\otimes1\otimes\eta_i+1\otimes\eta_i\otimes\gamma_i+\eta_i\otimes\gamma_i\otimes\gamma_i,$$

$$\Delta^2(e_j)=e_j\otimes1\otimes1+\omega_j\otimes e_j\otimes 1+\omega_j\otimes\omega_j\otimes e_j,\quad\text{and } S^{-1}(e_j)=-e_j\omega_j^{-1},$$

so that

$$\check{\eta}_i\check{e}_j=\delta_{i,j}(\check{\omega}_k+\check{e}_i\check{\eta}_i-\check{\gamma}_i),\quad\text{or } [\check{e}_i,\check{\eta}_j]=\delta_{i,j}(\check{\gamma}_i-\check{\omega}_i).$$

Under ψ, this corresponds to the equation

$$[e_i,(s-r)f_j]=\delta_{i,j}(\omega_i'-\omega_i),\quad\text{or } [e_i,f_j]=\delta_{i,j}\frac{\omega_i-\omega_i'}{r-s},$$

as desired. The remaining relations may be established similarly. The identity $\langle\epsilon_{i+1},\alpha_j\rangle = -\langle\epsilon_j,\alpha_i\rangle$ is particularly helpful for this purpose. □

Example 4.9 Let $\mathbb{K}=\mathbb{C}$, $n=3$, $\ell=3$, and $\theta=\exp(2\pi i/3)$. Then $\mathfrak{u}_{1,\theta}(\mathfrak{sl}_3)$ is a Drinfel'd double by Theorem 4.8: In this example, $y=0$ and $z=1$, so that $y^2-yz+z^2=1$ is relatively prime to $\ell=3$. Similarly, $\mathfrak{u}_{1,\theta^2}(\mathfrak{sl}_3)$ is a Drinfel'd double. In fact, $\mathfrak{u}_{1,\theta}(\mathfrak{sl}_n)$ is a Drinfel'd double for any n (and any primitive ℓth root of unity θ). By contrast, neither $\mathfrak{u}_{\theta,\theta^2}(\mathfrak{sl}_3)$ nor $\mathfrak{u}_{\theta^2,\theta}(\mathfrak{sl}_3)$ satisfies the hypothesis of Theorem 4.8.

5 Integrals

Let H be a finite-dimensional Hopf algebra. An element y in H is a *left integral* (resp. *right integral*) if $ay = \varepsilon(a)y$ (resp. $ya = \varepsilon(a)y$) for all $a \in H$. The left (resp. right) integrals form a one-dimensional ideal $\int_H^l$ (resp. $\int_H^r$) of H, and $S_H(\int_H^r) = \int_H^l$ under the antipode S_H of H (see for example, [33, Thm. 2.1.3]).

When $y \neq 0$ is a left integral of H, there exists a unique group-like element γ in the dual Hopf algebra H^* (the so-called *distinguished group-like element of* H^*) such that $ya = \gamma(a)y$. Had we begun instead with a right integral $y' \in H$, then we would have $ay' = \gamma^{-1}(a)y'$. This is an easy consequence of the fact that group-like elements are invertible, and the following calculation, which can be found in [33, p. 22]: When $y' \in \int_H^r$, $S_H(y')$ is a nonzero multiple of y, so that $S_H(y')S_H(a) = \gamma(S_H(a))S_H(y')$ for all $a \in H$. Applying S_H^{-1}, we find $ay' = \gamma(S_H(a))y'$. As γ is group-like, we have $\gamma(S_H(a)) = S_{H^*}(\gamma)(a) = \gamma^{-1}(a)$.

Now if $\lambda \neq 0$ is a right integral of H^*, then there exists a unique group-like element g of H (the *distinguished group-like element of* H) such that $\xi\lambda = \xi(g)\lambda$ for all $\xi \in H^*$. The algebra H is *unimodular* (i.e., $\int_H^l = \int_H^r$) if and only if $\gamma = \varepsilon$; and the dual algebra H^* is unimodular if and only if $g = 1$.

The left and right H^*-module actions on H are given by

$$\begin{aligned} \xi \rightharpoonup a &= \sum a_{(1)}\xi(a_{(2)}) \\ a \leftharpoonup \xi &= \sum \xi(a_{(1)})a_{(2)} \end{aligned}$$

for all $\xi \in H^*$ and $a \in H$. In particular, $\varepsilon \rightharpoonup a = a = a \leftharpoonup \varepsilon$ for all $a \in H$. Radford [36] found a remarkable expression relating the antipode, the distinguished group-like elements γ and g, and the H^*-action:

$$S^4(a) = g(\gamma \rightharpoonup a \leftharpoonup \gamma^{-1})g^{-1} \quad \text{for all } a \in H.$$

This formula is crucial in [26], where Kauffman and Radford determine a necessary and sufficient condition for a Drinfel'd double of a Hopf algebra to have a ribbon element.

In this section, we compute the left and right integrals in the Borel subalgebra $\mathfrak{b}$ of the restricted two-parameter quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and the distinguished group-like elements of $\mathfrak{b}$ and $\mathfrak{b}^*$. Then in the final section, we use this information to determine a necessary and sufficient condition for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ to have a ribbon element when $\mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong D(\mathfrak{b})$.

Proposition 5.1 *The element $y = tx$ is a left integral in $\mathfrak{b}$, where*

$$t = \prod_{i=1}^{n-1} \left(1 + \omega_i + \cdots + \omega_i^{\ell-1}\right),$$

$$x = \prod_{1 \le j \le i \le n-1} \mathcal{E}_{i,j}^{\ell-1},$$

and the factors in x are arranged so that the one corresponding to (i,j) is to the left of the one corresponding to (i',j') if $(i,j) < (i',j')$ in the lexicographic order.

Proof We need to argue that $by = \varepsilon(b)y$ for all $b \in \mathfrak{b}$. It suffices to show this for the generators ω_k and e_k, as the counit ε is an algebra homomorphism.

Observe that $\omega_k t = t = \varepsilon(\omega_k)t$ for all $k = 1, \ldots, n-1$, as the ω_i commute and $\omega_k\big(1 + \omega_k + \cdots + \omega_k^{\ell-1}\big) = 1 + \omega_k + \cdots + \omega_k^{\ell-1}$. From that, the relation $\omega_k y = \varepsilon(\omega_k)y$ is apparent for all k.

Next we compute $e_k y$. Because

$$e_k t = \prod_{i=1}^{n-1} \big(1 + r^{-\langle \epsilon_i, \alpha_k\rangle} s^{-\langle \epsilon_{i+1}, \alpha_k\rangle}\omega_i + \cdots + r^{-(\ell-1)\langle \epsilon_i, \alpha_k\rangle} s^{-(\ell-1)\langle \epsilon_{i+1}, \alpha_k\rangle}\omega_i^{\ell-1}\big) e_k,$$

it will suffice to show that $e_k x = 0 = \varepsilon(e_k)x$.

Now

$$x = \mathcal{E}_{1,1}^{\ell-1}\big(\mathcal{E}_{2,1}^{\ell-1}\mathcal{E}_{2,2}^{\ell-1}\big)\cdots\big(\mathcal{E}_{k-1,1}^{\ell-1}\cdots\mathcal{E}_{k-1,k-1}^{\ell-1}\big)\big(\mathcal{E}_{k,1}^{\ell-1}\cdots\mathcal{E}_{k,k}^{\ell-1}\big)\cdots\big(\mathcal{E}_{n-1,1}^{\ell-1}\cdots\mathcal{E}_{n-1,n-1}^{\ell-1}\big)$$

and from (2) of Proposition 2.8, we see that $e_k\mathcal{E}_{i,j} = \mathcal{E}_{i,j}e_k$ whenever $k > i+1$. So we may commute e_k past the terms with $i < k-1$. Now using (2.11) with k in place of $k+1$ and with $a = \ell - 1$, we obtain

$$e_k\mathcal{E}_{k-1,j}^{\ell-1} \quad = \quad r\mathcal{E}_{k-1,j}^{\ell-1}e_k - rs\mathcal{E}_{k-1,j}^{\ell-2}\mathcal{E}_{k,j}. \tag{5.2}$$

Let us consider the part of this expression with the factor $\mathcal{E}_{k,j}$. By (2) of Proposition 2.8, we have for $m > j$,

$$\mathcal{E}_{k,j}\mathcal{E}_{k-1,m} = \mathcal{E}_{k-1,m}\mathcal{E}_{k,j}.$$

Then by (3) of that proposition, we determine that for $l < j$,

$$\mathcal{E}_{k,j}\mathcal{E}_{k,l} = s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{k,j}.$$

Using these two relations, we may take the second expression in (5.2) and move the factor $\mathcal{E}_{k,j}$ across the remaining $\mathcal{E}_{k-1,m}^{\ell-1}$ with $m > j$ and the $\mathcal{E}_{k,l}^{\ell-1}$ with $l < j$ until $\mathcal{E}_{k,j}$ reaches $\mathcal{E}_{k,j}^{\ell-1}$. Then we obtain $\mathcal{E}_{k,j}^{\ell}$ which is 0.

Consequently, it suffices to treat the first term in the right-hand side of (5.2), which is $r\mathcal{E}_{k-1,j}^{\ell-1}e_k$. We have moved e_k past a term of the form $\mathcal{E}_{k-1,j}^{\ell-1}$, at the expense of adding a factor of r. We can keep doing that until e_k is next to $\mathcal{E}_{k,1}^{\ell-1}$. By (3) of Proposition 2.8, we see that $e_k\mathcal{E}_{k,j} = s^{-1}\mathcal{E}_{k,j}e_k$ for $j < k$. Applying that relation, we can move e_k next to $\mathcal{E}_{k,k}^{\ell-1}$ (at the expense of some s^{-1} factors) and get $\mathcal{E}_{k,k}^{\ell} = 0$. Thus, we have $e_k x = 0$, which implies the desired conclusion that y is a left integral in $\mathfrak{b}$. □

Proposition 5.3 *The element $y' = xt$ is a right integral in $\mathfrak{b}$, where x and t are as in Proposition 5.1.*

Proof Arguing as in Proposition 5.1, we see that $t\omega_j = t = \varepsilon(\omega_j)t$, and hence that $y'\omega_j = \varepsilon(\omega_j)y'$ for all j. As

$$te_j = e_j\prod_{i=1}^{n-1}\big(1 + r^{\langle \epsilon_i, \alpha_j\rangle}s^{\langle \epsilon_{i+1}, \alpha_j\rangle}\omega_i + \cdots + r^{(\ell-1)\langle \epsilon_i, \alpha_j\rangle}s^{(\ell-1)\langle \epsilon_{i+1}, \alpha_j\rangle}\omega_i^{\ell-1}\big),$$

it suffices to argue that $xe_j = 0$. Now $\mathcal{E}_{k,l}e_j = e_j\mathcal{E}_{k,l}$ for $l > j+1$ by Proposition 2.8 (2). Moreover, by (2.10) with $l = j+1$ and $a = \ell - 1$, we have

$$\mathcal{E}_{k,j+1}^{\ell-1}e_j \quad = \quad re_j\mathcal{E}_{k,j+1}^{\ell-1} - rs\mathcal{E}_{k,j}\mathcal{E}_{k,j+1}^{\ell-2}. \tag{5.4}$$

Therefore, we can move e_j past the terms $\mathcal{E}_{k,l}^{\ell-1}$ for $l > j+1$ until it is next to $\mathcal{E}_{k,j+1}^{\ell-1}$. Then we can apply (5.4). The second part of the expression on the right side of that relation gives 0 when it is combined with the term $\mathcal{E}_{k,j}^{\ell-1}$ on its left. To handle the first portion of (5.4), we can use the fact that $\mathcal{E}_{k,j}e_j = s^{-1}e_j\mathcal{E}_{k,j}$ for all $k > j$ to move e_j to the left until it is next to $\mathcal{E}_{j,j}^{\ell-1}$ and so gives 0. Thus, $xe_j = 0$ as desired. □

Larson and Sweedler [27] have shown that a finite-dimensional Hopf algebra H is semisimple if and only if $\varepsilon(\int_H^l) \neq 0$ if and only if $\varepsilon(\int_H^r) \neq 0$ (compare also [33, Thm. 2.2.1]). For the algebra $\mathfrak{b}$ above, y gives a basis for $\int_{\mathfrak{b}}^l$ and y' a basis for $\int_{\mathfrak{b}}^r$. As $\varepsilon(y) = 0 = \varepsilon(y')$, we have

Proposition 5.5 *The Hopf algebra $\mathfrak{b}$ generated by the elements ω_i, e_i, $1 \leq i < n$, is not semisimple.*

Next we compute the distinguished group-like elements of $\mathfrak{b}$ and $\mathfrak{b}^*$. The group-like elements of $\mathfrak{b}^*$ are exactly the algebra homomorphisms $\mathrm{Alg}_{\mathbb{K}}(\mathfrak{b}, \mathbb{K})$, so to verify that a particular homomorphism is the distinguished group-like element, it suffices to compute its values on the generators.

Proposition 5.6 *Suppose that $\Gamma = \sum_{j=1}^{n-1} j(n-j)\alpha_j$ in the root lattice of $\mathfrak{sl}_n$, and let $\gamma \in \mathrm{Alg}_{\mathbb{K}}(\mathfrak{b}, \mathbb{K})$ be defined by*

$$\gamma(e_k) = 0 \qquad \textit{and} \qquad \gamma(\omega_k) = r^{\langle \epsilon_k, \Gamma\rangle} s^{\langle \epsilon_{k+1}, \Gamma\rangle} \quad \textit{for} \;\; 1 \leq k < n. \tag{5.7}$$

Then γ is the distinguished group-like element of $\mathfrak{b}^*$.

Proof It suffices to argue that γ as in (5.7) satisfies $ya = \gamma(a)y$ for $a = e_k$ and $a = \omega_k$, $1 \leq k < n$, and for $y = tx$, the left integral of Proposition 5.1. Recall from the proof of Proposition 5.3 that $xe_k = 0$. Thus, $ye_k = txe_k = 0 = \gamma(e_k)y$. Now assuming that $\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq n$, we have

$$\begin{aligned} y\omega_k &= tx\omega_k \\ &= \prod_{1 \leq i < j \leq n} r^{-(\ell-1)\langle \epsilon_k, \alpha_{i,j}\rangle} s^{-(\ell-1)\langle \epsilon_{k+1}, \alpha_{i,j}\rangle} t\omega_k x \\ &= \prod_{1 \leq i < j \leq n} r^{\langle \epsilon_k, \alpha_{i,j}\rangle} s^{\langle \epsilon_{k+1}, \alpha_{i,j}\rangle} tx \\ &= r^{\langle \epsilon_k, \Gamma\rangle} s^{\langle \epsilon_{k+1}, \Gamma\rangle} y \\ &= \gamma(\omega_k)y. \end{aligned}$$

□

In Lemma 4.1 we have shown that under some assumptions, $\mathfrak{b}^*$ is isomorphic as a Hopf algebra to $(\mathfrak{b}')^{\text{coop}}$, where $\mathfrak{b}'$ is the subalgebra of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by the elements f_i and ω_i', $1 \leq i < n$, via the map $\phi : (\mathfrak{b}')^{\text{coop}} \to \mathfrak{b}^*$, $\phi(\omega_i') = \gamma_i$, $\phi(f_i) = \eta_i$. (The elements γ_i and η_i are defined in (4.2) and (4.3).) This allows us to define a Hopf pairing $\mathfrak{b}' \times \mathfrak{b} \to \mathbb{K}$ whose values on generators are given by

$$\begin{aligned} (f_i \mid e_j) &= \delta_{i,j} \\ (\omega_i' \mid \omega_j) &= r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle} = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle}, \end{aligned} \tag{5.8}$$

and on all other pairs of generators is 0. If we set $\omega_\gamma' := \prod_{i=1}^{n-1} (\omega_i')^{i(n-i)}$, then clearly, $(\omega_\gamma' \mid b) = \gamma(b)$ for all $b \in \mathfrak{b}$.

Suppose $\mathfrak{b}$ and $\mathfrak{b}'$ are the subalgebras above for the restricted quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$. Then we have

$$\mathfrak{b}^* \cong (\mathfrak{b}')^{\text{coop}} \cong \mathfrak{b}_{s^{-1},r^{-1}}, \tag{5.9}$$

where $\mathfrak{b}_{s^{-1},r^{-1}}$ is the subalgebra of $\mathfrak{u}_{s^{-1},r^{-1}}(\mathfrak{sl}_n)$ generated by its e_i and ω_i elements, and the second isomorphism $\psi : (\mathfrak{b}')^{\text{coop}} \to \mathfrak{b}_{s^{-1},r^{-1}}$ is given by $f_i \mapsto e_i$, $\omega_i' \mapsto \omega_i$. Under the isomorphism $\phi\psi^{-1}$, a nonzero left (resp. right) integral of $\mathfrak{b}_{s^{-1},r^{-1}}$ maps to a nonzero left (resp. right) integral of $\mathfrak{b}^*$. Thus, we have

Proposition 5.10 *Let $\lambda = \tau\eta$ and $\lambda' = \eta\tau \in \mathfrak{b}^*$, where*

$$\tau = \prod_{i=1}^{n-1} \left(1 + \gamma_i + \cdots + \gamma_i^{\ell-1}\right),$$

$$\eta = \prod_{1 \leq j \leq i \leq n} \eta_{i,j}^{\ell-1},$$

and the factors in η are arranged so that the one corresponding to (i,j) is to the left of the one corresponding to (i',j') if $(i,j) < (i',j')$ in the lexicographic order. Here $\eta_{j,j} = \eta_j$ and $\eta_{i,j} := [\eta_i, \eta_{i-1,j}]_s = \eta_i\eta_{i-1,j} - s\eta_{i-1,j}\eta_i$ for $1 \leq j < i < n$. Then λ is a left integral and λ' is a right integral of $\mathfrak{b}^*$.

Because

$$\begin{aligned} \omega_k' \left(\prod_{1 \leq j \leq i < n} \mathcal{F}_{i,j}^{\ell-1} \right) &= \left(\prod_{1 \leq i < j \leq n} r^{-(\ell-1)\langle \epsilon_{k+1}, \alpha_{i,j} \rangle} s^{-(\ell-1)\langle \epsilon_k, \alpha_{i,j} \rangle} \right) \left(\prod_{1 \leq j \leq i < n} \mathcal{F}_{i,j}^{\ell-1} \right) \omega_k' \\ &= r^{\langle \epsilon_{k+1}, \Gamma \rangle} s^{\langle \epsilon_k, \Gamma \rangle} \left(\prod_{1 \leq j \leq i < n} \mathcal{F}_{i,j}^{\ell-1} \right) \omega_k', \end{aligned}$$

where Γ is as in Proposition 5.6, and because

$$\phi^{-1}(\lambda') = \left(\prod_{1 \leq j \leq i < n} \mathcal{F}_{i,j}^{\ell-1} \right) \left(\prod_{i=1}^{n-1} \left(1 + \omega_i' + \cdots + (\omega_i')^{\ell-1}\right) \right),$$

it follows that $\gamma_k\lambda' = r^{\langle \epsilon_{k+1}, \Gamma \rangle} s^{\langle \epsilon_k, \Gamma \rangle} \lambda'$ and $\eta_k\lambda' = 0$. Thus, if $g := \prod_{j=1}^{n-1} \omega_j^{-j(n-j)}$, we have $\xi\lambda' = \xi(g)\lambda'$ for all $\xi \in \mathfrak{b}^*$. Consequently, we have established

Proposition 5.11 *The element* $g := \prod_{j=1}^{n-1} \omega_j^{-j(n-j)}$ *is the distinguished group-like element of* $\mathfrak{b}$*, and under the Hopf pairing in (5.8),* $(\omega_i' \mid g) = r^{\langle \epsilon_{i+1}, \Gamma\rangle} s^{\langle \epsilon_i, \Gamma\rangle} = \gamma_i(g)$*.*

6 Ribbon elements

A finite-dimensional Hopf algebra H is *quasitriangular* if there exists an invertible element $R = \sum x_i \otimes y_i$ in $H \otimes H$ such that $\Delta^{\mathrm{op}}(a)R = R\Delta(a)$ for all $a \in H$, and R satisfies the relations $(\Delta \otimes \mathrm{id})R = R_{1,3}R_{2,3}$, $(\mathrm{id} \otimes \Delta)R = R_{1,3}R_{1,2}$, where $R_{1,2} = \sum x_i \otimes y_i \otimes 1$, $R_{1,3} = \sum x_i \otimes 1 \otimes y_i$, and $R_{2,3} = \sum 1 \otimes x_i \otimes y_i$. Suppose $u = \sum S(y_i)x_i$. Then $c = uS(u)$ is central in H and is referred to as the *Casimir element.*

An element $v \in H$ is a *quasi-ribbon element* of a quasitriangular Hopf algebra (H, R) if

(i) $v^2 = c$,
(ii) $S(v) = v$,
(iii) $\varepsilon(v) = 1$,
(iv) $\Delta(v) = (R_{2,1}R_{1,2})^{-1}(v \otimes v)$, where $R_{2,1} = \sum y_i \otimes x_i$ and $R_{1,2} = R$.

If moreover v is central in H, then v is a *ribbon element*, and (H, R, v) is said to be a *ribbon Hopf algebra.* Ribbon elements provide a very effective means of constructing invariants of knots and links (see [38], [39], [26]).

The Drinfel'd double $D(A)$ of a finite-dimensional Hopf algebra A is quasitriangular, and Kauffman and Radford have provided a simple criterion for $D(A)$ to have a ribbon element.

Theorem 6.1 [26, Thm. 3] *Assume A is a finite-dimensional Hopf algebra, and let g and γ be the distinguished group-like elements of A and A^* respectively. Then:*

(i) *$(D(A), R)$ has a quasi-ribbon element if and only if there exist group-like elements $h \in A$, $\delta \in A^*$ such that $h^2 = g$ and $\delta^2 = \gamma$.*
(ii) *$(D(A), R)$ has a ribbon element if and only if there exist h and δ as in (i) such that*

$$S^2(a) = h(\delta \rightharpoonup a \leftharpoonup \delta^{-1})h^{-1}$$

for all $a \in A$.

By Proposition 5.11 we know that the distinguished group-like element of $\mathfrak{b}$ is $g = \prod_{j=1}^{n-1} \omega_j^{-j(n-j)}$. There exists a group-like element $h \in \mathfrak{b}$ such that $h^2 = g$ if and only if the equations $2a_j = -j(n-j)$ can be solved mod ℓ for $j = 1, \dots, n-1$. Solutions exist if and only if not both ℓ and n are even. Because γ corresponds to $\omega_\gamma' := \prod_{i=1}^{n-1} (\omega_i')^{i(n-i)}$ under the isomorphism $\phi^{-1} : \mathfrak{b}^* \to (\mathfrak{b}')^{\mathrm{coop}}$, precisely the same constraints hold for such a $\delta \in \mathfrak{b}^*$ to exist.

Suppose ℓ and n are not both even, and let $h \in \mathfrak{b}$, $\delta \in \mathrm{Alg}_{\mathbb{K}}(\mathfrak{b}, \mathbb{K})$ be given by

$$\begin{aligned} h &:= \prod_{j=1}^{n-1} \omega_j^{-\frac{1}{2}j(n-j)} \qquad \text{and} \\ \delta(e_k) &:= 0 \qquad \delta(\omega_k) := r^{\frac{1}{2}\langle \epsilon_k, \Gamma\rangle} s^{\frac{1}{2}\langle \epsilon_{k+1}, \Gamma\rangle} \quad (1 \le k < n), \end{aligned}$$

where Γ is as in Proposition 5.6. Then

$$h(\delta \rightharpoonup \omega_k \leftharpoonup \delta^{-1})h^{-1} = \delta(\omega_k)\delta^{-1}(\omega_k)h\omega_k h^{-1}$$
$$= \left(\delta\delta^{-1}\right)(\omega_k)\omega_k = \omega_k = S^2(\omega_k), \qquad \text{and}$$

$$\begin{aligned} h(\delta \rightharpoonup e_k \leftharpoonup \delta^{-1})h^{-1} &= \delta(1)\delta^{-1}(\omega_k)he_k h^{-1} \\ &= \left(r^{-\frac{1}{2}\langle\epsilon_k,\Gamma\rangle}s^{-\frac{1}{2}\langle\epsilon_{k+1},\Gamma\rangle}\right) he_k h^{-1} \\ &= \left(r^{-\frac{1}{2}\langle\epsilon_k,\Gamma\rangle}s^{-\frac{1}{2}\langle\epsilon_{k+1},\Gamma\rangle}\right)\left(\prod_{j=1}^{n-1} r^{-\frac{1}{2}j(n-j)\langle\epsilon_j,\alpha_k\rangle}s^{-\frac{1}{2}j(n-j)\langle\epsilon_{j+1},\alpha_k\rangle}\right) e_k \\ &= r^{-1}se_k = \omega_k^{-1}e_k\omega_k = S^2(e_k). \end{aligned}$$

As a result, we have

Theorem 6.2 *Assume r, s are ℓth roots of unity and $\mathfrak{b}$ is the subalgebra of the restricted quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by the elements $\omega_i, e_i,\ \ 1 \le i < n$. Then the following are equivalent:*

(i) *$D(\mathfrak{b})$ has a quasi-ribbon element;*
(ii) *$D(\mathfrak{b})$ has a ribbon element;*
(iii) *Not both ℓ and n are even.*

Under the hypothesis of Theorem 4.8, the restricted quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is isomorphic to the Drinfel'd double $D(\mathfrak{b})$ of its Hopf subalgebra $\mathfrak{b}$. Thus, we have the following

Corollary 6.3 *Assume that $r = \theta^y\ \ s = \theta^z$, where θ is a primitive ℓth root of unity and $(y^{n-1} - y^{n-2}z + - \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1$. Then the following are equivalent:*

(i) *$\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ has a quasi-ribbon element;*
(ii) *$\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ has a ribbon element;*
(iii) *Not both ℓ and n are even.*

Remarks 6.4 (1) These results are a direct consequence of the above calculations and Theorem 6.1. In proving their theorem, Kauffman and Radford show that the map $(\delta, h) \mapsto u(\delta^{-1} \otimes h^{-1})$, (where u is defined using the R-matrix as at the beginning of this section), is a bijection between the pairs (δ, h) such that $\delta^2 = \gamma$ and $h^2 = g$ and the quasi-ribbon elements. Additionally, the pairs satisfying $S^2(a) = h(\delta \rightharpoonup a \leftharpoonup \delta^{-1})h^{-1}$ for all a exactly correspond to the ribbon elements.

(2) Suppose that $\ell \ge 1$ and θ is a primitive ℓth root of unity. The well-known *Taft algebra A_ℓ* has generators a, x which satisfy $a^\ell = 1,\ x^\ell = 0,\ ax = \theta xa$, and $\Delta(a) = a\otimes a,\ \Delta(x) = x\otimes a + 1\otimes x$. In [26, Prop. 7], Kauffman and Radford extended a result of Hennings [22] to show that the Drinfel'd double $D(A_\ell)$ of A_ℓ has a unique ribbon element if and only if ℓ is odd. In the odd case, the ribbon element of $D(A_\ell)$ provides an invariant of 3-manifolds (see [22]). Also in the odd case, letting $q \in \mathbb{K}$ satisfy $q^2 = \theta$, we have $A_\ell^{\text{coop}} \cong \mathfrak{b}$, where $\mathfrak{b}$ is the subalgebra corresponding to $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_2)$. Thus, Theorem 6.2 should be regarded a generalization of their Proposition 7. Indeed, when ℓ is odd, $q = \theta^{1/2}$ is also a primitive ℓth root of unity, and $D(\mathfrak{b})$ has a ribbon element. In this case, we have $r = q^1,\ s = q^{-1}$ where q is a primitive ℓth root of 1, so $y = 1,\ z = -1$ and $(y - z, \ell) = (2, \ell) = 1$. Thus $D(\mathfrak{b}) \cong \mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_2)$. If instead ℓ is even and $\mathbb{K}$

contains an element q with $q^2 = \theta$, then q is a primitive 2ℓth root of unity, and $D(\mathfrak{b})$ does not have a ribbon element. In this case, $D(\mathfrak{b})$ and $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_2)$ differ by group-like elements.

(3) More generally, when $r = q$, $s = q^{-1}$, and q is a primitive ℓth root of unity, the hypothesis of Corollary 6.3 is simply that n and ℓ are relatively prime. Under this assumption, $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ has a ribbon element (cf. the criterion in [21] for $\mathfrak{u}_q(\mathfrak{sl}_n)$ to have a ribbon element).

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