REPRESENTATIONS OF TWO-PARAMETER QUANTUM GROUPS AND SCHUR-WEYL DUALITY

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ABSTRACT. We determine the finite-dimensional simple modules for two-parameter quantum groups corresponding to the general linear and special linear Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ and present a complete reducibility result. These quantum groups have a natural n-dimensional module V. We prove an analogue of Schur-Weyl duality in this setting: the centralizer algebra of the quantum group action on the k-fold tensor power of V is a quotient of a Hecke algebra for all n and is isomorphic to the Hecke algebra in case $n \geq k$.

INTRODUCTION

Two-parameter general linear and special linear quantum groups were introduced by Takeuchi [T] in 1990. Our interest in these quantum groups arose from our investigations [BW1] of down-up algebras and their embeddings into certain Hopf algebras. These Hopf algebras depend on two parameters r and s, and the Drinfel'd double of such a Hopf algebra is essentially the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_3)$ of Takeuchi (defined below). More generally, as shown in [BW2], $U_{r,s}(\mathfrak{sl}_n)$ is a Drinfel'd double of a Borel-type subalgebra, and there is an R-matrix which comes from the double construction and which reduces to the standard R-matrix for the one-parameter quantum group $U_q(\mathfrak{sl}_n)$ (a quotient of $U_{q,q^{-1}}(\mathfrak{sl}_n)$). In the analogous quantum function algebra setting, allowing two parameters unifies the Drinfel'd-Jimbo quantum groups ($r = q, s = q^{-1}$) [D, Ji1] with the Dipper-Donkin quantum groups ($r = 1, s = q^{-1}$) [DD].

In this work we study the representations of the two-parameter quantum groups $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and $U = U_{r,s}(\mathfrak{sl}_n)$, defined in Section 1. Our Hopf algebra $\widetilde{U}$ is isomorphic as an algebra to Takeuchi's quantum group $U_{r,s^{-1}}$ (see [T]), but as a Hopf algebra, it has the opposite coproduct. Our main goal is to prove a two-parameter analogue of Schur-Weyl duality, for which we need a result on complete reducibility of modules. To this end, in Sections 2 and 3 we adapt the methods of [Ja] and [L] to classify the finite-dimensional simple $\widetilde{U}$-modules when rs^{-1} is

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not a root of unity. We use a quantum Casimir operator to prove that all finite-dimensional $\widetilde{U}$-modules on which $\widetilde{U}^0$ (the subalgebra generated by the grouplike elements) acts semisimply are completely reducible. These results hold equally well for U. The hypothesis on $\widetilde{U}^0$ is necessary: In Remark 3.9, we present examples of finite-dimensional modules that are not completely reducible. These examples exist because the two-parameter quantum groups contain more grouplike elements than their one-parameter counterparts.

The construction of the R-matrix, which provides an isomorphism $R_{M',M} : M' \otimes M \to M \otimes M'$ for any two $\widetilde{U}$-modules M, M' in category $\mathcal{O}$ (defined in Section 3), is summarized in Section 4. On the tensor power $V^{\otimes k}$ of the natural module V (which is described in Section 1), the transformations $R_i = \mathrm{Id}^{\otimes(i-1)} \otimes R_{V,V} \otimes \mathrm{Id}^{\otimes(k-i-1)}$ $(1 \leq i < k)$ commute with the action of $\widetilde{U}$, and so they generate a subalgebra of $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$. This yields a map from the two-parameter Hecke algebra $H_k(r,s)$ to $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$. In the final section we prove a two-parameter analogue of Schur-Weyl duality: The transformations R_i generate the centralizer algebra $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$, and in case $n \geq k$, this centralizer algebra is isomorphic to $H_k(r,s)$. The proof of this result is elementary, relying only on basic facts about representations and explicit computations and is new in the one-parameter case as well (compare [DPS, Du, Ji2, KT, LR]). The proof in the $n \geq k$ case is similar to one for classical (nonquantum) Schur-Weyl duality given by De Concini and Procesi [DP]. It is a consequence of our result, Lemma 6.2 below, that $V^{\otimes k}$ is a cyclic $\widetilde{U}$-module in this case.

Throughout we will work over an algebraically closed field $\mathbb{K}$.

§1. Preliminaries

First we recall the definitions of the two-parameter quantum groups from [BW2], and some basics about their representations. Let $\epsilon_1, \ldots, \epsilon_n$ denote an orthonormal basis of a Euclidean space E with an inner product $\langle\ ,\ \rangle$. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \ldots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$. Then Φ is a finite root system of type A_{n-1} with Π a base of simple roots.

Fix nonzero elements r, s in $\mathbb{K}$ with $r \neq s$.

Let $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ be the unital associative algebra over $\mathbb{K}$ generated by elements $e_j,\ f_j, (1 \leq j < n)$, and $a_i^{\pm 1},\ b_i^{\pm 1}$ $(1 \leq i \leq n)$, which satisfy the following relations.

(R1) The $a_i^{\pm 1},\ b_j^{\pm 1}$ all commute with one another and $a_i a_i^{-1} = b_j b_j^{-1} = 1$,

(R2) $a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i \quad$ and $\quad a_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j a_i$,

(R3) $b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i \quad$ and $\quad b_i f_j = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j b_i$,

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(a_i b_{i+1} - a_{i+1} b_i)$,

(R5) $[e_i, e_j] = [f_i, f_j] = 0 \quad$ if $\quad |i-j| > 1$,

(R6) $e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2 = 0$,
$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i = 0$,

(R7) $f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0$,
$f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0$.

We will be interested in the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$ of $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements e_j, f_j, ω_j, and $\omega'_j \quad (1 \le j < n)$, where

$$\omega_j = a_j b_{j+1} \quad \text{and} \quad \omega'_j = a_{j+1} b_j. \tag{1.1}$$

These elements satisfy (R5)-(R7) along with the following relations:

(R1') The $\omega_i^{\pm 1}$, $\omega_j'^{\pm 1}$ all commute with one another and $\omega_i \omega_i^{-1} = \omega'_j (\omega'_j)^{-1} = 1$,

(R2') $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i$ and $\omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i$,

(R3') $\omega'_i e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega'_i$ and $\omega'_i f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega'_i$,

(R4') $[e_i, f_j] = \dfrac{\delta_{i,j}}{r - s} (\omega_i - \omega'_i)$.

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by the elements $b_i - a_i^{-1}$, $1 \le i \le n$, is just the quantum general linear group $U_q(\mathfrak{gl}_n)$, and $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega'_j - \omega_j^{-1}$, $1 \le j < n$, is $U_q(\mathfrak{sl}_n)$.

The algebras $\widetilde{U}$ and U are Hopf algebras, where the $a_i^{\pm 1}, b_i^{\pm 1}$ are group-like elements, and the remaining Hopf structure is given by

$$\begin{aligned} &\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i, \\ &\varepsilon(e_i) = \varepsilon(f_i) = 0, \qquad S(e_i) = -\omega_i^{-1} e_i, \qquad S(f_i) = -f_i (\omega'_i)^{-1}. \end{aligned} \tag{1.2}$$

Let $\Lambda = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, the weight lattice of $\mathfrak{gl}_n$, and $Q = \mathbb{Z}\Phi$ the root lattice. We assume Λ is equipped with the partial order in which $\nu \le \lambda$ if and only if $\lambda - \nu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0} \alpha_i$. Corresponding to $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra $\widetilde{U}^0$ of $\widetilde{U}$ generated by the elements $a_i^{\pm 1}, b_i^{\pm 1}$ $(1 \le i \le n)$ to $\mathbb{K}$ given by

$$\hat{\lambda}(a_i) = r^{\langle \epsilon_i, \lambda \rangle} \qquad \text{and} \qquad \hat{\lambda}(b_i) = s^{\langle \epsilon_i, \lambda \rangle}. \tag{1.3}$$

The restriction $\hat{\lambda} : U^0 \to \mathbb{K}$ of $\hat{\lambda}$ to the subalgebra U^0 of U generated by $\omega_j^{\pm 1}, (\omega'_j)^{\pm 1}$ $(1 \le j < n)$ satisfies

$$\hat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \qquad \text{and} \qquad \hat{\lambda}(\omega'_j) = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}. \tag{1.4}$$

Similarly for $U = U_{r,s}(\mathfrak{sl}_n)$, we let $\Lambda_{\mathfrak{sl}} = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_{n-1}$, the weight lattice of $\mathfrak{sl}_n$, where ϖ_i is the fundamental weight

$$\varpi_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n} \sum_{j=1}^{n} \epsilon_j.$$

If we fix nth roots $r^{1/n}$ and $s^{1/n}$ of r and s, respectively, then we may define algebra homomorphisms $\hat{\lambda} : U^0 \to \mathbb{K}$ by (1.4) for any $\lambda \in \Lambda_{\mathfrak{sl}}$.

Let M be a module for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ of dimension $d < \infty$. As $\mathbb{K}$ is algebraically closed, we have

$$M = \bigoplus_{\chi} M_\chi,$$

where each $\chi : \widetilde{U}^0 \to \mathbb{K}$ is an algebra homomorphism, and M_χ is the generalized eigenspace given by

$$M_\chi = \{m \in M \mid (a_i - \chi(a_i)\,1)^d m = 0 = (b_i - \chi(b_i)\,1)^d m, \quad \text{for all } i\}. \tag{1.5}$$

When $M_\chi \neq 0$ we say that χ is a *weight* and M_χ is the corresponding *weight space*. (If M decomposes into genuine eigenspaces relative to $\widetilde{U}^0$ (resp. U^0), then we say that $\widetilde{U}^0$ (resp. U^0) *acts semisimply on* M.)

From relations (R2) and (R3) we deduce that

$$\begin{aligned} e_j M_\chi &\subseteq M_{\chi \cdot \widehat{\alpha_j}} \\ f_j M_\chi &\subseteq M_{\chi \cdot (\widehat{-\alpha_j})}, \end{aligned} \tag{1.6}$$

where $\widehat{\alpha_j}$ is as in (1.3), and $\chi \cdot \psi$ is the homomorphism with values $(\chi \cdot \psi)(a_i) = \chi(a_i)\psi(a_i)$ and $(\chi \cdot \psi)(b_i) = \chi(b_i)\psi(b_i)$. In fact, if $(a_i - \chi(a_i)\,1)^k m = 0$, then applying relation (R2) yields $(a_i - \chi(a_i) r^{\langle \epsilon_i, \alpha_j \rangle}\,1)^k e_j m = 0$, and similarly for b_i and for f_j. Therefore, the sum of eigenspaces is a submodule of M, and if M is simple, this sum must be M itself. Thus in (1.5), we may replace the power d by 1 whenever M is simple, and $\widetilde{U}^0$ must act semisimply in this case. We also can see from (1.6) that for each simple M there is a homomorphism χ so that all the weights of M are of the form $\chi \cdot \hat{\zeta}$, where $\zeta \in Q$.

It is shown in [BW2, Prop. 3.5] that if $\hat{\zeta} = \hat{\eta}$, then $\zeta = \eta$ $(\zeta, \eta \in \Lambda)$ provided rs^{-1} is not a root of unity. As a result, we have the following proposition.

Proposition 1.7. *[BW2, Cor. 3.14] Let M be a finite-dimensional module for $U_{r,s}(\mathfrak{sl}_n)$ or for $U_{r,s}(\mathfrak{gl}_n)$. If rs^{-1} is not a root of unity, then the elements e_i, f_i $(1 \leq i < n)$ act nilpotently on M.*

When rs^{-1} is not a root of unity, a finite-dimensional simple module M is a *highest weight* module by Proposition 1.7 and (1.6). Thus there is some weight ψ and a nonzero vector $v_0 \in M_\psi$ such that $e_j v_0 = 0$ for all $j = 1, \dots, n-1$, and $M = \widetilde{U}.v_0$. It follows from the defining relations that $\widetilde{U}$ has a triangular decomposition: $\widetilde{U} = U^- \widetilde{U}^0 U^+$, where U^+ (resp., U^-) is the subalgebra generated by the elements e_i (resp., f_i). Applying this decomposition to v_0, we see that $M = \bigoplus_{\zeta \in Q^+} M_{\psi \cdot (\widehat{-\zeta})}$, where $Q^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$.

When all the weights of a module M are of the form $\hat{\lambda}$, where $\lambda \in \Lambda$, then for brevity we say that M has weights in Λ. Rather than writing $M_{\hat{\lambda}}$ for the weight space, we simplify the notation by writing M_λ. Note then (1.6) can be rewritten as $e_j M_\lambda \subseteq M_{\lambda+\alpha_j}$ and $f_j M_\lambda \subseteq M_{\lambda-\alpha_j}$. Any simple $\widetilde{U}$-module having one weight in Λ has all its weights in Λ.

Next we give an example of a simple $\widetilde{U}$-module with weights in Λ, which is the analogue of the natural representation for $\mathfrak{gl}_n$.

The natural representation for $U_{r,s}(\mathfrak{gl}_n)$ and $U_{r,s}(\mathfrak{sl}_n)$.

Consider an n-dimensional vector space V over $\mathbb{K}$ with basis $\{v_j \mid 1 \le j \le n\}$. We define an action of the generators of $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ by specifying their matrices relative to this basis:

$$\begin{aligned} e_j &= E_{j,j+1}, \qquad f_j = E_{j+1,j}, && (1 \le j < n) \\ a_i &= rE_{i,i} + \sum_{k \ne i} E_{k,k}, && (1 \le i \le n) \\ b_i &= sE_{i,i} + \sum_{k \ne i} E_{k,k} && (1 \le i \le n). \end{aligned}$$

It follows that $\omega_j = a_j b_{j+1} = rE_{j,j} + sE_{j+1,j+1} + \sum_{k \ne j,j+1} E_{k,k}$ and $\omega'_j = a_{j+1} b_j = sE_{j,j} + rE_{j+1,j+1} + \sum_{k \ne j,j+1} E_{k,k}$. It may be verified that this extends to an action of $\widetilde{U}$ (hence of $U = U_{r,s}(\mathfrak{sl}_n)$); that is, relations (R1)–(R7) hold.

It follows from the fact that $a_i v_j = r^{\langle \epsilon_i, \epsilon_j \rangle} v_j$ and $b_i v_j = s^{\langle \epsilon_i, \epsilon_j \rangle} v_j$ for all i, j that v_j corresponds to the weight $\epsilon_j = \epsilon_1 - (\alpha_1 + \cdots + \alpha_{j-1})$. Thus, $V = \bigoplus_{j=1}^n V_{\epsilon_j}$ is the natural analogue of the n-dimensional representation of $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, and it is a simple module for both $\widetilde{U}$ and U. When $r = q$ and $s = q^{-1}$, b_i acts as a_i^{-1} on V, and so V is a module for the quotient $U_q(\mathfrak{gl}_n)$ of $U_{q,q^{-1}}(\mathfrak{gl}_n)$ by the ideal generated by $b_i - a_i^{-1}$ $(1 \le i \le n)$. This is the natural module for the one-parameter quantum group $U_q(\mathfrak{gl}_n)$, and an analogous statement is true for $U_q(\mathfrak{sl}_n)$.

§2. Classification of finite-dimensional simple modules

Often results will be stated only for $\widetilde{U}$-modules, but generally everything holds as well for U-modules. We will indicate where there are differences in the theory.

Let $\widetilde{U}^{\ge 0}$ denote the subalgebra of $\widetilde{U}$ generated by a_i, b_i $(1 \le i \le n)$ and e_i $(1 \le i < n)$. Let ψ be any algebra homomorphism from $\widetilde{U}^0$ to $\mathbb{K}$ and V^ψ be the one-dimensional $\widetilde{U}^{\ge 0}$-module on which e_i acts as multiplication by 0 $(1 \le i < n)$, and $\widetilde{U}^0$ acts via ψ. We define the *Verma module* $M(\psi)$ with highest weight ψ to be the $\widetilde{U}$-module induced from V^ψ, that is

$$M(\psi) = \widetilde{U} \otimes_{\widetilde{U}^{\ge 0}} V^\psi.$$

Let $v_\psi = 1 \otimes v \in M(\psi)$, where v is any nonzero vector of V^ψ. Then $e_i.v_\psi = 0$ $(1 \le i < n)$ and $a.v_\psi = \psi(a) v_\psi$ for any $a \in \widetilde{U}^0$ by construction.

Notice that $\widetilde{U}^0$ acts semisimply on $M(\psi)$ by relations (R2) and (R3). If N is a $\widetilde{U}$-submodule of $M(\psi)$, then N is also a $\widetilde{U}^0$-submodule of the $\widetilde{U}^0$-module $M(\psi)$, and so $\widetilde{U}^0$ acts semisimply on N as well. If N is a *proper* submodule, it must be that $N \subset \sum_{\mu \in Q^+ \setminus \{0\}} M(\psi)_{\psi \cdot (\widehat{-\mu})}$ by (1.6), as $M(\psi)_\psi = \mathbb{K} v_\psi$ generates $M(\psi)$. Therefore $M(\psi)$ has a unique maximal submodule, namely the sum of all proper submodules, and a unique simple quotient, $L(\psi)$. In fact, all finite-dimensional simple $\widetilde{U}$-modules are of this form, as the following theorem demonstrates.

Theorem 2.1. *Let $\psi : \widetilde{U}^0 \to \mathbb{K}$ be an algebra homomorphism. Let M be a $\widetilde{U}$-module, on which $\widetilde{U}^0$ acts semisimply and which contains a nonzero element $m \in M_\psi$ such that $e_i.m = 0$ for all i $(1 \le i < n)$. Then there is a unique homomorphism of $\widetilde{U}$-modules $F : M(\psi) \to M$ with $F(v_\psi) = m$. In particular, if rs^{-1} is not a root of unity and M is a finite-dimensional simple $\widetilde{U}$-module, then $M \cong L(\psi)$ for some weight ψ.*

Proof. By the hypothesis on m, $\mathbb{K}m$ is a one-dimensional $\widetilde{U}^{\ge 0}$-submodule of M, considered as a $\widetilde{U}^{\ge 0}$-module by restriction. In fact, mapping v_ψ to m yields a $\widetilde{U}^{\ge 0}$-homomorphism from V^ψ to $\mathbb{K}m$. By the definition of $M(\psi)$, we have $\mathrm{Hom}_{\widetilde{U}}(M(\psi), M) \cong \mathrm{Hom}_{\widetilde{U}^{\ge 0}}(V^\psi, M)$, so there is a unique $\widetilde{U}$-module homomorphism $F : M(\psi) \to M$ with $F(v_\psi) = m$, namely $F(u \otimes v) = u.m$ for all $u \in \widetilde{U}$.

For the final assertion, note that $\widetilde{U}^0$ acts semisimply on any finite-dimensional simple module M, and by (1.6) and Proposition 1.7, there is some nonzero vector $m \in M_\psi$ such that $e_i.m = 0$ $(1 \le i < n)$. By the first part, M is a quotient of $M(\psi)$, and so $M \cong L(\psi)$, as $L(\psi)$ is the unique simple quotient of $M(\psi)$. □

As a special case, we will consider the modules $L(\lambda) = L(\hat{\lambda})$ where $\lambda \in \Lambda$. Let $\Lambda^+ \subset \Lambda$ be the subset of *dominant* weights, that is

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \alpha_i, \lambda \rangle \ge 0 \text{ for } 1 \le i < n\}.$$

Similarly, the set of dominant weights for $\mathfrak{sl}_n$ is

$$\Lambda^+_{\mathfrak{sl}} = \{\lambda \in \Lambda_{\mathfrak{sl}} \mid \langle \alpha_i, \lambda \rangle \ge 0 \text{ for } 1 \le i < n\} = \left\{ \sum_{i=1}^{n-1} \ell_i \varpi_i \mid \ell_i \in \mathbb{Z}_{\ge 0} \right\}.$$

We will show that if $L(\lambda)$ is finite-dimensional, then λ is dominant. This requires an identity for commuting e_i past powers of f_i. For $k \ge 1$, let

$$[k] = \frac{r^k - s^k}{r - s}. \tag{2.2}$$

Then the following lemma may be proven by induction.

Lemma 2.3. *If $k \ge 1$, then*

$$e_i f_i^k = f_i^k e_i + [k] f_i^{k-1} \frac{r^{1-k}\omega_i - s^{1-k}\omega_i'}{r - s}$$
$$e_i^k f_i = f_i e_i^k + [k] e_i^{k-1} \frac{s^{1-k}\omega_i - r^{1-k}\omega_i'}{r - s}.$$

Lemma 2.4. *Assume rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional module for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ on which $\widetilde{U}^0$ acts semisimply, and let $\lambda \in \Lambda$. Suppose there is some nonzero vector $v \in M_\lambda$ with $e_i.v = 0$ for all i $(1 \leq i < n)$. Then $\lambda \in \Lambda^+$. A similar statement is true for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda^+_{\mathfrak{sl}}$.*

Proof. Proposition 1.7 implies that for any given value of i there is some $k \geq 0$ such that $f_i^{k+1}.v = 0$ and $f_i^k.v \neq 0$. Applying e_i to $f_i^{k+1}.v = 0$ and using Lemma 2.3 and the fact that $e_i.v = 0$, we have

$$0 = [k+1] f_i^k \frac{r^{-k}\omega_i - s^{-k}\omega_i'}{r-s}.v = \frac{[k+1]}{r-s}(r^{-k}\hat{\lambda}(\omega_i) - s^{-k}\hat{\lambda}(\omega_i'))f_i^k.v.$$

Now $[k+1]/(r-s) \neq 0$ as rs^{-1} is not a root of unity. Therefore, since $f_i^k.v \neq 0$, we have $r^{-k}\hat{\lambda}(\omega_i) = s^{-k}\hat{\lambda}(\omega_i')$. Equivalently,

$$r^{-k} r^{\langle \epsilon_i, \lambda \rangle} s^{\langle \epsilon_{i+1}, \lambda \rangle} = s^{-k} r^{\langle \epsilon_{i+1}, \lambda \rangle} s^{\langle \epsilon_i, \lambda \rangle}, \quad \text{or} \quad r^{-k+\langle \alpha_i, \lambda \rangle} = s^{-k+\langle \alpha_i, \lambda \rangle}.$$

Again, because rs^{-1} is not a root of unity, this forces $\langle \alpha_i, \lambda \rangle = k \geq 0$, so $\lambda \in \Lambda^+$. □

Corollary 2.5. *When rs^{-1} is not a root of unity, any finite-dimensional simple $\widetilde{U}$-module with weights in Λ is isomorphic to $L(\lambda)$ for some $\lambda \in \Lambda^+$. An analogous result holds for U with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda^+_{\mathfrak{sl}}$.*

Next we will show that all modules $L(\lambda)$ with $\lambda \in \Lambda^+$ are indeed finite-dimensional, and that all other finite-dimensional simple $\widetilde{U}$-modules are shifts of these by one-dimensional modules. In doing this, it helps to consider first the special case of simple $U_{r,s}(\mathfrak{sl}_2)$-modules.

Highest weight modules for $U = U_{r,s}(\mathfrak{sl}_2)$.

For simplicity we drop the subscripts and just write e, f, ω, ω' for the generators of $U = U_{r,s}(\mathfrak{sl}_2)$. Any homomorphism $\phi : U^0 \to \mathbb{K}$ is determined by its values on ω and ω'. By abuse of notation, we adopt the shorthand $\phi = \phi(\omega)$ and $\phi' = \phi(\omega')$.

Corresponding to each such ϕ, there is a Verma module $M(\phi) = U \otimes_{U^{\geq 0}} \mathbb{K}v$ with basis $v_j = f^j \otimes v$ $(0 \leq j < \infty)$ such that the U-action is given by:

$$\begin{aligned}
f.v_j &= v_{j+1} \\
e.v_j &= [j]\frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r-s} v_{j-1} \qquad\qquad (v_{-1} := 0) \\
\omega.v_j &= \phi r^{-j\langle \epsilon_1, \alpha_1 \rangle} s^{-j\langle \epsilon_2, \alpha_1 \rangle} v_j = \phi r^{-j} s^j v_j \\
\omega'.v_j &= \phi' r^{-j\langle \epsilon_2, \alpha_1 \rangle} s^{-j\langle \epsilon_1, \alpha_1 \rangle} v_j = \phi' r^j s^{-j} v_j.
\end{aligned} \tag{2.6}$$

Note that $M(\phi)$ is a simple U-module if and only if $[j]\dfrac{\phi r^{-j+1} - \phi' s^{-j+1}}{r-s} \neq 0$ for any $j \geq 1$.

Suppose $[\ell+1]\dfrac{\phi r^{-\ell}-\phi' s^{-\ell}}{r-s}=0$ for some $\ell\geq 0$. Then either $r^{\ell+1}=s^{\ell+1}$, which implies rs^{-1} is a root of unity, or $\phi'=\phi r^{-\ell}s^{\ell}$. Assuming that rs^{-1} is not a root of unity and $\phi'=\phi r^{-\ell}s^{\ell}$, we see that the elements v_i, $i\geq \ell+1$, span a maximal submodule. The quotient is the $(\ell+1)$-dimensional simple module $L(\phi)$, which we can suppose is spanned by $v_0, v_1, \dots, v_\ell$ and has U-action given by

$$
\begin{aligned}
f.v_j &= v_{j+1}, \qquad (v_{\ell+1}=0)\\
e.v_j &= \phi r^{-\ell}[j][\ell+1-j]v_{j-1} \qquad (v_{-1}=0)\\
\omega.v_j &= \phi r^{-j}s^{j}v_j\\
\omega'.v_j &= \phi r^{-\ell+j}s^{\ell-j}v_j.
\end{aligned}
\tag{2.7}
$$

When $M(\phi)$ is not simple and rs^{-1} is not a root of unity, $j=\ell+1$ is the unique value such that $[j]\dfrac{\phi r^{-j+1}-\phi' s^{-j+1}}{r-s}=0$. In this case, $M(\phi)$ has a unique proper submodule, namely the maximal submodule generated by $v_{\ell+1}$ as above.

We now have the following classification of simple modules for $U_{r,s}(\mathfrak{sl}_2)$.

Proposition 2.8.

(i) *Assume $U=U_{r,s}(\mathfrak{sl}_2)$, where rs^{-1} is not a root of unity. Let $\phi: U^0\to\mathbb{K}$ be an algebra homomorphism such that $\phi(\omega')=\phi(\omega)r^{-\ell}s^{\ell}$ for some $\ell\geq 0$. Then there is an $(\ell+1)$-dimensional simple U-module $L(\phi)$ spanned by vectors $v_0, v_1, \dots, v_\ell$ and having U-action given by (2.7). Any $(\ell+1)$-dimensional simple U-module is isomorphic to some such $L(\phi)$.*

(ii) *If $\nu=\nu_1\epsilon_1+\nu_2\epsilon_2\in\Lambda^+$, then $\nu_1-\nu_2=\ell$ for some $\ell\in\mathbb{Z}_{\geq 0}$, and $\nu(\omega')=r^{\nu_2}s^{\nu_1}=r^{\nu_1-\ell}s^{\nu_2+\ell}=\nu(\omega)r^{-\ell}s^{\ell}$ in this case. Thus, the module $L(\nu)$ is $(\ell+1)$-dimensional and has U-action given by (2.7) with $\phi=r^{\nu_1}s^{\nu_2}=r^{\nu_1}s^{\nu_1-\ell}$.*

Finite-dimensionality of $L(\lambda)$ for $\lambda\in\Lambda^+$.

We show below that the simple modules $L(\lambda)$ for $\widetilde{U}=U_{r,s}(\mathfrak{gl}_n)$ with $\lambda\in\Lambda^+$ are finite-dimensional. For this it suffices to prove that $M(\lambda)$ has a $\widetilde{U}$-submodule of finite codimension, as $L(\lambda)$ is the quotient of $M(\lambda)$ by its unique maximal submodule.

As λ is dominant, $k_i=\langle\alpha_i,\lambda\rangle$ for $i=1,\dots,n-1$, are nonnegative integers. Define a $\widetilde{U}$-submodule $M'(\lambda)$ of $M(\lambda)$ by

$$
M'(\lambda)=\sum_{i=1}^{n-1}\widetilde{U}f_i^{k_i+1}.v_\lambda. \tag{2.9}
$$

Our goal is to prove that the module $L'(\lambda):=M(\lambda)/M'(\lambda)$ is nonzero and finite-dimensional.

By Lemma 2.3 we have $e_i f_i^{k_i+1}.v_\lambda=0$. If $j\neq i$, $e_j f_i^{k_i+1}.v_\lambda=f_i^{k_i+1}e_j.v_\lambda=0$ by the defining relations. Consequently, by Theorem 2.1, $\widetilde{U}f_i^{k_i+1}.v_\lambda$ is a homomorphic image of $M(\lambda-(k_i+1)\alpha_i)$, and so all its weights are less than or equal to $\lambda-(k_i+1)\alpha_i$. This implies that $v_\lambda\notin M'(\lambda)$, hence $L'(\lambda)\neq 0$.

Lemma 2.10. *The elements e_j, f_j $(1 \leq j < n)$ act locally nilpotently on $L'(\lambda)$.*

Proof. As the Verma module $M(\lambda)$ is spanned over $\mathbb{K}$ by all elements $x_1 \cdots x_t.v_\lambda$ where $x_1, \ldots, x_t \in \{f_1, \ldots, f_{n-1}\}$, $t \in \mathbb{Z}_{\geq 0}$, it is enough to argue by induction on t that a sufficiently high power of e_j (resp., f_j) takes such an element to $M'(\lambda)$. If $t = 0$, then $e_j.v_\lambda = 0 \in M'(\lambda)$, and $f_j^{k_j+1}.v_\lambda \in M'(\lambda)$ by construction. Now assume that there are positive integers N_j such that

$$e_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda) \text{ and } f_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda).$$

Suppose that $x_1 = f_i$. If $j \neq i$, then $e_j^{N_j} x_1 \cdots x_t.v_\lambda = f_i e_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$. Otherwise by Lemma 2.3,

$$e_i^{N_i+1} x_1 \cdots x_t.v_\lambda = f_i e_i^{N_i+1} x_2 \cdots x_t.v_\lambda + [N_i + 1] e_i^{N_i} \frac{s^{-N_i}\omega_i - r^{-N_i}\omega_i'}{r - s} x_2 \cdots x_t.v_\lambda.$$

Applying relation (R2') and the induction hypothesis, we see that these terms are both in $M'(\lambda)$.

Now $f_i^{N_i-1} x_1 \cdots x_t.v_\lambda = f_i^{N_i} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$, and if $|i - j| > 1$, we also have $f_j^{N_j} x_1 \cdots x_t.v_\lambda = f_i f_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$. Finally, we need to show that if $|i - j| = 1$, then $f_j^{N_j+1} x_1 \cdots x_t.v_\lambda \in M'(\lambda)$. This will follow from the induction hypothesis once we know that $f_j^{N_j+1} f_i \in \mathbb{K} f_j f_i f_j^{N_j} + \mathbb{K} f_i f_j^{N_j+1}$.

We argue by induction on $m \geq 1$ that

$$f_j^{m+1} f_i \in \mathbb{K} f_j f_i f_j^m + \mathbb{K} f_i f_j^{m+1}.$$

Indeed if $m = 1$, this follows from relation (R7); but if $m > 1$, then by induction and (R7),

$$f_j^{m+1} f_i \in f_j(\mathbb{K} f_j f_i f_j^{m-1} + \mathbb{K} f_i f_j^m) \subseteq \mathbb{K} f_j f_i f_j^m + \mathbb{K} f_i f_j^{m+1}. \quad \square$$

Lemma 2.11. *Assume rs^{-1} is not a root of unity, and let V be a module for $U = U_{r,s}(\mathfrak{sl}_2)$ on which U^0 acts semisimply. Suppose $V = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} V_{\lambda - j\alpha}$ for some weight $\lambda \in \Lambda$; each weight space of V is finite-dimensional; and e and f act locally nilpotently on V. Then V is finite-dimensional, and the weights of V are preserved under the simple reflection taking α to $-\alpha$.*

Proof. Let $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2$ be a weight of V, and $v \in V_\mu \setminus \{0\}$. As e acts locally nilpotently on V, there is a nonnegative integer k such that $e^{k+1}.v = 0$ and $e^k.v \neq 0$. By Theorem 2.1, $Ue^k.v$ is a homomorphic image of $M(\mu + k\alpha)$. But since f acts locally nilpotently on $Ue^k.v$, this image cannot be isomorphic to $M(\mu + k\alpha)$. Thus because $M(\mu + k\alpha)$ has a unique proper submodule, $Ue^k.v \cong L(\mu + k\alpha)$, and so it is finite-dimensional. Corollary 2.5 implies that $\mu + k\alpha$ is dominant. As there are only finitely many dominant weights less than or equal to the given weight λ (under the partial order $\nu \leq \lambda$ if and only if $\lambda - \nu \in \mathbb{Z}_{\geq 0}\alpha$), and each weight space is finite-dimensional, it must be that V itself is finite-dimensional.

In particular, V has a composition series with factors isomorphic to $L(\nu)$ for some $\nu \in \Lambda^+$. Any weight μ of V is a weight of some such $L(\nu)$ with $\nu = \nu_1\epsilon_1 + \nu_2\epsilon_2 \in \Lambda^+$. By (ii) of Proposition 2.8, $L(\nu)$ has weights $\nu, \nu - \alpha, \ldots, \nu - \ell\alpha$ where $\ell = \nu_1 - \nu_2$. Thus, $\mu = \nu - j\alpha$ for some $j \in \{0, 1, \ldots, \ell\}$. But then $\mu - \langle \mu, \alpha \rangle \alpha = \nu - (\ell - j)\alpha$ is a weight of $L(\nu)$ since $\ell - j \in \{0, 1, \ldots, \ell\}$, hence it is a weight of V. Thus, the weights of V are preserved under the simple reflection taking α to $-\alpha$. □

Lemma 2.12. *Assume rs^{-1} is not a root of unity, and let $\lambda \in \Lambda^+$. Then the module $L(\lambda)$ for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ is finite-dimensional. A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$, where Λ^+ is replaced by $\Lambda^+_{\mathfrak{sl}}$.*

Proof. This follows once we show that $L'(\lambda) = M(\lambda)/M'(\lambda)$, where $M'(\lambda)$ is as in (2.9), is finite-dimensional. We will prove that the set of weights of $L'(\lambda)$ is preserved under the action of the symmetric group S_n (the Weyl group of $\mathfrak{gl}_n$) on Λ which is generated by the simple reflections $s_i : \mu \to \mu - \langle \mu, \alpha_i \rangle \alpha_i$ $(1 \leq i < n)$. Each S_n-orbit contains a dominant weight, and there are only finitely many dominant weights less than or equal to λ. As the weights in $M(\lambda)$ are all less than or equal to λ, and the weight spaces are finite-dimensional, the same is true of $L'(\lambda)$. Therefore $L'(\lambda)$ is finite-dimensional.

To see that s_i preserves the set of weights of $L'(\lambda)$, let $\mu = \mu_1\epsilon_1 + \cdots + \mu_n\epsilon_n$ be a weight of $L'(\lambda)$. Consider $L'(\lambda)$ as a module for the copy U_i of $U_{r,s}(\mathfrak{sl}_2)$ generated by $e_i, f_i, \omega_i, \omega_i'$, and let $L_i'(\mu)$ be the U_i-submodule of $L'(\lambda)$ generated by $L'(\lambda)_\mu$. As all weights of $L'(\lambda)$ are less than or equal to λ, we have

$$L_i'(\mu) = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L_i'(\mu)_{\lambda' - j\alpha_i}$$

for some weight $\lambda' \leq \lambda$. By Lemmas 2.10 and 2.11, the simple reflection s_i preserves the weights of $L_i'(\mu)$, so in particular, $s_i(\mu)$ is also a weight of $L'(\lambda)$. □

Remark 2.13. It will follow from Lemma 3.7 in the next section that $L(\lambda) \cong L'(\lambda)$, since $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, $L'(\lambda)$ is a finite-dimensional quotient of $M(\lambda)$, and by that lemma, every finite-dimensional quotient is simple. Thus, we have

Corollary 2.14. *When rs^{-1} is not a root of unity, the finite-dimensional simple $\widetilde{U}$-modules having weights in Λ are precisely the modules $L(\lambda)$ where $\lambda \in \Lambda^+$. Moreover, $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$. Similar statements hold for $U = U_{r,s}(\mathfrak{sl}_n)$, where Λ is replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda^+_{\mathfrak{sl}}$.*

Proof. The first statement is a consequence of Corollary 2.5 and Lemma 2.12 (see also Remark 2.13). Assume there is an isomorphism of $\widetilde{U}$-modules from $L(\lambda)$ to $L(\mu)$. The highest weight vector of $L(\lambda)$ must be sent to a weight vector of $L(\mu)$, so $\lambda \leq \mu$. As a similar argument shows that $\mu \leq \lambda$, we have $\lambda = \mu$. □

Shifts by one-dimensional modules.

Suppose now that we have a one-dimensional module L for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$. Then by Theorem 2.1, $L = L(\chi)$ for some algebra homomorphism $\chi : \widetilde{U}^0 \to \mathbb{K}$, with the elements e_i, f_i $(1 \le i < n)$ acting as multiplication by 0. Relation (R4) yields

$$\chi(\omega_i) = \chi(a_i b_{i+1}) = \chi(a_{i+1} b_i) = \chi(\omega_i') \qquad (1 \le i < n). \tag{2.15}$$

Conversely, if an algebra homomorphism χ satisfies this equation, then $L(\chi)$ is one-dimensional by relation (R4). We will write $L_\chi = L(\chi)$ to emphasize that the module is one-dimensional.

Proposition 2.16. *Assume rs^{-1} is not a root of unity and $L(\psi)$ is the finite-dimensional simple module for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ with highest weight ψ. Then there exists a homomorphism $\chi : \widetilde{U}^0 \to \mathbb{K}$ such that (2.15) holds and an element $\lambda \in \Lambda^+$ so that $\psi = \chi \cdot \hat{\lambda}$. Thus, the weights of $L(\psi)$ belong to $\chi \cdot \hat{\Lambda}$. A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda^+_{\mathfrak{sl}}$.*

Proof. When $L(\psi)$ is viewed as a module for the copy U_i of $U_{r,s}(\mathfrak{sl}_2)$ generated by $e_i, f_i, \omega_i, \omega_i'$, it has a composition series whose factors are simple U_i-modules as described by Proposition 2.8. As the highest weight vector of $L(\psi)$ gives a highest weight vector of some composition factor, there is a weight ϕ_i of U_i and a nonnegative integer ℓ_i so that $\psi(\omega_i) = \phi_i(\omega_i)$ and $\psi(\omega_i') = \phi_i(\omega_i') = \phi_i(\omega_i) r^{-\ell_i} s^{\ell_i} = \psi(\omega_i) r^{-\ell_i} s^{\ell_i}$.

For the case $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$, set $\ell_n = 0$ and define $\lambda_i = \ell_i + \cdots + \ell_n$ for $i = 1, \ldots, n$. Let $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$, which belongs to Λ^+. Now define $\chi : \widetilde{U}^0 \to \mathbb{K}$ by the formulas

$$\chi(a_i) = \psi(a_i) r^{-\langle \epsilon_i, \lambda \rangle} = \psi(a_i) r^{-(\ell_i + \cdots + \ell_n)}$$
$$\chi(b_i) = \psi(b_i) s^{-\langle \epsilon_i, \lambda \rangle} = \psi(b_i) s^{-(\ell_i + \cdots + \ell_n)}.$$

Then it follows that $\chi(\omega_i') = \chi(\omega_i)$ for $i = 1, \ldots, n-1$, and $\psi = \chi \cdot \hat{\lambda}$ as desired.

For the case $U = U_{r,s}(\mathfrak{sl}_n)$, fix an nth root $(rs)^{1/n}$ of rs in $\mathbb{K}$. As above, there exist nonnegative integers ℓ_i $(1 \le i < n)$ so that $\psi(\omega_i') = \psi(\omega_i) r^{-\ell_i} s^{\ell_i}$. Let $\lambda = \sum_{i=1}^{n-1} \ell_i \varpi_i \in \Lambda^+_{\mathfrak{sl}}$. Define $\chi : U^0 \to \mathbb{K}$ by

$$\chi(\omega_i) = \psi(\omega_i) r^{-(\ell_i + \cdots + \ell_{n-1})} s^{-(\ell_{i+1} + \cdots + \ell_{n-1})} (rs)^{(c/n)}$$
$$\chi(\omega_i') = \psi(\omega_i') r^{-(\ell_{i+1} + \cdots + \ell_{n-1})} s^{-(\ell_i + \cdots + \ell_{n-1})} (rs)^{(c/n)}$$

where $c = \sum_{j=1}^{n-1} j \ell_j$. Then (2.15) holds, and $\psi = \chi \cdot \hat{\lambda}$. □

Remark 2.17. If M is any finite-dimensional module, then $M = \bigoplus_{i=1}^m \bigoplus_{\lambda \in \Lambda} M_{\psi_i \cdot \hat{\lambda}}$ for some weights ψ_i such that $\psi_i \cdot \hat{\Lambda}$ $(1 \le i \le m)$ are distinct cosets in $\mathrm{Hom}(\widetilde{U}^0, \mathbb{K}) / \hat{\Lambda}$ (viewed as a $\mathbb{Z}$-module under the action $k \cdot \psi = \psi^k$). Then $M_i := \bigoplus_{\lambda \in \Lambda} M_{\psi_i \cdot \hat{\lambda}}$ is a submodule, and $M = \bigoplus_{i=1}^m M_i$. Therefore, if M is an indecomposable $\widetilde{U}$-module, $M = \bigoplus_{\lambda \in \Lambda} M_{\psi \cdot \hat{\lambda}}$ for some $\psi \in \mathrm{Hom}(\widetilde{U}^0, \mathbb{K})$. A simple submodule S of M has weights in $\psi \cdot \hat{\Lambda}$. By replacing ψ with the homomorphism χ for S given by Proposition 2.16, we may assume that for any indecomposable module M, there is a χ satisfying (2.15) so that $M = \bigoplus_{\lambda \in \Lambda} M_{\chi \cdot \hat{\lambda}}$.

Lemma 2.18. *Let $\chi : \widetilde{U}^0 \to \mathbb{K}$ be an algebra homomorphism with $\chi(\omega_i) = \chi(\omega_i')$ $(1 \le i < n)$. Let M be a finite-dimensional $\widetilde{U}$-module whose weights are all in $\chi \cdot \hat{\Lambda}$. If $\widetilde{U}^0$ acts semisimply on M, then*

$$M \cong L_\chi \otimes N$$

for some $\widetilde{U}$-module N whose weights are all in Λ. A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$.

Proof. Let $\chi^{-1} : \widetilde{U}^0 \to \mathbb{K}$ be the algebra homomorphism defined by $\chi^{-1}(a_i) = \chi(a_i^{-1}) = (\chi(a_i))^{-1}$ and $\chi^{-1}(b_i) = \chi(b_i^{-1}) = (\chi(b_i))^{-1}$ for $1 \le i \le n$. Note that $L_\chi \otimes L_{\chi^{-1}}$ is isomorphic to the trivial module L_ε corresponding to the counit. Let

$$N = L_{\chi^{-1}} \otimes M.$$

Then $M \cong L_\chi \otimes N$ as L_ε is a multiplicative identity (up to isomorphism) for $\widetilde{U}$-modules. The weights of N are all in $\chi^{-1} \cdot \chi \cdot \hat{\Lambda} = \hat{\Lambda}$. □

We now have a classification of finite-dimensional simple modules for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and for $U = U_{r,s}(\mathfrak{sl}_n)$.

Theorem 2.19. *Assume rs^{-1} is not a root of unity. The finite-dimensional simple $\widetilde{U}$-modules are precisely the modules*

$$L_\chi \otimes L(\lambda),$$

where $\chi : \widetilde{U}^0 \to \mathbb{K}$ is an algebra homomorphism with $\chi(\omega_i) = \chi(\omega_i')$ $(1 \le i < n)$, and $\lambda \in \Lambda^+$. An analogous statement holds for U with Λ^+ replaced by $\Lambda^+_{\mathfrak{sl}}$.

Proof. Let M be a finite-dimensional simple $\widetilde{U}$-module. By Theorem 2.1, Proposition 2.16, and Lemma 2.18, $M \cong L_\chi \otimes N$ for some χ satisfying (2.15) and some simple module N with weights in Λ. By Corollary 2.5, $N \cong L(\lambda)$ for some $\lambda \in \Lambda^+$. Conversely, any $\widetilde{U}$-module of this form is finite-dimensional by Lemma 2.12 and simple by its construction. □

Remark 2.20. If $r = q$ and $s = q^{-1}$ for some $q \in \mathbb{K}$, the classification of finite-dimensional simple $U_q(\mathfrak{sl}_n)$-modules is a consequence of Theorem 2.19 applied to $U_{q,q^{-1}}(\mathfrak{sl}_n)$: The simple $U_q(\mathfrak{sl}_n)$-modules are precisely those simple $U_{q,q^{-1}}(\mathfrak{sl}_n)$-modules on which ω_i' acts as ω_i^{-1}, so that

$$\chi(\omega_i) = \chi(\omega_i') = \chi(\omega_i^{-1}).$$

This implies $\chi(\omega_i) = \pm 1$ $(1 \le i < n)$. Each choice of algebra homomorphism $\chi : U^0 \to \mathbb{K}$ with $\chi(\omega_i) = \chi(\omega_i') = \pm 1$ yields a one-dimensional $U_{q,q^{-1}}(\mathfrak{sl}_n)$-module L_χ, and so the simple $U_q(\mathfrak{sl}_n)$-modules are the $L_\chi \otimes L(\lambda)$ with $\lambda \in \Lambda^+_{\mathfrak{sl}}$ and χ as above. We have

$$\hat{\lambda}(\omega_i) = q^{\langle \epsilon_i, \lambda \rangle} q^{-\langle \epsilon_{i+1}, \lambda \rangle} = q^{\langle \alpha_i, \lambda \rangle}.$$

Thus, we recover the results of [Ja, 5.2 and 5.10].

Remark 2.21. We can interpret Proposition 2.8 in light of Theorem 2.19: Let $L(\phi)$ be the simple $U_{r,s}(\mathfrak{sl}_2)$-module described in the proposition. Let $\lambda = (\ell/2)\alpha \in \Lambda^+_{\mathfrak{sl}}$ and define $\chi : U^0 \to \mathbb{K}$ by $\chi(\omega) = \phi(\omega) r^{-\ell/2} s^{\ell/2}$, $\chi(\omega') = \phi(\omega') r^{\ell/2} s^{-\ell/2} = \phi(\omega) r^{-\ell/2} s^{\ell/2} = \chi(\omega)$. Then $\phi = \chi \cdot \hat{\lambda}$ and $L(\phi) \cong L_\chi \otimes L(\lambda)$.

§3. Complete reducibility

In this section we will establish complete reducibility of all finite-dimensional modules for $\widetilde{U}$ (resp., U) on which $\widetilde{U}^0$ (resp., U^0) acts semisimply. Statements made for $\widetilde{U}$ hold equally well for U, and we will point out where there are differences. It is helpful to work in a more general context.

Let $\mathcal{O}$ denote the category of modules M for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ which satisfy the conditions:

($\mathcal{O}1$) $\widetilde{U}^0$ acts semisimply on M, and the set $\mathrm{wt}(M)$ of weights of M belongs to Λ: $M = \bigoplus_{\lambda\in\mathrm{wt}(M)} M_\lambda$, where $M_\lambda = \{m \in M \mid a_i.m = r^{\langle\epsilon_i,\lambda\rangle},\ \ b_i.m = s^{\langle\epsilon_i,\lambda\rangle}$ for all $i\}$;
($\mathcal{O}2$) $\dim_{\mathbb{K}} M_\lambda < \infty$ for all $\lambda \in \mathrm{wt}(M)$;
($\mathcal{O}3$) $\mathrm{wt}(M) \subseteq \bigcup_{\mu\in F}(\mu - Q^+)$ for some finite set $F \subset \Lambda$.

The morphisms in $\mathcal{O}$ are $\widetilde{U}$-module homomorphisms. In defining category $\mathcal{O}$ for $U = U_{r,s}(\mathfrak{sl}_n)$, we replace Λ by the weight lattice $\Lambda_{\mathfrak{sl}}$ of $\mathfrak{sl}_n$.

All finite-dimensional $\widetilde{U}$-modules which satisfy ($\mathcal{O}1$) belong to category $\mathcal{O}$, as do all highest weight modules with weights in Λ such as the Verma modules $M(\lambda)$.

We recall the definition of the quantum Casimir operator [BW2, Sec. 4]. It is a consequence of (R2) and (R3) that the subalgebra U^+ of $\widetilde{U}$ (or of $U = U_{r,s}(\mathfrak{sl}_n)$) generated by 1 and e_i $(1 \le i < n)$ has the decomposition $U^+ = \bigoplus_{\zeta\in Q^+} U^+_\zeta$ where

$$U^+_\zeta = \{z \in U^+ \mid a_i z = r^{\langle\epsilon_i,\zeta\rangle} z a_i,\ b_i z = s^{\langle\epsilon_i,\zeta\rangle} z b_i\ (1 \le i < n)\}.$$

The weight space U^+_ζ is spanned by all the monomials $e_{i_1}\cdots e_{i_\ell}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_\ell} = \zeta$. Similarly, the subalgebra U^- generated by 1 and the f_i has the decomposition $U^- = \bigoplus_{\zeta\in Q^+} U^-_{-\zeta}$. The spaces U^+_ζ and $U^-_{-\zeta}$ are nondegenerately paired by the Hopf pairing specified by

$$\begin{aligned}
&(f_i, e_j) = \frac{\delta_{i,j}}{s-r} \\
&(\omega_i', \omega_j) = r^{\langle\epsilon_j,\alpha_i\rangle} s^{\langle\epsilon_{j+1},\alpha_i\rangle} \qquad (3.1)\\
&(b_n, a_n) = 1, \quad (b_n, \omega_j) = s^{-\langle\epsilon_n,\alpha_j\rangle}, \quad (\omega_i', a_n) = r^{\langle\epsilon_n,\alpha_i\rangle}.
\end{aligned}$$

(See [BW2, Sec. 2].) The Hopf algebras $\widetilde{U}$ and U are Drinfel'd doubles of certain Hopf subalgebras with respect to this pairing [BW2, Thm. 2.7]. Let $d_\zeta = \dim_{\mathbb{K}} U^+_\zeta$. Assume $\{u^\zeta_k\}_{k=1}^{d_\zeta}$ is a basis for U^+_ζ, and $\{v^\zeta_k\}_{k=1}^{d_\zeta}$ is the dual basis for $U^-_{-\zeta}$ with respect to the pairing.

Now let

$$\Omega = \sum_{\zeta\in Q^+} \sum_{k=1}^{d_\zeta} S(v^\zeta_k) u^\zeta_k, \qquad (3.2)$$

where S denotes the antipode. All but finitely many terms in this sum will act as multiplication by 0 on any weight space M_λ of $M \in \mathcal{O}$. Therefore Ω is a well-defined operator on such a module M.

The second part of the Casimir operator involves a function $g : \Lambda \to \mathbb{K}^{\#}$ defined as follows. If ρ denotes half the sum of the positive roots, then $2\rho = \sum_{j=1}^{n}(n+1-2j)\epsilon_j \in \Lambda$. For $\lambda \in \Lambda$, set

$$g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho,\lambda\rangle}. \tag{3.3}$$

When M is a $\widetilde{U}$-module in $\mathcal{O}$, we define the linear operator $\Xi : M \to M$ by

$$\Xi(m) = g(\lambda)m$$

for all $m \in M_\lambda$, $\lambda \in \Lambda$. Then Ξ is well-defined, as $\hat{\lambda} = \hat{\mu}$ if and only if $\lambda = \mu$ $(\lambda, \mu \in \Lambda)$ [BW2, Prop. 3.5]. (When $U = U_{r,s}(\mathfrak{sl}_n)$, it is necessary to first fix roots $r^{1/2n}$ and $s^{1/2n}$ of r and s in $\mathbb{K}$.) Then Ξ, as given above, is well-defined. We have the following result from [BW2].

Proposition 3.4. *[BW2, Thm. 4.20] The operator $\Omega\Xi : M \to M$ commutes with the action of $\widetilde{U}$ on any $\widetilde{U}$-module $M \in \mathcal{O}$.*

We require the next lemma in order to prove complete reducibility.

Lemma 3.5. *Assume rs^{-1} is not a root of unity, and let $\lambda, \mu \in \Lambda^+$. If $\lambda \geq \mu$ and $g(\lambda) = g(\mu)$, then $\lambda = \mu$.*

Proof. Because $\lambda \geq \mu$, we may suppose $\lambda = \mu + \beta$ where $\beta = \sum_{i=1}^{n-1} k_i\alpha_i$ and $k_i \in \mathbb{Z}_{\geq 0}$. By assumption we have

$$(rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho,\lambda\rangle} = g(\lambda) = g(\mu) = (rs^{-1})^{\frac{1}{2}\langle \mu+2\rho,\mu\rangle},$$

and as rs^{-1} is not a root of unity, it must be that $\langle \lambda + 2\rho, \lambda\rangle = \langle \mu + 2\rho, \mu\rangle$, or equivalently, $2\langle \mu + \rho, \beta\rangle + \langle \beta, \beta\rangle = 0$. Since $\mu \in \Lambda^+$, $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_n\epsilon_n$ where $\mu_i \in \mathbb{Z}$ for all i and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. A calculation yields

$$0 = \langle 2\mu+2\rho,\beta\rangle + \langle \beta,\beta\rangle = \sum_{i=1}^{n-1} 2k_i\big(\mu_i - \mu_{i+1} + 1\big) + \sum_{i=1}^{n}(k_i - k_{i-1})^2, \quad (k_0 = 0 = k_n).$$

The only way this can happen is if $k_i = 0$ for all i and $\lambda = \mu$. $\square$

Lemma 3.6. *Assume rs^{-1} is not a root of unity.*

(i) *$\Omega\Xi$ acts as multiplication by the scalar $g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho,\lambda\rangle}$ on the Verma module $M(\lambda)$ with $\lambda \in \Lambda$, hence on any submodule or quotient of $M(\lambda)$.*

(ii) *The eigenvalues of the operator $\Omega\Xi : M \to M$ are integral powers of $(rs^{-1})^{\frac{1}{2}}$ on any finite-dimensional $M \in \mathcal{O}$. (For $U = U_{r,s}(\mathfrak{sl}_n)$, the eigenvalues are integral powers of $(rs^{-1})^{1/2n^2}$.)*

Proof. By its construction, $\Omega\Xi$ acts by multiplication by $g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle\lambda+2\rho,\lambda\rangle}$ on the maximal vector v_λ of $M(\lambda)$. But since $M(\lambda) = \widetilde{U}.v_\lambda$ and $\Omega\Xi$ commutes with $\widetilde{U}$ on modules in $\mathcal{O}$, $\Omega\Xi$ acts as multiplication by $(rs^{-1})^{\frac{1}{2}\langle\lambda+2\rho,\lambda\rangle}$ on all of $M(\lambda)$.

If $M \in \mathcal{O}$ is finite-dimensional, it has a composition series. Each factor is a finite-dimensional simple $\widetilde{U}$-module with weights in Λ, and in particular, is a quotient of $M(\lambda)$ for some $\lambda \in \Lambda$. On such a factor, $\Omega\Xi$ acts as multiplication by $g(\lambda)$. Therefore the action of $\Omega\Xi$ on M may be expressed by an upper triangular matrix with each diagonal entry equal to $g(\lambda)$ for some $\lambda \in \Lambda$. □

Lemma 3.7. *Assume rs^{-1} is not a root of unity. Let $\lambda \in \Lambda$ and M be a nonzero finite-dimensional quotient of the Verma module $M(\lambda)$. Then M is simple.*

Proof. First observe that by Lemma 2.4, $\lambda \in \Lambda^+$. Assume M' is a proper submodule of M. As M is generated by its one-dimensional subspace M_λ, we must have $M'_\lambda = 0$. Let $\mu \in \Lambda$ be maximal such that $M'_\mu \neq 0$, and note that $\mu < \lambda$. Let m' be a nonzero vector of M'_μ. It follows from the maximality of μ that $e_i.m' = 0$ for all i $(1 \leq i < n)$. Letting $M'' = \widetilde{U}.m'$, a nonzero finite-dimensional quotient of $M(\mu)$, we see that $\mu \in \Lambda^+$ as well. By Lemma 3.6 (i), $\Omega\Xi$ acts as multiplication by $g(\lambda)$ on M, and by $g(\mu)$ on M''. This forces $g(\lambda) = g(\mu)$, which contradicts Lemma 3.5 as $\mu < \lambda$. □

Finally we state the needed complete reducibility result, whose proof parallels that of [L, Thm. 6.2.2].

Theorem 3.8. *Assume rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional $\widetilde{U}$-module on which $\widetilde{U}^0$ acts semisimply. Then M is completely reducible.*

Proof. Suppose first that M has weights in Λ. Then M is a direct sum of generalized eigenspaces for $\Omega\Xi$, which by Proposition 3.4, is a direct sum decomposition of M as a $\widetilde{U}$-module. Therefore, we may assume M is itself a generalized eigenspace of $\Omega\Xi$, so that $(\Omega\Xi - (rs^{-1})^c)^d(M) = 0$ for some $c \in \frac{1}{2}\mathbb{Z}$, $d = \dim_{\mathbb{K}} M$, by Lemma 3.6 (ii).

Let $P = \{m \in M \mid e_i.m = 0 \ (1 \leq i < n)\}$, and note that $P = \bigoplus_{\lambda\in\Lambda} P_\lambda$, $P_\lambda = P \cap M_\lambda$. If $m \in P_\lambda - \{0\}$, the $\widetilde{U}$-submodule $\widetilde{U}.m$ of M is a nonzero quotient of $M(\lambda)$ by Theorem 2.1. By Lemma 3.7, each such $\widetilde{U}.m$ is a simple $\widetilde{U}$-module, and so the $\widetilde{U}$-submodule M' of M generated by P is a sum of simple $\widetilde{U}$-modules. That is, M' is completely reducible. Let $M'' = M/M'$.

Assuming $M'' \neq 0$, there is a weight $\mu \in \Lambda$ maximal such that $M''_\mu \neq 0$. It follows from the maximality of μ that $e_i.m'' = 0$ for $m'' \in M''_\mu - \{0\}$ and for all i $(1 \leq i < n)$. By Lemma 2.4, $\mu \in \Lambda^+$, and by Theorem 2.1 and Lemma 3.6, $\Omega\Xi$ acts as multiplication by $g(\mu)$ on the U-module $U.m''$ generated by m''. This implies $g(\mu) = (rs^{-1})^c$.

Let $m \in M_\mu$ be a representative for $m'' \in (M/M')_\mu$, and $M_1 = \widetilde{U}.m$. Then there is a weight $\eta \in \Lambda$ maximal such that $M_1 \cap M_\eta \neq 0$. Let $m_1 \in M_1 \cap M_\eta - \{0\}$, so that $e_i.m_1 = 0$ for all i $(1 \le i < n)$. Again applying Theorem 2.1 and Lemmas 2.4 and 3.6, we have $\eta \in \Lambda^+$ and $\Omega\Xi(m_1) = g(\eta)m_1$. Therefore $g(\eta) = (rs^{-1})^c$.

We now have $g(\mu) = g(\eta)$, where $\eta, \mu \in \Lambda^+$, and $\eta \ge \mu$ by construction. By Lemma 3.5, $\eta = \mu$, so M_1 is the one-dimensional space spanned by m, and $e_i.m = 0$ $(1 \le i < n)$, that is $m \in P$. This implies $m'' = 0$, a contradiction to the assumption that $M'' \ne 0$. Therefore $M'' = 0$, and $M = M'$ is completely reducible.

Finally, when M does not have weights in Λ, we may assume that M is indecomposable. By Remark 2.17, M has all its weights in $\chi \cdot \hat{\Lambda}$ for some χ satisfying (2.15). By Lemma 2.18, $M \cong L_\chi \otimes N$ for some $\widetilde{U}$-module N whose weights are all in Λ. Note that $\widetilde{U}^0$ acts semisimply on N as well ($N = L_{\chi^{-1}} \otimes M$), and so N is completely reducible by the above argument. This implies that M itself is completely reducible. □

Remark 3.9. It is necessary to include the hypothesis that $\widetilde{U}^0$ acts semisimply in Theorem 3.8, as the next examples illustrate. (Recall that $\widetilde{U}^0$ does indeed act semisimply on any simple $\widetilde{U}$-module, as remarked in the text following (1.6).) Let $V = \mathbb{K}^m$ for $m \ge 2$ and $\xi, \xi' \in \mathbb{K} \setminus \{0\}$. We define a $\widetilde{U}$-module structure on V by requiring that e_i, f_i act as multiplication by 0 and a_i, b_i act via the $m \times m$ Jordan blocks with diagonal entries ξ, ξ', respectively. Then relations (R1)-(R7) of $\widetilde{U}$ hold on V. The scalars ξ, ξ' may be chosen so that V has weights in Λ, for example choose an integer c, let $\lambda = c(\epsilon_1 + \cdots + \epsilon_n)$, and set $\xi = r^c = \hat{\lambda}(a_i)$, $\xi' = s^c = \hat{\lambda}(b_i)$. Clearly V is not completely reducible. A related example for $U_{r,s}(\mathfrak{sl}_n)$ is given by sending each ω_i, ω_i' to the Jordan block with 1s on the diagonal, thus corresponding to the weight 0.

§4. The R-matrix

In this section we recall the definition of the R-matrix from [BW2, Sec. 4] and use it to prove a more general result on the commutativity of the tensor product of finite-dimensional modules than was given there (compare [BW2, Thm. 4.11] with Theorem 4.2 below). Let M, M' be $\widetilde{U}$-modules in category $\mathcal{O}$. We define an isomorphism of $\widetilde{U}$-modules $R_{M',M} : M' \otimes M \to M \otimes M'$ as follows. If $\lambda = \sum_{i=1}^n \lambda_i \alpha_i \in \Lambda$, where $\alpha_n = \epsilon_n$, set

$$\omega_\lambda = \omega_1^{\lambda_1} \cdots \omega_{n-1}^{\lambda_{n-1}} a_n^{\lambda_n} \quad \text{and} \quad \omega_\lambda' = (\omega_1')^{\lambda_1} \cdots (\omega_{n-1}')^{\lambda_{n-1}} b_n^{\lambda_n}.$$

Also let

$$\Theta = \sum_{\zeta \in Q^+} \sum_{k=1}^{d_\zeta} v_k^\zeta \otimes u_k^\zeta,$$

where the notation is as in the paragraph following (3.1). Define

$$R_{M',M} = \Theta \circ \widetilde{f} \circ P$$

where $P(m' \otimes m) = m \otimes m'$, $\widetilde{f}(m \otimes m') = (\omega_\mu', \omega_\lambda)^{-1}(m \otimes m')$ when $m \in M_\lambda$ and $m' \in M_\mu'$, and the Hopf pairing (,) is defined in (3.1). (There is an equivalent definition of $\widetilde{f}$ that works in the case $U = U_{r,s}(\mathfrak{sl}_n)$, given in [BW2] after (4.3).) Then $R_{M',M}$ is an isomorphism of $\widetilde{U}$-modules that satisfies the quantum Yang-Baxter equation and the hexagon identities [BW2, Thms. 4.11, 5.4, and 5.7].

We will show that the tensor product of *any* two finite-dimensional $\widetilde{U}$-modules in $\mathcal{O}$ is commutative (up to module isomorphism), starting first with the special case that one of the modules is a one-dimensional module $L_\chi = L(\chi)$, as defined in Section 2.

Lemma 4.1. *Let M be a $\widetilde{U}$-module in category $\mathcal{O}$, and let L_χ be a one-dimensional $\widetilde{U}$-module. Then*

$$L_\chi \otimes M \cong M \otimes L_\chi.$$

Proof. Fix a basis element v of L_χ. Define a linear function $F : L_\chi \otimes M \to M \otimes L_\chi$ as follows. If $m \in M_\lambda$ and $\lambda = -\sum_{i=1}^n c_i\alpha_i$, then

$$F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n} m \otimes v,$$

where $\chi_i = \chi(\omega_i) = \chi(\omega_i')$ $(1 \le i < n)$ and $\chi_n = \chi(a_n)$. Clearly F is bijective, and we check that F is a $\widetilde{U}$-homomorphism:

$$e_i.F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n}(e_i \otimes 1 + \omega_i \otimes e_i)(m \otimes v) = \chi_1^{c_1} \cdots \chi_n^{c_n} e_i.m \otimes v$$

On the other hand, as $e_i.m \in M_{\lambda+\alpha_i}$, we have

$$F(e_i.(v \otimes m)) = \chi_i F(v \otimes e_i m) = \chi_i(\chi_1^{c_1} \cdots \chi_i^{c_i-1} \cdots \chi_n^{c_n}) e_i.m \otimes v = e_i.F(v \otimes m).$$

Similarly, F commutes with f_i. As the action by a_i, b_i preserves the weight spaces, F commutes with a_i, b_i $(1 \le i \le n)$ as well. Therefore F is an isomorphism of $\widetilde{U}$-modules. □

Theorem 4.2. *Let M and M' be finite-dimensional modules for $\widetilde{U}$ (resp., U) with $\widetilde{U}^0$ (resp., U^0) acting semisimply. Then*

$$M \otimes M' \cong M' \otimes M.$$

Proof. As the tensor product distributes over direct sums, we may assume that M and M' are indecomposable. Therefore the weights of M are all in $\chi \cdot \hat{\Lambda}$ for some algebra homomorphism $\chi : \widetilde{U}^0 \to \mathbb{K}$ with $\chi(\omega_i) = \chi(\omega_i')$. (See Remark 2.17.) By Lemma 2.18, $M \cong L_\chi \otimes N$ for some module N with weights in Λ. Similarly, $M' \cong L_{\chi'} \otimes N'$ for some χ'. By Lemma 4.1 and [BW2, Thm. 4.11],

$$\begin{aligned} M \otimes M' \cong (L_\chi \otimes N) \otimes (L_{\chi'} \otimes N') &\cong (L_\chi \otimes L_{\chi'}) \otimes (N \otimes N') \\ &\cong (L_{\chi'} \otimes L_\chi) \otimes (N' \otimes N) \\ &\cong (L_{\chi'} \otimes N') \otimes (L_\chi \otimes N) \cong M' \otimes M. \quad \square \end{aligned}$$

§5. Tensor powers of the natural module

In this section we consider tensor powers $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ (k factors) of the natural module V for $\widetilde{U}$ (defined in Section 1). Set $R = R_{V,V}$, and for $1 \leq i < k$, let R_i be the $\widetilde{U}$-module isomorphism on $V^{\otimes k}$ defined by

$$R_i(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = z_1 \otimes \cdots \otimes z_{i-1} \otimes R(z_i \otimes z_{i+1}) \otimes z_{i+2} \otimes \cdots \otimes z_k.$$

Then it is a consequence of the quantum Yang-Baxter equation that the braid relations hold:

$$\begin{aligned} R_i \circ R_{i+1} \circ R_i &= R_{i+1} \circ R_i \circ R_{i+1} && \text{for} \quad 1 \leq i < k \\ R_i \circ R_j &= R_j \circ R_i && \text{for} \quad |i-j| \geq 2. \end{aligned} \tag{5.1}$$

We would like to argue that

$$R_i^2 = (1 - rs^{-1})R_i + rs^{-1}\mathrm{Id} \tag{5.2}$$

for all $i = 1, \ldots, k-1$. For this it suffices to work with the 2-fold tensor product $V \otimes V$.

Proposition 5.3. *Whenever $s \neq -r$, the $\widetilde{U}$-module $V \otimes V$ decomposes into two simple $\widetilde{U}$-submodules, $S^2_{r,s}(V)$ (the (r,s)-symmetric tensors) and $\wedge^2_{r,s}(V)$ (the (r,s)-antisymmetric tensors). These modules are defined as follows:*

(i) *$S^2_{r,s}(V)$ is the span of $\{v_i \otimes v_i \mid 1 \leq i \leq n\} \cup \{v_i \otimes v_j + s v_j \otimes v_i \mid 1 \leq i < j \leq n\}$.*

(ii) *$\Lambda^2_{r,s}(V)$ is the span of $\{v_i \otimes v_j - r v_j \otimes v_i \mid 1 \leq i < j \leq n\}$.*

Proof. A computation shows that $S^2_{r,s}(V)$ and $\Lambda^2_{r,s}(V)$ are $\widetilde{U}$-submodules of $V \otimes V$. Note that each weight space of $S^2_{r,s}(V)$ is one-dimensional and is spanned by one of the weight vectors listed in (i). Therefore any submodule of $S^2_{r,s}(V)$ must contain one of these vectors. It may be checked that any of these vectors generates all of $S^2_{r,s}(V)$ in case $s \neq -r$. In particular, $v_1 \otimes v_1$ is a highest weight vector, and given any other vector in (i), there is an element of U taking it to $v_1 \otimes v_1$. Therefore $S^2_{r,s}(V)$ is simple. A similar argument proves that $\Lambda^2_{r,s}(V)$ is simple, with highest weight vector $v_1 \otimes v_2 - r v_2 \otimes v_1$. □

Remark 5.4. The $s = -r$ case is "nongeneric," and in this exceptional case, $V \otimes V$ need not be completely reducible. For example, when $n = 2$ what happens is that $v_1 \otimes v_2 - r v_2 \otimes v_1$ spans a one-dimensional module (as it does for $n = 2$ generic) that is not complemented in $V \otimes V$. Modulo that submodule, $v_1 \otimes v_1$ spans a one-dimensional module. Modulo the resulting two-dimensional module, $v_1 \otimes v_2 + r v_2 \otimes v_1$ and $v_2 \otimes v_2$ span a two-dimensional module.

Proposition 5.5. *The minimum polynomial of* $R = R_{V,V}$ *on* $V \otimes V$ *is* $(t-1)(t+rs^{-1})$ *if* $s \neq -r$.

Proof. It follows from the definition of R that $R(v_1 \otimes v_1) = v_1 \otimes v_1$ and $R(v_1 \otimes v_2 - rv_2 \otimes v_1) = -rs^{-1}(v_1 \otimes v_2 - rv_2 \otimes v_1)$. By Proposition 5.3, $S^2_{r,s}(V)$ and $\Lambda^2_{r,s}(V)$ are simple, and in fact, $v_1 \otimes v_1$ and $v_1 \otimes v_2 - rv_2 \otimes v_1$ are the highest weight vectors. In particular, each is a cyclic module generated by its highest weight vector. As $Ra(v_1 \otimes v_1) = aR(v_1 \otimes v_1) = a(v_1 \otimes v_1)$ for all $a \in \widetilde{U}$, this implies that $S^2_{r,s}(V)$ is in the eigenspace of R corresponding to eigenvalue 1. Analogously, $\wedge^2_{r,s}(V)$ corresponds to the eigenvalue $-rs^{-1}$, and since $V \otimes V$ is the direct sum of those submodules, we have the desired result. □

From Proposition 5.5 it follows that R acts as

$$(5.6)\quad r\sum_{i<j} E_{j,i}\otimes E_{i,j} + s^{-1}\sum_{i<j} E_{i,j}\otimes E_{j,i} + (1-rs^{-1})\sum_{i<j} E_{j,j}\otimes E_{i,i} + \sum_{i} E_{i,i}\otimes E_{i,i}$$

on $V \otimes V$. Indeed, (5.6) is a linear operator that acts on $S^2_{r,s}(V)$ as multiplication by 1, and on $\Lambda^2_{r,s}(V)$ as multiplication by $-rs^{-1}$. By Proposition 5.5, R has the same properties, and so R is equal to this sum on $V \otimes V$.

§6. Quantum Schur-Weyl duality

Assume $r, s \in \mathbb{K}$. Let $H_k(r,s)$ be the unital associative algebra over $\mathbb{K}$ with generators T_i, $1 \leq i < k$, subject to the relations:

(H1) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad 1 \leq i < k$
(H2) $T_iT_j = T_jT_i, \qquad |i-j| \geq 2$
(H3) $T_i^2 = (s-r)T_i + rs1.$

When $r \neq 0$, the elements $t_i = r^{-1}T_i$ satisfy the braid relations (H1), (H2), along with the relation

$$\text{(H3')} \qquad t_i^2 = (q-1)t_i + q1,$$

where $q = r^{-1}s$. The *Hecke algebra* $H_k(q)$ (of type A_{k-1}) is generated by elements t_i, $1 \leq i < k$, which satisfy (H1), (H2), (H3'). It has dimension $k!$ and is semisimple whenever q is not a root of unity. At $q = 1$, the Hecke algebra $H_k(q)$ is isomorphic to $\mathbb{K}S_k$, the group algebra of the symmetric group S_k, where we may identify t_i with the transposition $(i\ i+1)$.

The two-parameter Hecke algebra $H_k(r,s)$ defined above is isomorphic to $H_k(r^{-1}s)$ whenever $r \neq 0$. Thus, it is semisimple whenever $r^{-1}s$ is not a root of unity. For any $\sigma \in S_k$, we may define $T_\sigma = T_{i_1} \cdots T_{i_\ell}$ where $\sigma = t_{i_1} \ldots t_{i_\ell}$ is a reduced expression for σ as a product of transpositions. It follows from (H1) and (H2) that T_σ is independent of the reduced expression and these elements give a basis.

The results of Section 5 show that the $\widetilde{U}$-module $V^{\otimes k}$ affords a representation of the Hecke algebra $H_k(r,s)$:

$$\begin{aligned} H_k(r,s) &\to \mathrm{End}_{\widetilde{U}}(V^{\otimes k}) \\ T_i &\mapsto sR_i \qquad (1 \le i < k). \end{aligned} \tag{6.1}$$

When $k = 2$ and $s \neq -r$, $V^{\otimes 2} = S^2_{r,s}(V) \oplus \wedge^2_{r,s}(V)$ is a decomposition of $V^{\otimes 2}$ into simple $\widetilde{U}$-modules by Proposition 5.3. The maps $p_1 = (sR_1 + r)/(s + r)$ and $p_2 = (s - sR_1)/(s + r)$, $(R_1 = R_{V,V})$, are the corresponding projections onto the simple summands. Thus, the map in (6.1) is an isomorphism for $k = 2$. More generally, we will show next that it is surjective whenever rs^{-1} is not a root of unity, and it is an isomorphism when $n \ge k$. This is the two-parameter version of the well-known result of Jimbo [Ji] that $H_k(q) \cong \mathrm{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes k})$ and is the analogue of classical Schur-Weyl duality, $\mathbb{K}S_k \cong \mathrm{End}_{\mathfrak{gl}_n}(V^{\otimes k})$ for $n \ge k$. We will apply the following lemma. The case $n < k$ is dealt with separately, and it uses the isomorphism $H_k(r,s) \cong \mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ in the $n = k$ case.

Lemma 6.2. *If $n \ge k$ and V is the natural representation of $\widetilde{U}$, then $V^{\otimes k}$ is a cyclic $\widetilde{U}$-module generated by $v_1 \otimes \cdots \otimes v_k$.*

Proof. Let $\underline{v} = v_1 \otimes \cdots \otimes v_k$. We begin by showing that if $\sigma \in S_k$, then $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \widetilde{U}.\underline{v}$.

Suppose we have an arbitrary permutation $x_1 \otimes \cdots \otimes x_k$ ($x_i \in \{v_1, \ldots, v_k\}$ for all i) of the factors of $\underline{v}$. For some $\ell < m$, assume that $x_\ell = v_j$ and $x_m = v_{j+1}$. Then because of the formulas

$$\begin{aligned} \Delta^{k-1}(e_j) &= \sum_{i=1}^{k} \underbrace{\omega_j \otimes \cdots \otimes \omega_j}_{i-1} \otimes e_j \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-i} \\ \Delta^{k-1}(f_j) &= \sum_{i=1}^{k} \underbrace{1 \otimes \cdots \otimes 1}_{k-i} \otimes f_j \otimes \underbrace{\omega'_j \otimes \cdots \otimes \omega'_j}_{i-1}, \end{aligned} \tag{6.3}$$

there are nonzero scalars c and c' such that

$$(ce_jf_j + c').(x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k.$$

Similarly, there are nonzero scalars d and d' such that

$$(de_jf_j + d').(x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_k.$$

As the transpositions $(j\ j+1)$ generate S_k, $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \widetilde{U}.\underline{v}$ for all $\sigma \in S_k$.

Next we will use induction on k to establish the following. For any k elements $i_1, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 \le i_2 \le \cdots \le i_k$, there is a $u \in \widetilde{U}$ such that $u.\underline{v} = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and u does not contain any terms with factors of $e_m, e_{m+1}, \ldots, e_{n-1}, f_{m+1}, f_{m+2}, \ldots, f_{n-2}$, or f_{n-1} where $m = \max\{i_k, k\}$. If $k = 1$, we may apply $f_{m-1} \cdots f_1$ to $\underline{v} = v_1$ to obtain v_m for any $m \in \{1, \ldots, n\}$. If

$k > 1$, let ℓ be such that $i_\ell < i_k$, $i_{\ell+1} = i_{\ell+2} = \cdots = i_k$. (If no such ℓ exists, that is if $i_1 = \cdots = i_k$, then set $\ell = 0$ and apply u' from (6.5) below to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$.) By induction, there is an element $u \in \widetilde{U}$ such that

$$u.(v_1 \otimes \cdots \otimes v_\ell) = v_{i_1} \otimes \cdots \otimes v_{i_\ell}, \tag{6.4}$$

where u has no terms with factors of $e_{m'}, e_{m'+1}, \ldots, e_{n-1}, f_{m'+1}, \ldots, f_{n-1}$ ($m' = \max\{i_\ell, \ell\}$).

Suppose initially that $i_\ell \leq \ell$. Then $m' = \ell$, and so $u.(v_1 \otimes \cdots \otimes v_k)$ is a nonzero scalar multiple of $(v_{i_1} \otimes \cdots \otimes v_{i_\ell}) \otimes (v_{\ell+1} \otimes \cdots \otimes v_k)$. Now apply

(6.5)
$$u' = \begin{cases} (f_{i_k-1} f_{i_k-2} \cdots f_{\ell+1}) \cdots (f_{i_k-1} f_{i_k-2})(f_{i_k-1})(e_{i_k} e_{i_k+1} \cdots e_{k-1}) \cdots (e_{i_k} e_{i_k+1})(e_{i_k}) & \text{if } i_k < k \\ (f_{i_k-1} f_{i_k-2} \cdots f_{\ell+1}) \cdots (f_{i_k-1} f_{i_k-2} \cdots f_{k-1})(f_{i_k-1} f_{i_k-2} \cdots f_k) & \text{if } i_k \geq k \end{cases}$$

to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired. (Note that we did not use any factors of $e_m, e_{m+1}, \ldots, e_{n-1}, f_{m+1}, \ldots, f_{n-1}$ for $m = \max\{i_k, k\}$.)

If on the other hand, $i_\ell > \ell$ (so that $m' = i_\ell$ and $i_k > \ell + 1$), first apply u' from (6.5) to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of

$$(v_1 \otimes \cdots \otimes v_\ell) \otimes (v_{i_k} \otimes \cdots \otimes v_{i_k}),$$

and then apply u from (6.4) to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired.

Finally, if $i_1, \ldots, i_k \in \{1, \ldots, n\}$ are *any* k elements (not necessarily in nondecreasing numerical order), let $\sigma \in S_k$ be a permutation such that

$$i_{\sigma(1)} \leq i_{\sigma(2)} \leq \cdots \leq i_{\sigma(k)}.$$

By the first paragraph of the proof, there is an element of $\widetilde{U}$ taking $\underline{v}$ to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$. Now we may apply u from (6.4) and u' from (6.5) in the appropriate order (as above) to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$. □

This leads to the two-parameter analogue of Schur-Weyl duality.

Theorem 6.6. *Assume rs^{-1} is not a root of unity. Then:*
- (i) *$H_k(r,s)$ maps surjectively onto* $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$.
- (ii) *If $n \geq k$, then $H_k(r,s)$ is isomorphic to* $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$.

Proof. We establish part (ii) first. Assume $F \in \mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ and $\underline{v} = v_1 \otimes \cdots \otimes v_k$. As F commutes with the action of $\widetilde{U}$, $F(\underline{v})$ must have the same weight as $\underline{v}$, that is,

$\epsilon_1 + \cdots + \epsilon_k$. The only vectors of $V^{\otimes k}$ with this weight are the linear combinations of the permutations of $v_1 \otimes \cdots \otimes v_k$, so that

$$F(\underline{v}) = \sum_{\sigma \in S_k} c_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \tag{6.7}$$

for some scalars $c_\sigma \in \mathbb{K}$. We will show that there is an element R^σ in the image of $H_k(r,s)$ in $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ such that $R^\sigma(\underline{v}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. (Previously we constructed an element $u \in \widetilde{U}$ with this property.)

We begin with the transposition $\tau = t_j = (j\ j+1)$. For any tensor product $v_{i_1} \otimes \cdots \otimes v_{i_k}$ of distinct basis vectors, we have by (5.6) that

$$v_{i_{\tau(1)}} \otimes \cdots \otimes v_{i_{\tau(k)}} = \begin{cases} r^{-1} R_j (v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if} \quad i_j < i_{j+1} \\ (sR_j + (r-s)\mathrm{Id})(v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if} \quad i_j > i_{j+1}. \end{cases}$$

Therefore, if $\sigma = t_{j_1} \cdots t_{j_m}$, a product of such transpositions, we can set $R^{t_{j_\ell}} := r^{-1} R_{j_\ell}$ or $R^{t_{j_\ell}} := sR_{j_\ell} + (r-s)\mathrm{Id}$, depending on the numerical order of the appropriate indices i_{j_ℓ} and $i_{j_\ell+1}$ in $R^{t_{j_{\ell-1}}} \circ \cdots \circ R^{t_{j_1}} \underline{v}$. Then defining $R^\sigma = R^{t_{j_m}} \circ \cdots \circ R^{t_{j_1}} \in \mathrm{End}_{\widetilde{U}}(V^{\otimes k})$, we have the desired map such that $R^\sigma(\underline{v}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Now let $F_0 = F - \sum_{\sigma \in S_k} c_\sigma R^\sigma \in \mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ (with the c_σ coming from (6.7)). By Lemma 6.2, $F_0(V^{\otimes k}) = F_0(\widetilde{U}.\underline{v}) = \widetilde{U}.F_0(\underline{v}) = 0$. Therefore $F = \sum_{\sigma \in S_k} c_\sigma R^\sigma$ is in the image of $H_k(r,s)$. Consequently, the map $H_k(r,s) \to \mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ in (6.1) is a surjection, and $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ is the $\mathbb{K}$-linear span of $\{R^\sigma \mid \sigma \in S_k\}$. Now suppose that $\sum_{\sigma \in S_k} c_\sigma R^\sigma = 0$ for some scalars $c_\sigma \in \mathbb{K}$. Then in particular, $0 = \sum_{\sigma \in S_k} c_\sigma R^\sigma(\underline{v}) = \sum_{\sigma \in S_k} c_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$, which implies $c_\sigma = 0$ for all $\sigma \in S_k$. Therefore $\{R^\sigma \mid \sigma \in S_k\}$ is a basis for the vector space $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ and $\dim_{\mathbb{K}} \mathrm{End}_{\widetilde{U}}(V^{\otimes k}) = k! = \dim_{\mathbb{K}} H_k(r,s)$. It follows that $H_k(r,s)$ is isomorphic to $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ for $n \geq k$, as asserted.

Next we turn to the proof of (i) and assume here that $n < k$. For $i = n, k$, let $\widetilde{U}_i = U_{r,s}(\mathfrak{gl}_i)$, let Λ_i be the weight lattice of $\mathfrak{gl}_i$, and let V_i be the natural $\widetilde{U}_i$-module. By (ii), we may identify $H_k(r,s)$ with $\mathrm{End}_{\widetilde{U}_k}(V_k^{\otimes k})$. We will show that $H_k(r,s)$ maps surjectively onto $\mathrm{End}_{\widetilde{U}_n}(V_n^{\otimes k})$.

Consider $V_k^{\otimes k}$ as a $\widetilde{U}_n$-module via the inclusion of $\widetilde{U}_n$ into $\widetilde{U}_k$, and regard $V_n^{\otimes k}$ as a $\widetilde{U}_n$-submodule of $V_k^{\otimes k}$ in the obvious way. Now $V_n^{\otimes k}$ is a finite-dimensional $\widetilde{U}_n$-module on which $\widetilde{U}_n^0$ acts semisimply, so by Theorem 3.8, it is completely reducible. Therefore,

$$V_n^{\otimes k} = L_1 \oplus \cdots \oplus L_t \tag{6.8}$$

for simple $\widetilde{U}_n$-modules L_i. It suffices to show that the projections onto the simple summands L_i can be obtained from $H_k(r,s)$.

Consider

$$\widetilde{U}_k.V_n^{\otimes k} = \widetilde{U}_k.L_1 + \cdots + \widetilde{U}_k.L_t, \tag{6.9}$$

the $\widetilde{U}_k$-submodule of $V_k^{\otimes k}$ generated by $V_n^{\otimes k}$. By Corollary 2.5, each L_i is isomorphic to some $L(\lambda_i)$, $\lambda_i \in \Lambda_n^+$, and in particular is generated by a highest weight vector m_i with $e_j.m_i = 0$ for all j $(1 \le j < n)$. We claim that $e_j.m_i = 0$ as well when $n \le j < k$. This follows from the expression for $\Delta^{k-1}(e_j)$ in (6.3) and the action of e_j on the natural module V_k for $\widetilde{U}_k$ given by $e_j.v_i = \delta_{i,j+1}v_j$, because m_i must be some linear combination of vectors $v_{i_1} \otimes \cdots \otimes v_{i_k}$ with $i_1, \dots, i_k \in \{1, \dots, n\}$. Therefore m_i is also a highest weight vector for the finite-dimensional $\widetilde{U}_k$-module $\widetilde{U}_k.L_i$. By Theorem 2.1 and Lemma 3.7, $\widetilde{U}_k.L_i = \widetilde{U}_k.m_i$ is a simple $\widetilde{U}_k$-module. Therefore (6.9) must be a direct sum:

$$\widetilde{U}_k.V_n^{\otimes k} = \widetilde{U}_k.L_1 \oplus \cdots \oplus \widetilde{U}_k.L_t.$$

Because $V_k^{\otimes k}$ is a completely reducible $\widetilde{U}_k$-module, there is some complementary $\widetilde{U}_k$-submodule W such that

$$V_k^{\otimes k} = \widetilde{U}_k.L_1 \oplus \cdots \oplus \widetilde{U}_k.L_t \oplus W. \tag{6.10}$$

Let $\pi_i \in H_k(r,s)$ be the projection of $V_k^{\otimes k}$ onto $\widetilde{U}_k.L_i$. Then, π_i commutes with the $\widetilde{U}_k$-action, and acts as the identity map on $\widetilde{U}_k.L_i$ and as 0 on the other summands in (6.10). Since $L_j \subseteq \widetilde{U}_k.L_j$ for all j, the map π_i restricted to $V_n^{\otimes k}$ commutes with the $\widetilde{U}_n$-action and is the projection onto L_i. Thus, $H_k(r,s) \to \mathrm{End}_{\widetilde{U}_n}(V_n^{\otimes k})$ is onto. □

Finally, we note that because $V^{\otimes k}$ is a semisimple $\widetilde{U}$-module, it follows from the double commutant theorem (see for example, [GW, Thm. 3.3.7]) that $\mathrm{End}_{H_k(r,s)}(V^{\otimes k})$ is isomorphic to the image of $\widetilde{U}$ in $\mathrm{End}_{\mathbb{K}}(V^{\otimes k})$.

References

[BW1] G. Benkart and S. Witherspoon, *A Hopf structure for down-up algebras*, Math. Z. **238** (2001), 523-553.

[BW2] G. Benkart and S. Witherspoon, *Two-parameter quantum groups and Drinfel'd doubles*, Algebr. Represent. Theory (to appear).

[D] V.G. Drinfel'd, *Quantum groups*, in Proceedings of the International Congress of Mathematicians, Berkeley, 1986, A.M. Gleason (ed.), pp. 798-820, Amer. Math. Soc. 1987.

[DD] R. Dipper and S. Donkin, *Quantum GL_n*, Proc. London Math. Soc. **63** (1991), 165-211.

[DP] C. De Concini and C. Procesi, *A characteristic free approach to invariant theory*, Adv. Math. **21** (1976), 330-354.

[DPS] J. Du, B. Parshall, and L. Scott, *Quantum Weyl reciprocity and tilting modules*, Comm. Math. Phys. **195** (1998), 321-352.

[Du] J. Du, *A note on quantized Weyl reciprocity at roots of unity*, Algebra Colloq. **2:4** (1995), 363-372.

[GW] R. Goodman and N.R. Wallach, *Representations and Invariants of the Classical Groups, Encyclopedia of Mathematics and Its Applications*, vol. 68, Cambridge University Press, 1998.

[Ja] J. C. Jantzen, *Lectures on Quantum Groups*, vol. 6, Graduate Studies in Math., Amer. Math. Soc., 1993.

[Ji1] M. Jimbo, *A q-difference analog of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63-69.

[Ji2] M. Jimbo, *A q-analogue of $U(\mathfrak{gl}_n(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247-252.

[Jo] A. Joseph, *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1995.

[KT] D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions V: A degenerate version of $U_q(\mathfrak{gl}_n)$*, Internat. J. Algebra Comput. **9** (1999), 405-430.

[LR] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras*, Adv. Math. **125** (1997), 1-94.

[L] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, 1993.

[T] M. Takeuchi, *A two-parameter quantization of $GL(n)$*, Proc. Japan Acad. Ser. A Math. Sci. **66** (1990), 112-114.

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