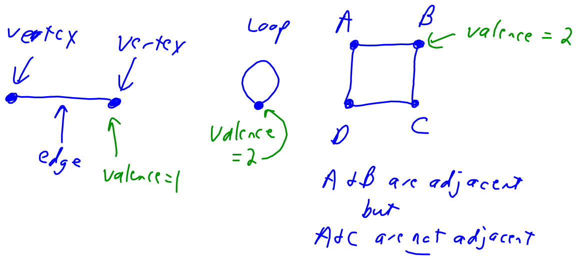
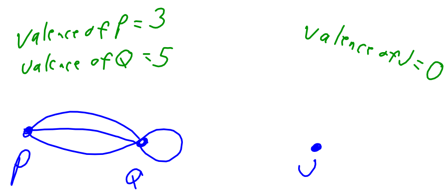
CHAPTER 1 – URBAN SERVICES

A ***graph*** is a collection of one or more points (***vertices***). ***Edges*** connect vertices. A ***loop*** is an edge that connects a vertex to itself. A ***simple graph*** contains no loops. Two different vertices are ***adjacent*** if they are connected by an edge. The ***valence*** or ***degree*** of a vertex is the number of ends of edges at the vertex.

Examples of graphs:



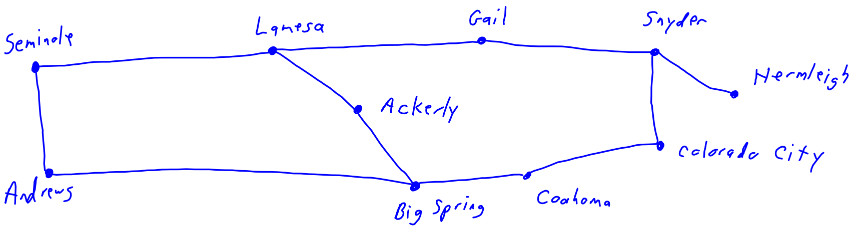


Use the map below to represent the cities you see and the main roads that connect them as a simple graph. What cities are adjacent to Lamesa?

Seminole
Gail
Ackerly

(Map from Google Maps https://www.google.com/maps/@31.9565757,-101.2671841,8.25z)

The edge between Lamesa and Gail is a judgement call of whether or not the road directly connects to Lamesa

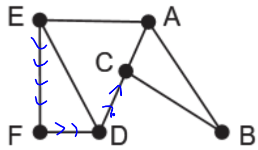


pronounced "oiler"

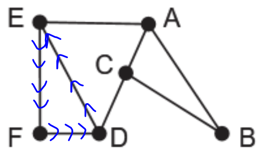
A ***path*** is a connected sequence of edges showing a route on a graph. It is named using the list of adjacent vertices that create the edges.
A path that uses every edge exactly once is an ***Euler path***.
If the path ends at the same vertex it started at, it is a ***circuit***.
A circuit that uses every edge exactly once is an ***Euler circuit***.

Classify the list of vertices for the graphs below with the most precise term that is appropriate.

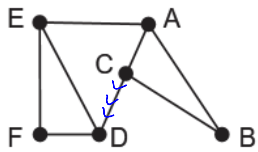
Decision tree
path? → No → Not a path
↓ Yes
Circuit?
Yes ↙ ↘ No
Special Circuit? Special path?



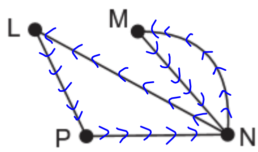
EFDC
Path



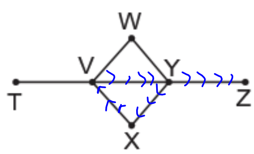
FDEF
Circuit



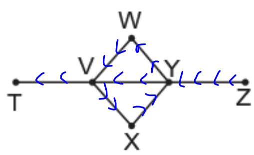
CDBC
Not adjacent
Not a path



NLPNMN
Euler circuit



YXVYZ
Path

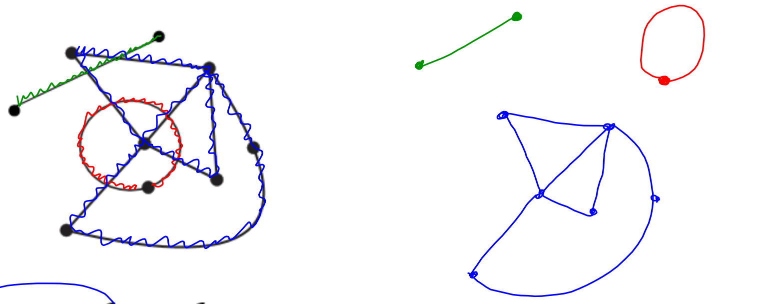


ZYWVXYVT
Euler path

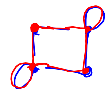
A graph is ***connected*** if for every pair of vertices there is a path that connects them.

Is the graph below connected? If not, how many components (sub-graphs) are there? Note that a component could consist of a vertex or vertices connected by edges

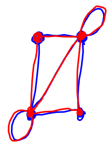
Graph is NOT Connected.
It has 3 Components



Euler's Theorem for a connected graph



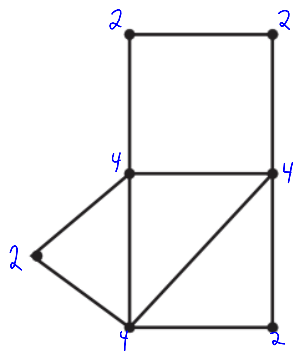
1. If the graph has no vertices of odd degree, then it has at least one Euler circuit and if a graph has an Euler circuit, then it has no vertices of odd degree. Even degree because every time you enter a vertex you also leave the vertex.



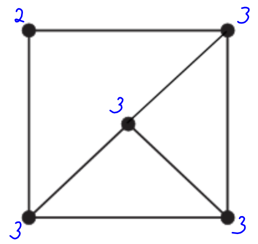
2. If a graph has 2 vertices of odd degree, then there is at least one Euler path, but no Euler circuit. Any Euler path must start at a vertex with an odd degree and end at the other vertex of odd degree. Except for where you start and finish every time you enter a vertex, you also leave the vertex
3. If the graph has more than two vertices of odd degree, then it does not have an Euler path.

Determine whether the following graphs contain an Euler path, or Euler circuit, or neither.

Count valences

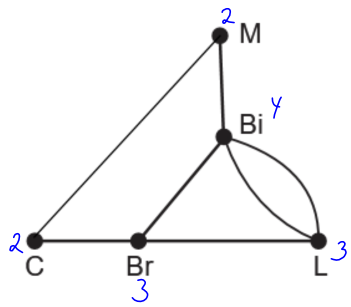


Connect with all vertices of even degree, so Euler circuit



Connected with more than two vertices of odd degree so Neither Euler circuit nor Euler path

Note: The same would be true if there was exactly 1 vertex of odd degree



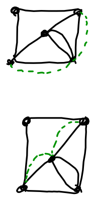
Connected with exactly two vertices of odd degree, so Euler path

Chinese postman problem: Cover all the edges at least once with the minimum cost.

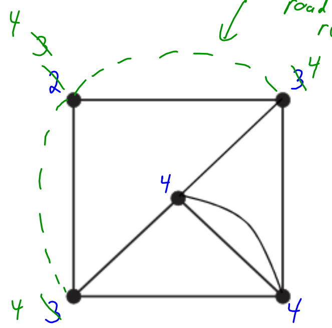
When you ***Eulerize*** a graph, you reuse edges as necessary to form an Euler circuit.

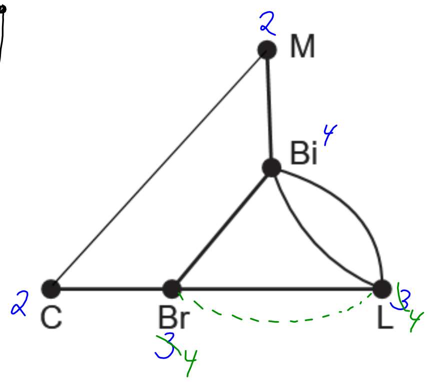
Other Eulerizations

Count Valence (degree)



indicates a road that is reused





Originally, have Euler Path, but Not an Euler circuit. Want Euler circuit, so reuse edges as necessary to create a graph with an Euler circuit.

And other Eulerizations are also possible

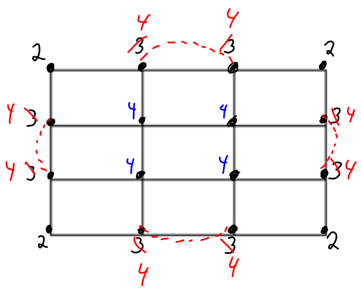
If a graph is ***rectangular***, it is a group of rectangular blocks that form a larger rectangle.

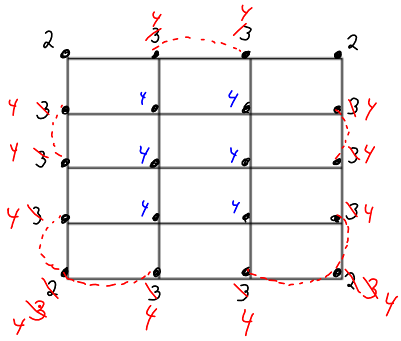
only work with edges

One way to Eulerize a rectangular graph is to use an ***edge-walker algorithm***. Starting at one vertex of the outer rectangle, reuse edges to each odd vertex that connects to the next vertex.

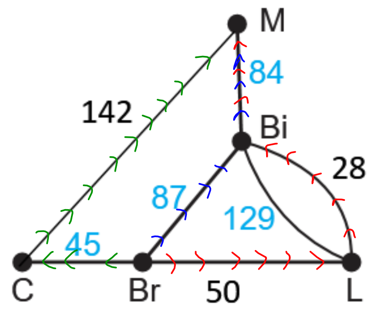
Could draw in the vertices at every intersection because we told you this is a rectangular graph.

Notice that all interior vertices are even degree, so we do not bother the interior.





Edges of a graph may have an associated cost for traversing the edge. The graph has the distance between the cities in miles. What is the least cost to go from City Br to City M?



$Br \xrightarrow{87} Bi \xrightarrow{84} M$ 171 mi

$Br \xrightarrow{45} C \xrightarrow{142} M$ 187 mi

$\boxed{Br \xrightarrow{50} L \xrightarrow{28} Bi \xrightarrow{84} M \quad 162 \text{ mi}}$

Choose cheaper one

Shortest in terms of miles

The cost of an edge may have any value – just watch the units given.

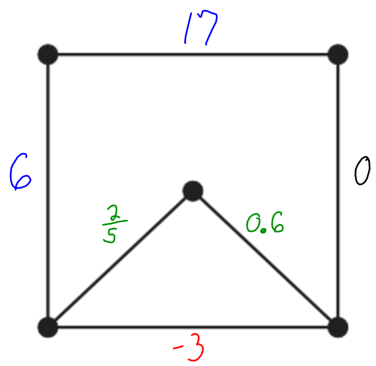
must be consistent

Counting Number

Zero

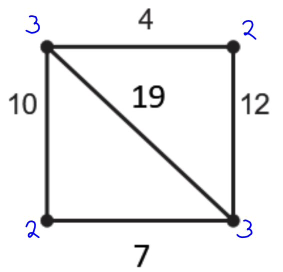
Fractions, Decimals

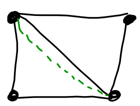
Negatives (rare in real life)



Eulerize the graph below at the lowest cost. The cost of an edge is the time to travel between the vertices in minutes.

Count valence





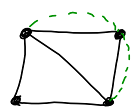
Reuse 19

so

Cost = Fixed cost + 19

= 71 min

Minimum cost Eulerization

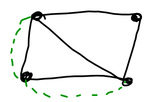


Reuse 4+12

so

Cost = Fixed cost + 16

= 68 min



Reuse 10+7

so

Cost = Fixed cost + 17

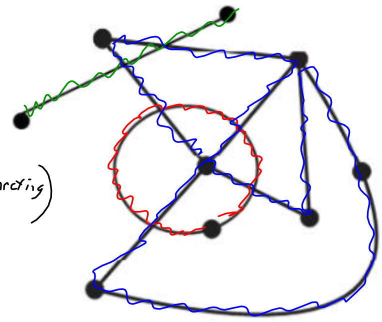
= 69 min

Fixed cost = cost of using every edge exactly once = 10 + 4 + 12 + 7 + 19 = 52 min

but graph is not yet an Euler circuit, so need to repeat one or more edges

SAMPLE EXAM QUESTIONS FROM CHAPTER 1

1. Mark all true statements about the graph on the right:



(A) The graph is connected. NO. It has 3 components

(B) The graph has 7 vertices. F

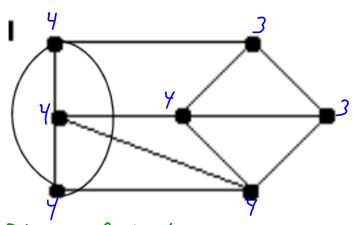
(C) The graph has 8 edges! F 10 edges (Count the connecting lines)

(D) The graph has 9 vertices T count the points

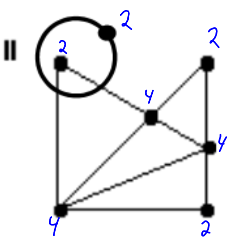
(E) None of these statements are true.

2. Which of the graphs below have Euler circuits?

Need: Connected graph with all vertices of even degree



Euler Path but Not an Euler Circuit



Not Connected, so No Euler circuit

(A) Only graph I

(B) Only graph II

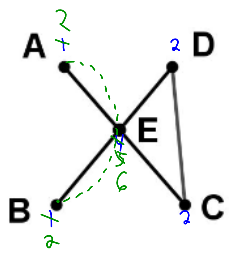
(C) Both graph I and graph II

(D) Neither graph have an Euler circuit

(E) Need more information to determine the answer

3. In order to Eulerize the graph below, give the fewest number of edges that need to be duplicated.

Can't connect A & B because we cannot "created a new road"



(A) More than 4

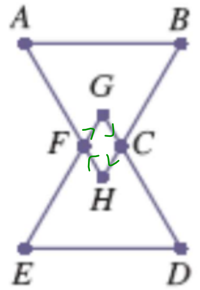
(B) 4

(C) 3

(D) 2

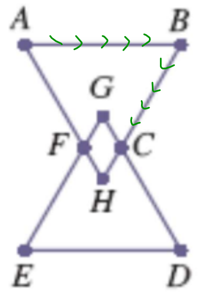
(E) 1

4. Classify the following: FGCHF.
   (A) Not a path
   (B) A path
   (C) A circuit
   (D) An Euler path
   (E) An Euler circuit

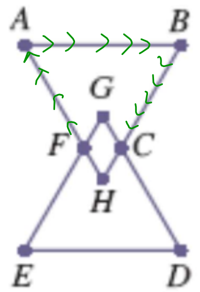


5. Classify the following: ABCFEDCBA.
   (A) Not a path
   (B) A path
   (C) A circuit
   (D) An Euler path
   (E) An Euler circuit

Not adjacent

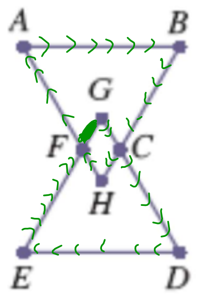


6. Classify the following: FABC.
   (A) Not a path
   (B) A path
   (C) A circuit
   (D) An Euler path
   (E) An Euler circuit



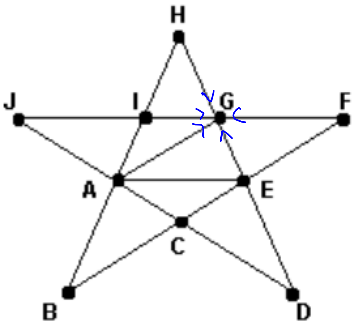
7. Classify the following: FABCDEFGCHFGF.
   (A) Not a path
   (B) A path
   (C) A circuit
   (D) An Euler path
   (E) An Euler circuit

Not an Euler circuit because we did not use each edge EXACTLY once



8. What is the valence of vertex G in the graph below? 5

degree



9. After a major natural disaster, such as a flood, hurricane, or tornado, many tasks need to be completed as efficiently as possible. For which situation below would finding an Euler circuit or an efficient Eulerization of a graph be the appropriate mathematical technique to apply?

vertices

(A) Rescuers visit every home to make sure no one is trapped.

vertices

(B) Road crews check all the bridges to make sure they are structurally sound.

edges

(C) Oil company crews travel every pipeline checking for leaks.

vertices

(D) The internet provider responds to reports of outages.