CHAPTER 14: Apportionment
14.1 The Apportionment Problem

An ***apportionment problem*** is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an ***apportionment method***.

The total population, p, divided by the house size, h, is called the ***standard divisor, s***.

$$s = \frac{p}{h}$$

A group's ***quota*** q_i is the group's population, p_i, divided by the standard divisor, s.

$$q_i = \frac{p_i}{s}$$

Different apportionment methods will use different rounding rules.

When q is not already an integer, there are multiple ways to round.
- Round q up to the next integer, $\lceil q \rceil$.
- Round q down to the previous integer, $\lfloor q \rfloor$.
- Round to the nearest integer, $[q]$. If q is halfway to the next integer or larger, round up to the next integer. Otherwise, round down to the previous integer.
- Round according to the geometric mean. The geometric mean of $\lfloor q \rfloor$ and $\lceil q \rceil$ is $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$. If q is equal to or larger than q^*, round up to the next integer. Otherwise, round down to the previous integer.

Example
Complete the following chart.

q	$\lceil q \rceil$	$\lfloor q \rfloor$	$[q]$	q^*	Round according to q^*
6					
4.6					
4.5					
4.48					
4.47					
0.2					

14.2 Hamilton Method
Step 1 Compute the standard divisor.
Step 2 Compute the quota for each "state" (group).
Step 3 Round each quota *down*.
Step 4 Calculate the number of seats left to be assigned.
Step 5 Assign the remaining seats to the states with the *largest fractional part* of *q*.

Example

Three friends, Amy, Ben, and Cathy own businesses and decided to pool their resources to buy a "box" with 38 seats at a local sporting event. Use the Hamilton method to apportion the 38 seats if Amy pays $6200, Ben pays $1200, and Cathy pays $10,300.

$s =$

Person	Contribution	q	Rounded quota	Hamilton Apportionment
Amy	$6200			
Ben	$1200			
Cathy	$10,300			
TOTAL				

Example

After Amy, Ben, and Cathy apportioned the tickets, they found out that there are actually 39 seats in the box. Reapportion the 39 tickets using the Hamilton method.

$s =$

Person	Contribution	q	Rounded quota	Hamilton Apportionment
Amy	$6200			
Ben	$1200			
Cathy	$10,300			
TOTAL				

Example

A committee was forming to represent all four towns in the county. The population of each town is given below. If there are 79 representatives, how many representatives does each town receive?

$s =$

Town	Population	q	Rounded quota	Hamilton Apportionment
Town A	32,300			
Town B	18,640			
Town C	14,300			
Town D	200			
TOTAL				

14.3 and 14.4 Divisor Methods and Which Method is Best

We have used the standard divisor, *s*, to represent the average district population. We will use *s* for all apportionment methods to calculate the quota.

The divisor methods will also use an adjusted divisor, *d*, to calculate an adjusted quota. The adjusted quota combined with the appropriate rounding rules for each method will give the final apportionment for divisor methods.

Jefferson Method
Step 1   Compute the standard divisor.
Step 2   Compute the quota for each "state" (group).
Step 3   Round each quota *down*.
Step 4   If the total number of seats is not correct, call the current apportionment N, and find new divisors, $d_i = \frac{p_i}{N_i+1}$, that correspond to giving each state one more seat.
Step 5   Assign a seat to the state with the *largest d*. (Notice that divisor methods look at the entire number of *d* rather than the fractional part of the number.)
Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, *d*.

Example

Let's use a different apportionment method to split the original 38 seats in the box. Use the Jefferson method to distribute the seats.

$$s = \frac{17{,}700}{38} \approx 465.789474$$

Person	Cont.	q	Rounded quota	d_i	Jefferson App.
Amy	$6200	13.3107			
Ben	$1200	2.5763			
Cathy	$10,300	22.1130			
TOTAL	**$17,700**				

$d =$

Example

Also, use the Jefferson method to apportion the 79 representatives to the towns.

$s =$

Town	Pop.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
A	32,300						
B	18,640						
C	14,300						
D	200						
TOTAL							

$d =$

Webster Method

Step 1   Compute the standard divisor.

Step 2   Compute the quota for each "state" (group).

Step 3   Round each quota *to the nearest integer.*

Step 4   If the total number of seats is not correct, call the current apportionment N, and find new divisors.

If the number of seats needs to increase, use $d_i^+ = \dfrac{p_i}{N_i + 0.5}$.

If the number of seats needs to decrease, use $d_i^- = \dfrac{p_i}{N_i - 0.5}$.

Step 5   Adjust the seats according to d.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, d.

Example

Use the Webster method to distribute the 39 box seats.

$$s = \frac{17{,}700}{39} \approx 453.846154$$

Person	Cont.	q	Rounded quota	d_i	Webster App.
Amy	$6200	13.6610			
Ben	$1200	2.6441			
Cathy	$10,300	22.6949			
TOTAL	**$17,700**				

$$d =$$

Example
Use the Webster method to apportion the 79 representatives.

$s =$

Region	Pop.	q	Rounded quota	d_i	Webster App.
Town A	32,300				
Town B	18,640				
Town C	14,300				
Town D	200				
TOTAL					

$d =$

Example

Use the Webster method to apportion the representatives if they decided to only have 78 representatives.

$$s = \frac{65,440}{78} \approx 838.974359$$

Region	Pop.	q	Rounded quota	d_i	Webster App.
Town A	32,300	38.4994			
Town B	18,640	22.2176			
Town C	14,300	17.0446			
Town D	200	0.2384			
TOTAL	**65,440**				

$$d =$$

Hill-Huntington Method

The Hill-Huntington method does a great job of keeping the relative differences of representative share $\left(\text{i.e., } \frac{\text{apportionment}}{\text{population}}\right)$ and district population $\left(\text{i.e., } \frac{\text{population}}{\text{apportionment}}\right)$ stable between states. It also ensures that every group gets at least one representative, so it favors small states. Since 1941, the Hill-Huntington method with a house size of 435 has been used to apportion the House of Representatives.

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *according to the geometric mean* of $\lfloor q \rfloor$ and $\lceil q \rceil$, $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$.

Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.

If the number of seats needs to increase, use $d_i^+ = \dfrac{p_i}{\sqrt{N_i(N_i+1)}}$.

If the number of seats needs to decrease, use $d_i^- = \dfrac{p_i}{\sqrt{N_i(N_i-1)}}$.

Step 5 Adjust the seats according to d.

If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.

If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.

Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, d.

Example

Use the Hill-Huntington method to distribute the 39 box seats.

$$s = \frac{17,700}{39} \approx 453.846154$$

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Amy	$6200	13.6610				
Ben	$1200	2.6441				
Cathy	$10,300	22.6949				
TOTAL	**$17,700**					

$$d =$$

Example

The friends decided to give 4 tickets to mutual friends. Use the Hill-Huntington method to distribute remaining 35 box seats.

$$s = \frac{17{,}700}{35} \approx 505.714$$

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Amy	$6200	12.2599				
Ben	$1200	2.3729				
Cathy	$10,300	20.3672				
TOTAL	**$17,700**					

$$d =$$

Example
Use Hill-Huntington to distribute the 78 county representatives.

$$s = \frac{65{,}440}{78} \approx 838.974359$$

Region	Pop.	q	q^*	Rounded quota	d_i	HH App.
Town A	32,300	38.4994				
Town B	18,640	22.2176				
Town C	14,300	17.0446				
Town D	200	0.2384				
TOTAL	**65,140**					

$$d =$$

A *paradox* is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

Possible Issues - Alabama Paradox (Section 14.2)

The *Alabama paradox* occurs when a state loses a seat as the result of an increase in the house size.

Example

Use the information from pages 3 and 4 to see how many seats Amy, Ben, and Cathy received when they thought there were 38 tickets and when they thought there were 39 tickets in the box using the Hamilton method.

Person	Contribution	38 tickets Hamilton Apportionment	39 ticket Hamilton Apportionment
Amy	$6200		
Ben	$1200		
Cathy	$10,300		
TOTAL			

What information tells you that the Alabama paradox occurred in this example?

<u>Possible Issues - Population Paradox (Section 14.2)</u>

Consider two numbers, A and B, where $A > B$.

The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\dfrac{A - B}{B} \times 100\%$

The ***population paradox*** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Example

We have 100 council members to apportion to four districts. The population and the Hamilton Apportionment are given for the previous census and for the latest census.

District	Prev. Pop.	Latest Pop.	Prev. Ham. App.	Latest Ham App.	Absolute Difference	Relative Difference
North	27,460	28,140	42	43		
South	17,250	17,450	27	26		
East	19,210	19,330	30	29		
West	1000	990	1	2		

Did the population paradox occur?

Explain what information helped you determine whether or not the population paradox occurred.

Possible Issues – New States Paradox (Section 14.2)

The **new states paradox** occurs in a reapportionment in which an increase in the total number of states (with a proportionate increase in representatives) causes a shift in the apportionment of existing states.

Example
A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress

$s =$

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899			
Liquid	59,096			
TOTAL				

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

Use Hamilton's method to apportion the seats for their congress (the 12 original seats plus the additional seats that were added when Plasma joined).

$s =$

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899			
Liquid	59,096			
Plasma	38,240			
TOTAL				

What information tells you that the new states paradox occurred in this example?

Possible Issues – Quota Condition (Section 14.3)

Example

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired according to an apportionment using Jefferson's method. Determine who gets the new teachers.

$s =$

Class	Enrollment	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
Ceramics	785						
Painting	152						
Dance	160						
Theatre	95						
TOTAL							

The **quota condition** says that the number assigned to each represented unit must be the standard quota, q, rounded up or rounded down.

What information tells you that the quota condition was violated in this example?

Comparing Methods

Balinski and Young found that no apportionment method that satisfies the quota condition is free of paradoxes.

- Divisor methods are free of the paradoxes, but they can violate the quota condition.
- Hamilton's method may have paradoxes but does not violate the quota condition.

Sample Exam questions

Sample exam questions are likely to focus on performing all four apportionment methods and recognizing each of the four issues (three paradoxes and the quota condition).