CHAPTER 14: Apportionment
14.1 The Apportionment Problem

An ***apportionment problem*** is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an ***apportionment method***.

The total population, p, divided by the house size, h, is called the ***standard divisor, s***. $s = \frac{p}{h}$

A group's ***quota*** q_i is the group's population, p_i, divided by the standard divisor, s. $q_i = \frac{p_i}{s}$

Different apportionment methods will use different rounding rules.

When q is not already an integer, there are multiple ways to round.

- Round q up to the next integer, $\lceil q \rceil$.
- Round q down to the previous integer, $\lfloor q \rfloor$.
- Round to the nearest integer, $[q]$. If q is halfway to the next integer or larger, round up to the next integer. Otherwise, round down to the previous integer.
- Round according to the geometric mean. The geometric mean of $\lfloor q \rfloor$ and $\lceil q \rceil$ is $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$. If q is equal to or larger than q^*, round up to the next integer. Otherwise, round down to the previous integer.

Example

Complete the following chart.

4.48 4.6
4 4.5 5

q	Ceiling $\lceil q \rceil$	$\lfloor q \rfloor$ Floor	Nearest Int. $[q]$	$\sqrt{\lfloor q \rfloor \lceil q \rceil}$ q^*	Round according to q^*
6 (Already an integer, so no rounding needed)	6	6	6	$\sqrt{6 \cdot 6} = \sqrt{36} = 6$	6
4.6	5	4	5	$\sqrt{4 \cdot 5} \approx 4.4721$	5 $q > q^*$
4.5	5	4	5	$\sqrt{4 \cdot 5} \approx 4.4721$	5
4.48	5	4	4	$\sqrt{4 \cdot 5} \approx 4.4721$	5
4.47	5	4	4	$\sqrt{4 \cdot 5} \approx 4.4721$	4 $q < q^*$
0.2	1	0	0	$\sqrt{0 \cdot 1} = 0$	1 $q > q^*$

4.6 4.4721
4.5 4.4721
4.48 4.4721
4.47 4.4721

if $q > q^*$, round to $\lceil q \rceil$
if $q < q^*$, round to $\lfloor q \rfloor$

Steps 1, 2 same for all methods

14.2 Hamilton Method

Step 1 Compute the standard divisor.
Step 2 Compute the quota for each "state" (group).
Step 3 Round each quota *down.*
Step 4 Calculate the number of seats left to be assigned.
Step 5 Assign the remaining seats to the states with the *largest fractional part* of q.

Example
Three friends, Amy, Ben, and Cathy own businesses and decided to pool their resources to buy a “box” with 38 seats at a local sporting event. Use the Hamilton method to apportion the 38 seats if Amy pays \$6200, Ben pays \$1200, and Cathy pays \$10,300.

$s = \frac{\$17,700}{38 \text{ seats}} = \$465.79$ per seat

Largest fractional part (closest to next integer)

Person	Contribution	q	Rounded quota $\lfloor q \rfloor$	Hamilton Apportionment
Amy	\$6200	$\frac{6200}{465.79} \approx 13.3107$	13	13
Ben	\$1200	$\frac{1200}{465.79} \approx 2.5763$	2	+1 3
Cathy	\$10,300	$\frac{10,300}{465.79} \approx 22.1130$	22	22
TOTAL	17,700		37	38

38 − 37 = 1 seat left to give

Example

After Amy, Ben, and Cathy apportioned the tickets, they found out that there are actually 39 seats in the box. Reapportion the 39 tickets using the Hamilton method.

$s = \frac{\$17700}{39 \text{ seats}} \approx \$453.85/\text{seat}$

$\lfloor q \rfloor$

Person	Contribution	q	Rounded quota	Hamilton Apportionment
Amy	\$6200	$\frac{6200}{453.85} \approx 13.6610$	13	+1 14
Ben	\$1200	$\frac{1200}{453.85} \approx 2.6441$	2	2
Cathy	\$10,300	$\frac{10300}{453.85} \approx 22.6949$	22	+1 23
TOTAL	17,700		37	39

39-37=
2 seats left to assign

Example

A committee was forming to represent all four towns in the county. The population of each town is given below. If there are 79 representatives, how many representatives does each town receive?

$s = \frac{65440 \text{ people}}{79 \text{ rep}} \approx 828.3544$ people/rep

Largest Fractional part

Town	Population	$\frac{pop}{s}$ q	Rounded quota $\lfloor q \rfloor$	Hamilton Apportionment
Town A	32,300	$\frac{32,300}{828.3544} \approx 38.9930$	38	+1 39
Town B	18,640	22.5024	22	+1 23
Town C	14,300	17.2631	17	17
Town D	200	0.2414	0	0
TOTAL	65,440		77	79

79 - 77 = 2 rep left to distribute

14.3 and 14.4 Divisor Methods and Which Method is Best

We have used the standard divisor, s, to represent the average district population. We will use s for all apportionment methods to calculate the quota.

The divisor methods will also use an adjusted divisor, d, to calculate an adjusted quota. The adjusted quota combined with the appropriate rounding rules for each method will give the final apportionment for divisor methods.

Jefferson Method

Step 1 Compute the standard divisor.
Step 2 Compute the quota for each "state" (group).
Step 3 Round each quota *down*.
Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors, $\boxed{d_i = \frac{p_i}{N_i+1}}$, that correspond to giving each state one more seat.
Step 5 Assign a seat to the state with the *largest* d. (Notice that divisor methods look at the entire number of d rather than the fractional part of the number.)

Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, d.

Example
Let's use a different apportionment method to split the original 38 seats in the box. Use the Jefferson method to distribute the seats.

$$s = \frac{17{,}700}{38} \approx 465.789474$$

N

$\lfloor q \rfloor$ $d = \frac{pop}{N+1}$

Largest d_i (paying most per seat)

Person	Cont.	q	Rounded quota	d_i	Jefferson App.
Amy	\$6200	13.3107	13	$\frac{6200}{13+1} \approx 442.86$	13
Ben	\$1200	2.5763	2	$\frac{1200}{2+1} \approx 400.00$	2
Cathy	\$10,300	22.1130	22	$\frac{10,300}{22+1} \approx 447.83$	+1 23
TOTAL	\$17,700		37		38

38-37=1 seat left to distribute

$d =$ 447.83
If you use this adjusted divisor instead of s to compute $\frac{pop}{d}$, all of the quotas would round to the final apportionment

			$\lfloor q \rfloor$
Ex:	Amy	$6200 \div 447.83 \approx 13.8445$	13
	Ben	$1200 \div 447.83 \approx 2.6795$	2
	Cathy	$10,300 \div 447.83 \approx 23$	23

using all digits of d

Example

Also, use the Jefferson method to apportion the 79 representatives to the towns.

$s = \frac{65400}{79} = 828.354430$

N $\lfloor q \rfloor$ $d = \frac{pop}{N+1}$ Largest d_i N $d = \frac{pop}{\cdot}$ Largest d_i

Town	Pop.	$\frac{pop}{s}$ q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
A	32,300	38.9930	38	$\frac{32300}{38+1} \approx$ 828.21	+1 39	$\frac{32300}{39+1} \approx$ 807.50	39
B	18,640	22.5024	22	$\frac{18640}{22+1} \approx$ 810.43	22	810.43	+1 23
C	14,300	17.2631	17	$\frac{14300}{17+1} \approx$ 794.44	17	794.44	17
D	200	0.2414	0	$\frac{200}{0+1} =$ 200	0	200	0
TOTAL	65400		77		78		79

79 − 77 = 2 seats left to distribute
Must distribute one at a time because it is possible for same town to get both seats.

$d =$ 810.43

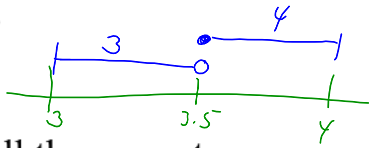
Webster Method

Same for all methods

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *to the nearest integer.*



Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.

If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{N_i+0.5}$.

If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{N_i-0.5}$.

Step 5 Adjust the seats according to *d*.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, d.

Example
Use the Webster method to distribute the 39 box seats.

$s = \frac{17,700}{39} \approx 453.846154$

N
[q]

$d^- = \frac{pop}{N - 0.5}$

remove from smallest d_i^-

Person	Cont.	q	Rounded quota	d_i	Webster App.
Amy	$6200	13.6610	14	$\frac{6200}{14-0.5} \approx 459.26$	14
Ben	$1200	2.6441	3	$\frac{1200}{3-0.5} \approx 480$	3
Cathy	$10,300	22.6949	23	$\frac{10,300}{23-0.5} \approx 457.78$	-1 22
TOTAL	**$17,700**		40		39

39 - 40 = 1
So remove one seat

$d =$ 457.78

Example

Use the Webster method to apportion the 79 representatives.

$s = \frac{65{,}440}{79} \approx 828.354430$

[q]

Region	Pop.	q	Rounded quota	d_i	Webster App.
Town A	32,300	$\frac{32300}{828.35} \approx$ 38.9930	39		
Town B	18,640	22.5024	23		
Town C	14,300	17.2631	17		
Town D	200	0.2414	0		
TOTAL	65,440		79		

No adjustment Needed

$d =$ 828.354430

Example
Use the Webster method to apportion the representatives if they decided to only have 78 representatives.

$$s = \frac{65{,}440}{78} \approx 838.974359$$

[q]

$d_i^+ = \frac{pop}{N + 0.5}$

Extra seat goes to town w/ largest d_i^+

Region	Pop.	q	Rounded quota	d_i	Webster App.
Town A	32,300	38.4994	38	$\frac{32,300}{38+0.5} \approx$ 838.9610	+1 39
Town B	18,640	22.2176	22	828.4444	22
Town C	14,300	17.0446	17	817.1429	17
Town D	200	0.2384	0	400	0
TOTAL	65,440		77		78

78 - 77 = 1
Need one more rep to apportion

$d =$ 838.9610

Hill-Huntington Method

The Hill-Huntington method does a great job of keeping the relative differences of representative share $\left(\text{i.e., } \frac{\text{apportionment}}{\text{population}}\right)$ and district population $\left(\text{i.e., } \frac{\text{population}}{\text{apportionment}}\right)$ stable between states. It also ensures that every group gets at least one representative, so it favors small states. Since 1941, the Hill-Huntington method with a house size of 435 has been used to apportion the House of Representatives.

same for all methods

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *according to the geometric mean* of $\lfloor q \rfloor$ and $\lceil q \rceil$, $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$.

Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.
If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{\sqrt{N_i(N_i+1)}}$.
If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{\sqrt{N_i(N_i-1)}}$.

Step 5 Adjust the seats according to d.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, d.

Example
Use the Hill-Huntington method to distribute the 39 box seats.

$$s = \frac{17{,}700}{39} \approx 453.846154$$

$\sqrt{\lfloor q \rfloor \cdot \lceil q \rceil}$ N $d_i = \frac{pop}{\sqrt{N(N-1)}}$

remove from person w/ smallest d_i

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Amy	\$6200	13.6610	$\sqrt{13 \cdot 14} \approx 13.4907$	$q > q^*$ so $\lceil q \rceil$ 14	$\frac{6200}{\sqrt{14(14-1)}} \approx 459.57$	14
Ben	\$1200	2.6441	$\sqrt{2 \cdot 3} \approx 2.4495$	$q > q^*$ so $\lceil q \rceil$ 3	$\frac{1200}{\sqrt{3(3-1)}} \approx 489.90$	3
Cathy	\$10,300	22.6949	$\sqrt{22 \cdot 23} \approx 22.4944$	$q > q^*$ so $\lceil q \rceil$ 23	$\frac{10{,}300}{\sqrt{23(23-1)}} \approx 457.89$	−1 22
TOTAL	**\$17,700**			40		39

39 − 40 = −1
so need to remove a seat

$d =$ 457.89

Example

The friends decided to give 4 tickets to mutual friends. Use the Hill-Huntington method to distribute remaining 35 box seats.

$s = \frac{17{,}700}{35} \approx 505.714$

$\sqrt{\lfloor q \rfloor \lceil q \rceil}$

$d_i^{+} = \frac{pop}{\sqrt{N(N+1)}}$

Extra seat to person with largest d_i^{+}

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Amy	$6200	12.2599	$\sqrt{12 \cdot 13} \approx 12.4900$	$q < q^*$ so $\lfloor q \rfloor$ 12	$\frac{6200}{\sqrt{12(12+1)}} \approx 496.40$	12
Ben	$1200	2.3729	$\sqrt{2 \cdot 3}$ 2.4495	$q < q^*$ so $\lfloor q \rfloor$ 2	$\frac{1200}{\sqrt{2(2+1)}} \approx 489.90$	2
Cathy	$10,300	20.3672	$\sqrt{20 \cdot 21}$ 20.4939	$q < q^*$ so $\lfloor q \rfloor$ 20	$\frac{10{,}300}{\sqrt{20(20+1)}} \approx 502.59$	+1 21
TOTAL	$17,700			34		35

35 - 34 = 1 ticket left to apportion

$d = 502.59$

Example
Use Hill-Huntington to distribute the 78 county representatives.

$$s = \frac{65{,}440}{78} \approx 838.974359$$

$\sqrt{\lfloor q \rfloor \cdot \lceil q \rceil}$ N $d_i = \frac{pop}{\sqrt{N(N-1)}}$

remove from town with smallest d_i

Region	Pop.	q	q^*	Rounded quota	d_i	HH App.
Town A	32,300	38.4994	$\sqrt{38\cdot 39} \approx$ 38.4968	$q > q^*$ so $\lceil q \rceil$ 39	$\frac{32{,}300}{\sqrt{39(39-1)}} \approx 839.03$	−1 38
Town B	18,640	22.2176	$\sqrt{22\cdot 23} \approx$ 22.4944	$q < q^*$ so $\lfloor q \rfloor$ 22	$\frac{18{,}640}{\sqrt{22(22-1)}} \approx 867.21$	22
Town C	14,300	17.0446	$\sqrt{17\cdot 18} \approx$ 17.4929	$q < q^*$ so $\lfloor q \rfloor$ 17	$\frac{14300}{\sqrt{17(17-1)}} \approx 867.06$	17
Town D	200	0.2384	$\sqrt{0\cdot 1} \approx$ 0	$q > q^*$ so $\lceil q \rceil$ 1	$\frac{200}{\sqrt{1(1-1)}}$ is undefined	1
TOTAL	65,140			79		78

78 − 79 = −1
so need to remove a rep

$d =$ 839.03

A ***paradox*** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

Possible Issues - Alabama Paradox (Section 14.2)

The ***Alabama paradox*** occurs when a state loses a seat as the result of an increase in the house size.

Example

Use the information from pages 3 and 4 to see how many seats Amy, Ben, and Cathy received when they thought there were 38 tickets and when they thought there were 39 tickets in the box using the Hamilton method.

Person	Contribution	38 tickets Hamilton Apportionment	39 ticket Hamilton Apportionment
Amy	$6200	13	14
Ben	$1200	3	2
Cathy	$10,300	22	23
TOTAL		38	39

What information tells you that the Alabama paradox occurred in this example?

when they got an "extra" ticket to distribute,
Ben recived fewer tickets than he had before.

<u>Possible Issues - Population Paradox (Section 14.2)</u>

Consider two numbers, A and B, where $A > B$.
The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\frac{A-B}{B} \times 100\%$

The ***population paradox*** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Example

We have 100 council members to apportion to four districts. The population and the Hamilton Apportionment are given for the previous census and for the latest census.

$\frac{\text{Abs diff}}{\text{smaller}} \times 100\%$

Seats	District	Prev. Pop.	Latest Pop.	Prev. Ham. App.	Latest Ham App.	Absolute Difference	Relative Difference
+1	North	27,460	28,140	42	43	28140 - 27460 = 680	$\frac{680}{27460} \times 100\% \approx$ 2.476 %
−1	South	17,250	17,450	27	26	17450 − 17250 = 200	$\frac{200}{17250} \times 100\% \approx$ 1.159 %
−1	East	19,210	19,330	30	29	19330 − 19210 = 120	$\frac{120}{19210} \times 100\% \approx$ 0.6247 %
+1	West	1000	990	1	2	1000 − 990 = 10	$\frac{10}{990} \times 100\% \approx$ 1.0101 %

West actually lost people

Did the population paradox occur? Yes

Explain what information helped you determine whether or not the population paradox occurred.

South and East both lost a seat even though they both had a larger relative increase than West who gained a seat.
(In fact, West even lost population but gained a seat).

Possible Issues – New States Paradox (Section 14.2)

The ***new states paradox*** occurs in a reapportionment in which an increase in the total number of states (with a proportionate increase in representatives) causes a shift in the apportionment of existing states.

Example

A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress

$s = \frac{203,995}{12} \approx 16999.58$

$\lfloor q \rfloor$

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899	$\frac{144,899}{16999.58} \approx 8.524$	8	+1 9
Liquid	59,096	$\frac{59,096}{16999.58} \approx 3.476$	3	3
TOTAL	203,995		11	12

12 − 11 = 1 seat left to apportion

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

$\frac{pop}{s} = \frac{38,240}{16999.58} \approx 2.25$ so we will add 2 rep.

Use Hamilton's method to apportion the seats for their congress (the 12 original seats plus the 2 additional seats that were added when Plasma joined).

$$s = \frac{242{,}235}{\underbrace{12}_{\text{original}} + \underbrace{2}_{\text{new}}} = \frac{242{,}235}{14} \approx 17302.5$$

State	Population	$\frac{pop}{S}$ q	Rounded quota (L q J)	Hamilton Apportionment
Solid	144,899	$\frac{144{,}899}{17302.5} \approx 8.374$	8	8
Liquid	59,096	3.415	3	+1 4
Plasma	38,240	2.210	2	2
TOTAL	242,235		13	14

14 − 13 = 1
left to apportion

What information tells you that the new states paradox occurred in this example?

We added a new state (Plasma) and the proportionate number of representatives, but a state (Solid) lost a seat in the process. The existing apportionment shifted when the new state was added.

Possible Issues – Quota Condition (Section 14.3)

Example

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired according to an apportionment using Jefferson's method. Determine who gets the new teachers.

$s = \frac{1192}{10} = 119.2$ N $\lfloor q \rfloor$ $d^+ = \frac{pop}{N+1}$

Lower / Upper $\lfloor q \rfloor$ $\lceil q \rceil$ # seats expected by Quota condition

Lower/Upper	Class	Enrollment	q ($\frac{pop}{s}$)	Rounded quota	d_i	Next App. (Largest d_i^+)	Next d_i	Jefferson App. (Largest d_i^+)
6 or 7	Ceramics	785	$\frac{785}{119.2} \approx 6.586$	6	$\frac{785}{6+1} \approx 112.14$	+1 7	$\frac{785}{7+1} \approx 98.13$	+1 8
1 or 2	Painting	152	1.275	1	$\frac{152}{1+1} = 76$	1	76	1
1 or 2	Dance	160	1.342	1	$\frac{160}{1+1} = 80$	1	80	1
0 or 1	Theatre	95	0.797	0	$\frac{95}{0+1} = 95$	0	95	0
	TOTAL	1192		8		9		10

$10 - 8 = 2$ seats left to apportion

The ***quota condition*** says that the number assigned to each represented unit must be the standard quota, q, rounded up or rounded down.

What information tells you that the quota condition was violated in this example? The number of seats for Ceramics is larger than $\lceil q \rceil$.

Comparing Methods

Balinski and Young found that no apportionment method that satisfies the quota condition is free of paradoxes.

- Divisor methods are free of the paradoxes, but they can violate the quota condition. That is why they can only add one seat at a time.
- Hamilton's method may have paradoxes but does not violate the quota condition.

Sample Exam questions

Sample exam questions are likely to focus on performing all four apportionment methods and recognizing each of the four issues (three paradoxes and the quota condition).